Introduction to Probability Theory and Stochastic Processes
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Lecture – 79

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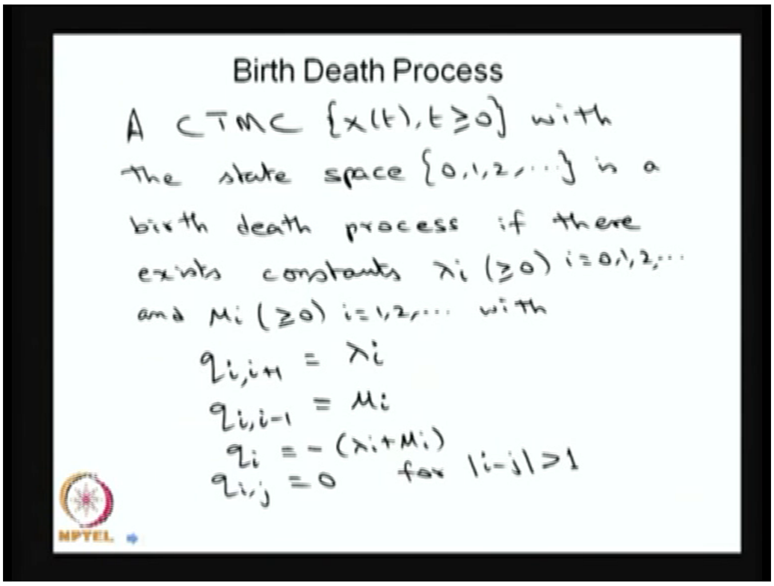

Now I am moving into the special case of a Continuous time Markov Chain that is a birth death process. This is a very important time homogeneous a continuous time Markov chain, because many of the scenario can be mapped with the birth death process either with the finite state, or infinite state. Let me first give the definition of a birth death process.

I started with continuous time Markov chain it is a time homogeneous continuous time Markov chain, with the states space countably infinite it can be a finite also, that CTMC is going to be call it as a birth death process, if there exists a constant lambda is and mu is such that and these are all nothing, but the infinite decimal generator matrix elements. And this is i 2 i plus 1 that rate is always lambda i and the rate in which the system is moving from the state i to i minus 1 that rate is a mu i.

And the diagonal elements are minus of lambda i plus mu i whereas, all the other rate rates the system is moving from the state i to j, other than i to i plus 1 i to i minus 1 and i to i and all other rates are it is always 0, absolute of i minus j is greater than 1. That means, you will have the infinite decimal generator matrix in which, you will have only have a diagonal, tridiagonal matrix and all other elements are going to be 0.

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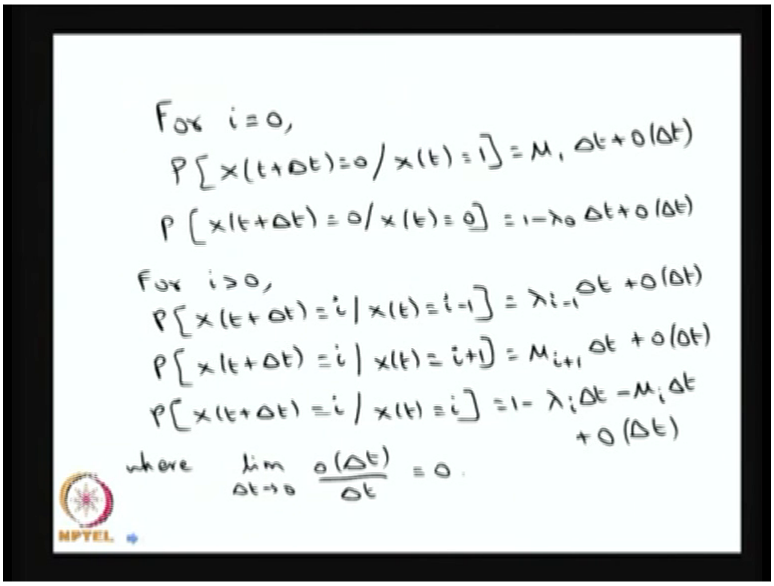

I can write down the condition so, that it land up a the rates are going to be only lambda is and mu is so on not all other rates are going to be 0. So, if I start with the i is equal to 0 the system is moving from the state 1 to 0 in the interval of delta t, because it is a time homogeneous model. So, this is nothing, but this probability the system is moving from the state 1 to 0 in the interval of delta t that is nothing, but the rate is a mu 1 times delta t plus order of delta t.

Similarly, the system is moving from the state 0 to 0 from the time t to t plus delta t or during the interval delta t that is nothing, but 1 minus lambda naught times delta t plus order of delta t. So, these mu is and the lambda naught and so on these values are always going to be greater or equal to 0 strictly greater than 0 also. For i is greater than 0 the system is moving from the state i to i that is 1 minus lambda i times delta t minus mu i times delta t plus order of delta t, whereas, the system is moving from i plus 1 to i one step backward that is mu i plus 1 delta t.

The system is moving from the state i minus 1 to i for i is greater than 0 that is a forward one step move, that is lambda times i minus 1 delta t plus order of delta t, these order of delta t it may be a function of delta t it need not be the same, as a t tends as delta t tends to 0 this quantities are going to be 0, order of delta t divided by delta t is going to be 0.

Therefore, this is the way the system is moving from the one state to either one step forward, or either one step backward, or move anywhere. So, these are all the only three possibilities with these probabilities.

Therefore we land up the q matrix is going to be the system is moving from the state i to i plus 1 forward one move, that rate is lambda is and the system is moving from the i to i minus 1, one step backward that is mu i, or the system being in the same state that rate is minus lambda i plus mu i. Therefore, there is no other move from the system from one state to all other states, either one step forward or one step backward.

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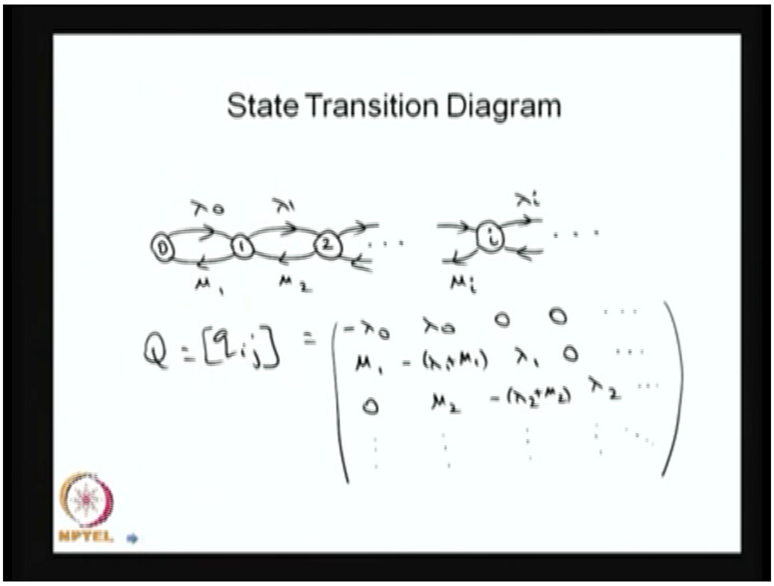

So, this can be visualized in the state transition diagram, since I started with the state space 0 to infinity, there is a possibility you can have a label from a some a negative integers to the positive integer. So, you can always a transform into something therefore, default scenario or the simplest 1 I discussed from 0 to infinity therefore, you can visualize whatever be the label that can be transfer in a one to one fashion.

So, this is the rate in which the system is moving from the state 0 to 1, that rate is lambda naught, the system is moving from the state 1 to 2, that rate is lambda 1 or the system is moving from the state 1 to 0 that rate is mu 1. Therefore, the time spent in the state 1, before moving into any other states that is a minimum of the time spending in the state 1 before moving into the state 2, or the system time spending in the state 1 before moving into the state 0.

So, both are exponentially distributed with the parameters lambda 1 and mu 1 and the minimum of that time is the spending time, or the waiting time in the state 1, that is going to be exponential distribution with the parameter lambda 1 plus mu 1, because both are independent. The time spending in the state 1 before moving into the state 2 and similarly the time spending in the state 1 before moving into the state 0 and both the random variables are independent that is the assumption. Therefore, it is going to be a exponentially distributed random in the time spending in the state 1, that is exponential distributed with the parameter lambda 1 plus mu 1, like that you can discuss for all other states.

So, whenever you have a see birth death process the system, either move one step forward or one step backward, then it is called a birth death process. Therefore, here these are lambda is are called the system is moving from one state to forward one step therefore, this lambda is are called birth rates. The system is moving from one state to the previous one state and the corresponding rates mu is mu 1 mu 2 mu 3 and so on and these rates are going to be call it as a death rates.

So, lambda is are nothing, but the lambda is are nothing, but the birth rates; that means, the rate in which the system is moving from the state i to i plus 1 that depends on i therefore, that rate is lambda i. The system is moving from the state i to i minus 1 that is related to the death by 1, that is a function of i therefore, that death rate is mu i. So, the lambda is are the birth rates and the mu i's are the death rates.

Therefore suppose example the system moving from the state 2 to 1, the death rate will be mu suffix 2. So, you can fill up the q matrix if you see the q matrix, it is a tridiagonal matrix.

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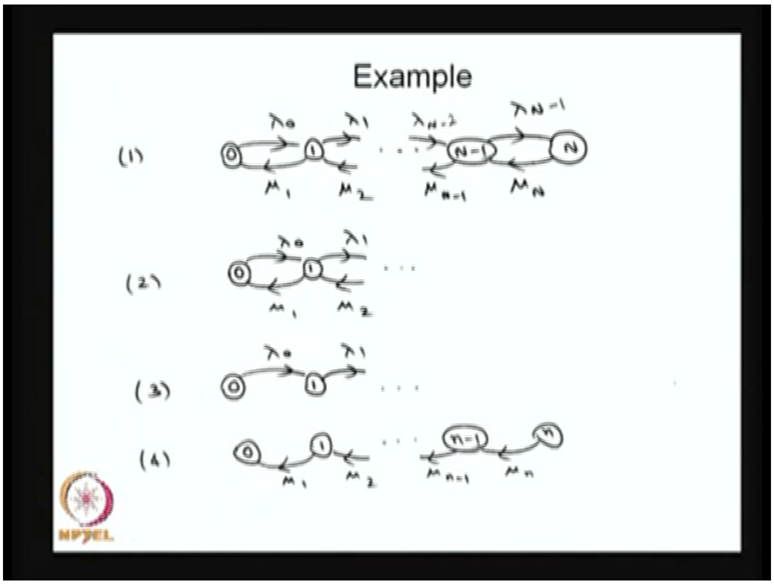

So, here I am giving few examples of for the birth death process, the first example consist of a, the first example is a finite state model. The birth rates are lambda naught lambda 1 till lambda n minus 1, in the death rates are mu 1 mu 2 and mu n, it is a finite state birth death process.

The second example is the infinite state birth death process, the third example the all the death rates are 0 that is also possible, the fourth example all the birth rates are 0 that is also possible, but you can discuss the one can discuss the state classification also. The first one all it is a finite state model all the states are communicating with all other states. Therefore, it is a irreducible positive recurrent birth death process.

The second one say infinite state all the states are communicating with all other states which are reducible, but one cannot conclude without knowing the values about the lambda naughts and lambda i's and the mu i's one cannot conclude it is a positive recurrent, or null recurrent. If the mean recurrence time that is going to be a finite one then you can conclude it is a positive recurrent otherwise it is null recurrent. So, as such we one cannot discuss now, the positive recurrent or null recurrent, but you can conclude it is a recurrent state.

The third example the system is keep moving forward therefore, all the states are transient states, it is not a irreducible it is a reducible model all the states are transient states; that means, has a t tends to infinity the system will be in the some infinite state.

So, one cannot define a infinite state therefore, the limiting distribution would not exist in this situation.

The fourth example it is a finite model, but all the states are not communicating with all other states therefore, it is a not a irreducible it is a reducible model, whenever the system starts from some state other than 0 over the time the system is keep moving backward and once it reaches the state 0 it will be forever therefore, state 0 is a observing barrier.