Introduction to Probability Theory and Stochastic Processes
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Lecture – 78

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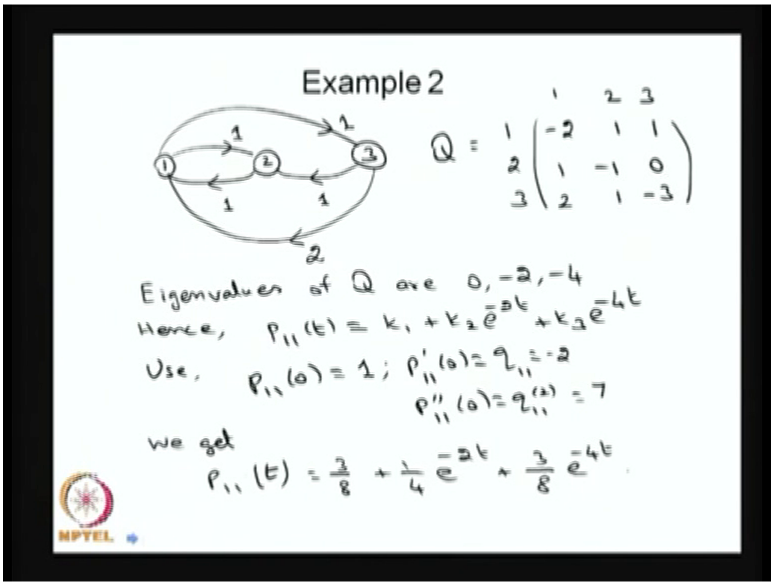

I am going to give one more example, this has three states and this is a state transition diagram and the values are nothing, but the rates in which the system is moving from one state to other states. So, that is the difference between the state transition diagram of DTMC and the CTMC. So, this is the rate in which the system is moving from one state to another state and some arcs are not there; that means, there is no way the system is moving from the state 2 to 1, 2 to 3 in a small interval of time whereas, all the other possibilities are I have given.

So, the corresponding Q matrix it is a 3 cross 3 matrix and, you can make out all the row sums are going to be 0 and, the diagonal elements are minus of sum of other values in the same rows. And the other than the diagonal elements with the values are greater than or equal to 0, my interest is to find out the time dependent solution for this example also, I can make a forward Kolmogorov equation P dash of t is equal to P of t times q it is a 3 cross 3 matrix therefore, I will have a 3 equations.

And I have one equation I can have a summation of probability is equal to 1 and, I can start with the initial condition the system being in the state 1 at time 0 that probability is 1, I can start with that and I can solve those 3 equations, with the initial condition and I can get the solution that is a one way.

Since it is a finite state CTMC, there are many ways to get the time dependent solution basically you have to solve the system of a difference or differential equations, with the initial condition. Here I am using the eigenvalue method; that means, a find the eigenvalues for the q matrix therefore, use the eigenvalue and eigenvector concept and get the P 1 1 of t with the unknown k 1 k 2 k 3.

And to find the unknowns of k 1 k 2 k 3 use the initial condition, here I am using the initial condition as well as the q matrix values the q 1 1 that means, the element corresponding to the 1 comma 1 that is nothing, but the P dash of 1 comma 1 of 0; similarly if I go for q square matrix and q 1 1 of 2 the element in the 1 comma 1, in the q square matrix that is nothing, but P double dash of 1 comma 1 0.

Therefore, now I can use these 3 initial conditions to get the unknowns value k 1 and k 2 k 3 k 1 comma k 2 and k 3 so, once I know the k 1 k 2 k 3, I can substitute therefore, the P 1 1 of t is equal to this much. Similarly I can go for finding the P 1 comma 2 of t and P 1 comma 3 of t, I do not want to P 1 comma 3 in the same way, because once I know the P 1 comma 1 of t and P 1 1 comma 2 comma t so, 1 comma 3 comma t is nothing, but 1 minus of that those two probabilities, because the summation of probability is equal to 1. So, this is the other way of getting the time dependent solution in the transition probability of system being in the state j, given that it was in the state I at time 0.

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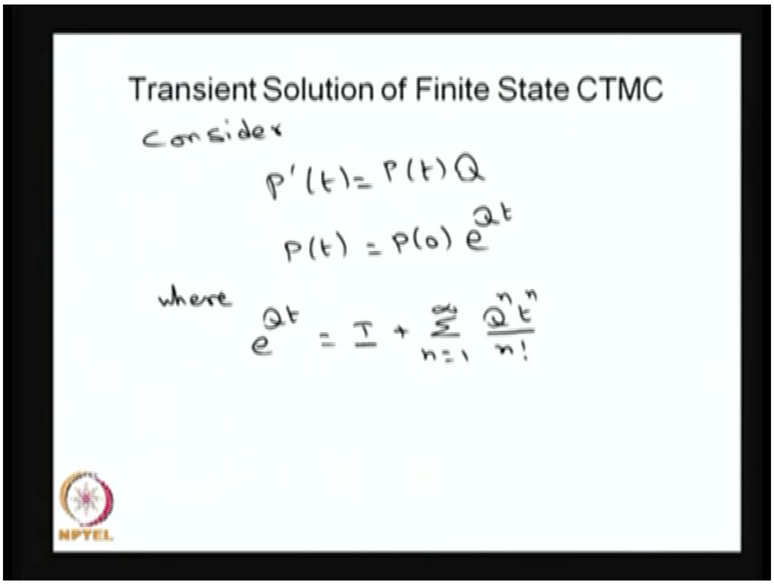

Suppose the CTMC has the finite state space, then I can use the exponential matrix also to get the time dependent solution that is what I have given this way.

So, start with the forward equation therefore, the solution is going to be P of t is equal to P of 0 e power Q of t P of t is a matrix, P of 0 is the matrix e power Q t that is also again going to be a matrix exponential matrix therefore, I am writing a e power Q t is nothing, but Q is the matrix and the t is the real value. So, greater than or equal to 0 therefore, e power Q t is going to be the I matrix, I matrix is nothing, but the identical matrix of order, whatever the state space number plus the summation I is equal to n is equal to 1 to infinity of Q power n times t power n divided by n factorial.

So, that the whole thing is going to be the exponential matrix and, using that you can get the P of t means, I am not going detailed for how to compute this e power Q t and so, on but whenever you have a CTMC with a finite space through this method also, one can get the time dependent solution. So, with this I have completed the examples for the CTMC, to find out the time dependant or transition probabilities.

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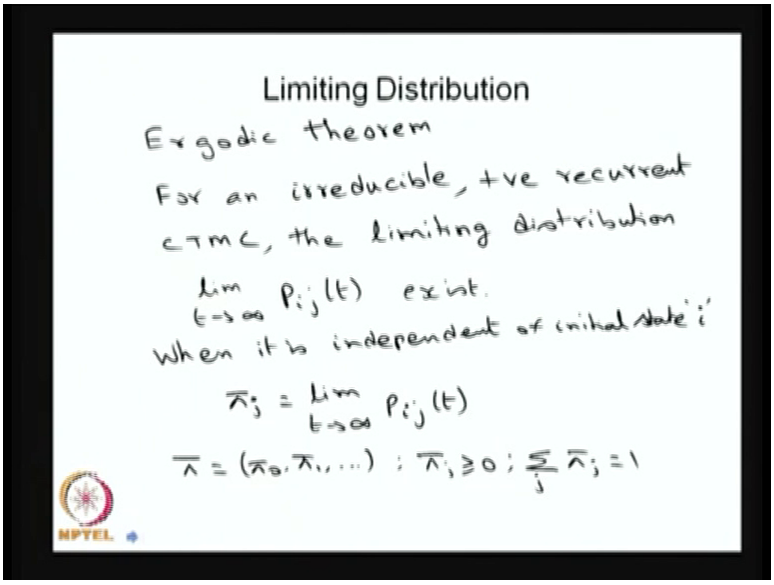

Now, I am moving into the limiting distribution, the way we discussed the limiting distribution for the CTMC, the same concept can be used to further CTMC also. The change is a instead of the one step transition probability matrix here, we have to use the infinitesimal generator matrix in a different way.

So, I am first giving the Ergodic theorem, whenever the CTMC is irreducible; that means, all the states are communicating with the all other states. Since all the states are communicating with all other states, if 1 is of the particular type its opposed to recurrent, then all the other states are going to be a positive recurrent, if 1 is going to be a null recurrent, then all the other states also going to be a null recurrent.

So, here I am making the assumption the CTMC is a irreducible, as well as all the states are positive recurrent. Then the limiting distribution always exists, suppose it is a independent of a initial state, it need not be a independent of initial state suppose the same thing is independent of initial state, then I can write that limiting probability is P i j of t, since it is independent of i, I can write it as the i j, then I can form a vector.

And since it is a limiting distribution it is a probability distribution therefore, the probabilities are these probabilities are always greater than or equal to 0 and the summation of probability is going to be 1, it would not be defective it would not be less than 1, that is a Ergodic theorem says, whenever you have a irreducible CTMC with all the states are positive recurrent. Then has a t tends to infinity the system, as the distribution limiting distribution. If it is independent of initial state, then you can label with the pi j as a probabilities and this probability distribution satisfies it is a probability mass function therefore, it satisfies the probability mass function conditions.

That means whenever you have a dynamical system, in which it is a irreducible model and all the states are positive recurrent; that means, the mean recurrence time is going to be a finite value, then that system is call it as a ergodic system, or the ergodic concept can be used therefore, as a t tends to infinity you can get the limiting distribution.

If it is independent of initial state means whatever be the seed you are going to do it for the discrete even simulation, for the dynamical system that is a ergodic for a ergodic system, then the initial condition c does not matter to get the limiting distribution. Later we are going to give some few examples how to find out the limiting distribution.

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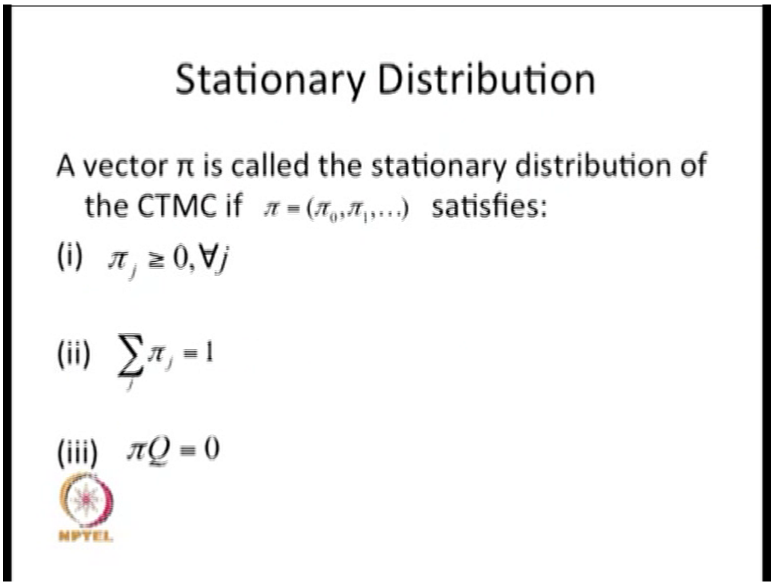

I am explaining the stationary distribution also, the stationary distribution the way I have explained the DTMC sorry, in the way I have discussed the DTMC the CTMC also same. So, I have a vector if the vector satisfies these three conditions probabilities therefore, greater than or equal to 0 summation is equal to 1 and, you should able to solve this equation and get the pi’s it is a homogeneous equation.

So, you need a second condition to have the non-zero probabilities. So, if you solve pi Q is equal to 0 along with the summation of pi j is equal to 1 and, if this pi j’s exist, then the CTMC has the stationary distribution. The similar way I have discussed the stationary distribution for the DTMC model also, instead of pi Q is equal to 0 we had a pi P is equal to pi.

So, if any vector satisfies that pi P is equal to pi and summation of pi a is equal to 1 and all the pi's are greater than or equal to 0, then that is going to be a stationary distribution for DTMC, the same way if pi q is equal to 0 and pi summation of a pi j is equal to 1 pi js are greater than or equal to 0, if this is satisfied by any vector, then that is going to be the stationary distribution for a time homogeneous CTMC, every time we are discussing the default CTMC that is a time homogeneous CTMC.

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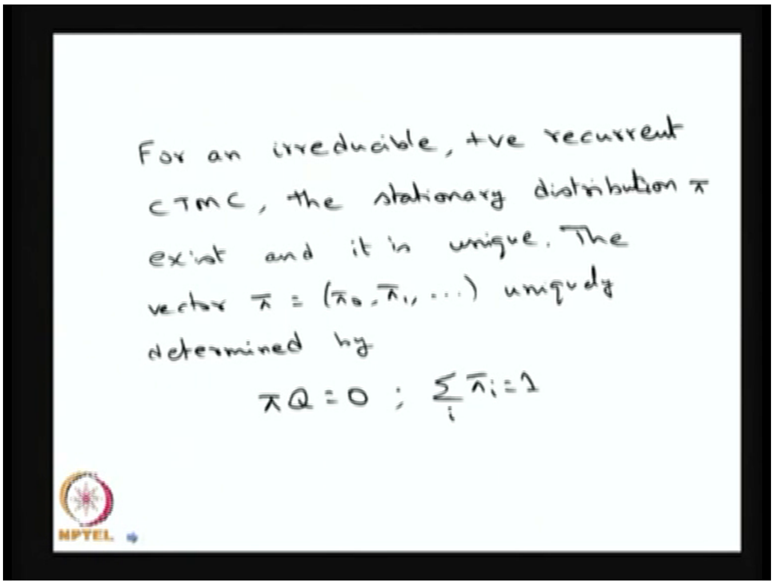

The main result for the stationary distribution, whenever you have a irreducible positive recurrent CTMC. The stationary distribution exists and that is going to be unique, whenever the CTMC is a positive recurrent as well as a irreducible. There is no need of a periodicity in the CTMC whereas, the same stationary distribution the stationary distribution for the DTMC, we have included one more condition that is a periodic.

But for the CTMC there is no periodicity for the state therefore, as long as the system has a system is irreducible and a positive recurrent 1, then the stationary distribution exists and it is unique. And by solving these equations you can get the unique stationary distribution.