Introduction to Probability Theory and Stochastic Processes
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Lecture – 77

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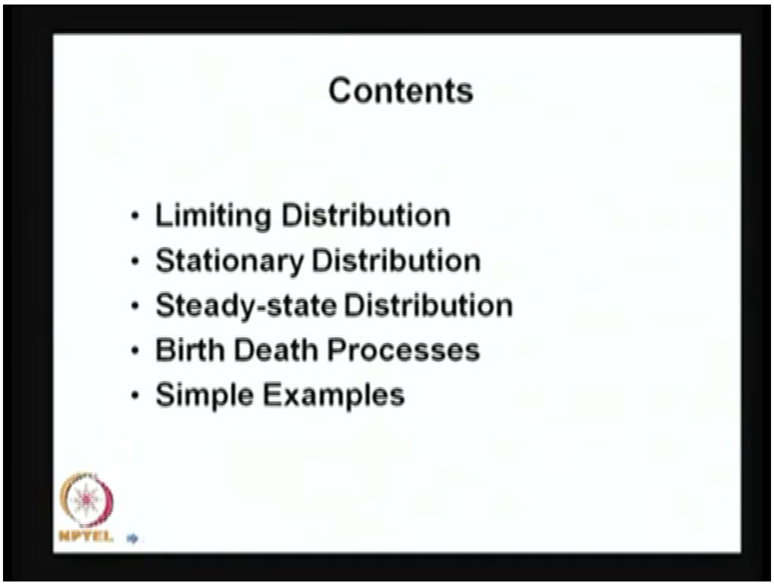

In the lecture 2, I am planning to discuss the Limiting Distribution, Stationary Distribution, and a Steady state Distribution; followed by that I am planning to give a description about the birth death processes, and also some Simple Examples for the limiting distribution, stationary, steady state distributions and birth death processes.

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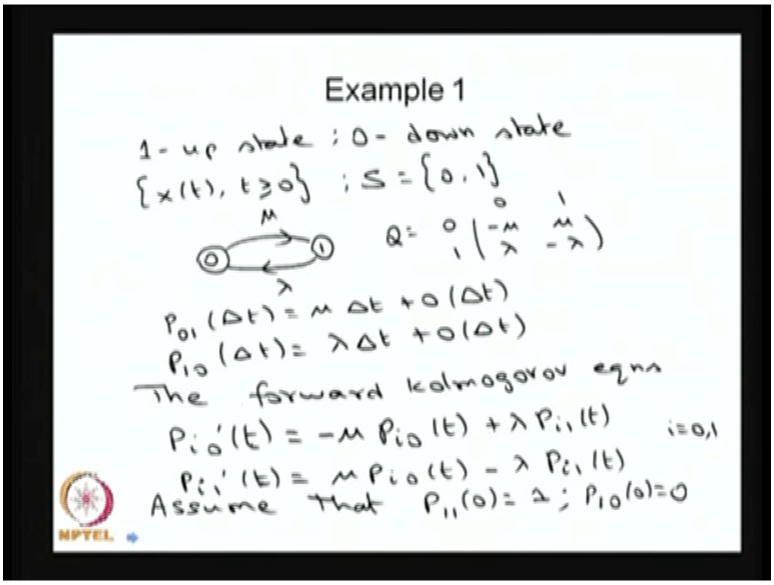

Before I go to the limiting distribution, let me explain the let me give the example for the continuous time Markov chain to get the time dependent solution.

This example is the very simplest example that is, a 2 states continuous time Markov chain, the default one is a time homogeneous. The state space are 1 and 0, 1 you can consider as a up state or a operational state and 0 is a down state non operation state. So, this can be visualized for the any model in which the whole dynamics can be described with the 2 state and the Markov property is satisfied.

The system going from the state 1 to 0 or the time spending in the state 1 before moving into the state 0 that is, a exponentially distributed with the parameter lambda. Once it is failed; that means, the system is in the down state, the time spent in the repair time that is exponentially distributed with the parameter mu. So, once the repair is over the system is operation state therefore, it is in the upstate. So, is the 0 is related to the down state and 1 is related to the upstate.

And the mu is nothing but the mean 1 by mu is the mean time for the repair and 1 by lambda is the mean time of a failure. And the failure time is exponentially distributed with the parameter lambda and the repair time is exponentially distributed with the parameter mu. This is a state transition diagram for the 2 state CTMC. The corresponding a Q matrix, the infinitesimal generator matrix; that it consists of a it is a 2 cross 2 matrix, the system going from the state 0 to 1 that rate is mu. The system is going from the state 1 to 0 that rate is lambda. And the diagonal values are minus of summation of other values that row rows.

So, 0 to 0 is minus mu and a1 to 1 is a minus lambda therefore, the rates are in the other than diagonal elements and the diagonal elements are minus of sum of the row values other than that a diagonal element. So, this is nothing but in a very small interval of time delta t the system is moving from the state 0 to 1, that probability the probability of system moving from the state 0 to 1, that this is nothing but the downstate to the upstate in a very small interval of time delta t, why you are finding; why you are finding the probability of delta t, since the model is a time homogeneous only the interval is matter not the actual time or you can visualize this as the sometime t to t plus delta t also.

So, this is the interval of a delta small negligible interval delta t, the system is moving from the state a 0 to 1 that probability is nothing but the rate mu is a rate the rate is nothing but the repair rate. So, the mean rate a mu times the delta t plus order of delta t. It is a small o; order of delta t means; as a delta t tends to 0 the order of delta t will be 0. Similarly, you can visualize the probability of system moving from the state 1 to 0 in the interval delta t in a small interval delta, t that is; same as the failure rate lambda times the delta t that is a small interval of time plus order of delta t.

So, this order of delta t also tends to 0 as delta t tends to 0. So, using this I can make the forward Kolmogorov equation, I can go for writing a forward Kolmogorov equation or backward Kolmogorov equation, but forward Kolmogorov equation is easy to make out. So, if the system is in the state I at time 0, what is the net rate? The system will be in the state 1 at the time t. That net rate is nothing but what are all the inflow that probability rate minus what are all the outflows. So, that is the way you can visualize the right hand side.

So, all the positive plane terms are related to the incoming rates and the all the negative terms related to the outgoing rates. So, since it is a 2 state model, if the system is in the state 0 at time t there is a possibility it, it is not moved anywhere from the state 0 or it would have come from the state 1 therefore, the incoming will be the state one therefore, the system will be in the state one at a time t. And a starting from given that the starting from the state i that probability multiplied by the rate sort of inflow minus.

Because we are writing the equation for the state 0 therefore, it is not moved from the state 0 that is, a with the rate mu it can move to the state 0 to 1 therefore, minus mu times it does not move from the state 0 and therefore, minus mu times the probability of being in the state 0 at time t given that, it was in the state i at time 0, that probability multiplied by minus mu that is a outflow and a lambda times a Pi 1 t that is a inflow. Therefore, in the left hand side it is a derivative of the function t it is a probability function.

So, Pi 0 dash t that is nothing but the net rate being in the system at time 0, sorry at time t in a state 0, given that it was in the state I at time 0 that; net rate is same as a inflow minus outflow with the corresponding rates. Similarly, you can write the equation for the state 1 ; that means, you start from the state 1 either you would have move you would have come from the state 0 to the 1 or in you didn't move from the state 1.

Therefore, minus lambda times Pi 1 of t plus mu times Pi 0 of t that is the net rate are corresponding to the state 1. So now, we are able to write the forward Kolmogorov equation. So, this is the interpretation of the forward Kolmogorov equation. You can write easily by making a matrix a Pij of t dash that is equal to P of t times q, where q is the infinitesimal generator matrix, then also you will get the same thing.

So, I am just giving the interpretation, now my interest is to find out the time dependent or transient solution for the these 2 states CTMC, for that this is a difference of differential equation. We need a initial condition to solve these equations. So, I make the assumption at time 0 the system is in the state 1. Therefore, the transition probability of system the Pi P11 of 0 that is equal to 1. Since, I made the assumption the system was in the state 0 at sorry, the state 1 at time 0 therefore, that the being in the state 0 that is going to be 0.

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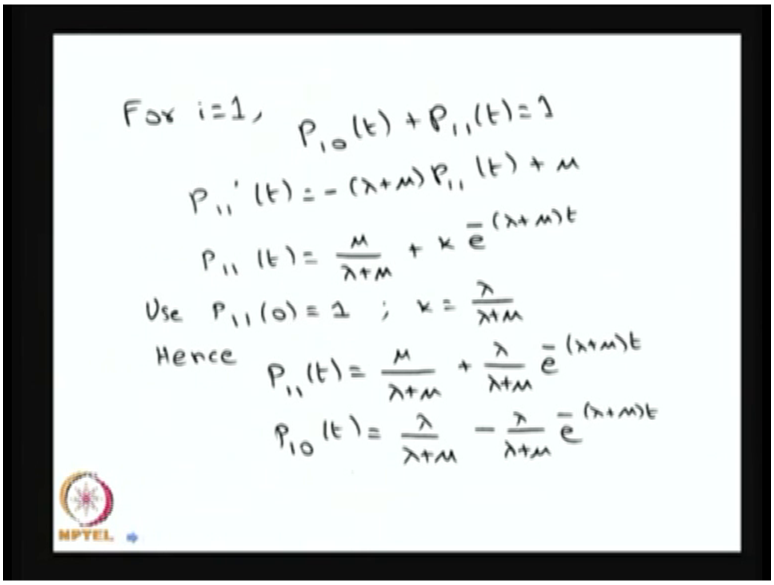

So, I need this both the initial conditions to solve the equation. So, let me start since I made the initial condition state is 1 therefore, i is equal to 1. So, I will have the first equation that is a I always have the summation of the probability at time t, these are transition probabilities are going to be 1 the summation. And also I have 2 difference of differential equations. So, what I can do; I can take the second equation in this then instead of a P 1 0 of t, I can use the summation of probability is equal to 1 therefore, the instead of a P 1 0 of t I can use the P 1 0 of t is nothing but a 1 minus a P 11 of t, I can substitute in the second equation.

Therefore, I will get a P 1 1 dash of t is equal to minus lambda plus mu times P 1 1 of t plus mu substituting P 1 0 of t is equal to 1 minus P 1 1 of t in the second equation the previous slide. Now I have to solve these a differential equation, the unknown is a P 1 1 of t, conditional probability. I have to use the initial condition P 1 1 of 0 is equal to 1. Using that I can get I will get a P 1 1 of t is equal to mu divided by lambda plus mu plus some constant, e power minus lambda plus mu times t. That constant I can find out using this initial condition therefore, k is equal to lambda divided by lambda plus mu.

So, the P 1 1 of t is equal to substituting k is equal to lambda divided by lambda plus mu in this equation, I will get the P 1 1 of t. Once I know the P 1 1 of t use the first equation. So, I will get P 1 0 of t is equal to 1 minus P 1 1 of t therefore, P 1 0 of t that is equal to this expression. You can cross check now if you add both the equations you will get a 1. And if you put a t equal to 0 you will get the initial condition also correctly.

And if you put a t tends to infinity that we are going to discuss in the limiting distribution, if you put t tends to infinity in this expression you will get a mu divided by lambda plus mu lambda divided by lambda plus mu. So, this is for the t tends to infinity therefore, if you make a matrix the limit n tends to infinity of a limit, if you find out the limiting distribution of a limit t tends to infinity of P of t.

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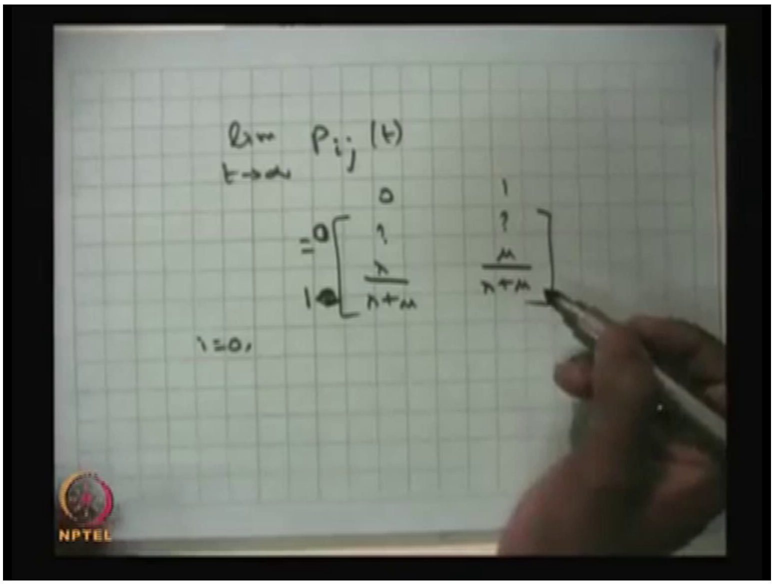

So, you will get the matrix and the this matrix as a t tends to infinity for this example it is a 2 cross 2 matrix. And that consists of for different values you will have for now we are doing for the second row therefore, that is equal to lambda divided by lambda plus mu ah sorry, and this is equal to mu divided by lambda plus mu. So, if the system start from the state 1, at a t tends to infinity the system will be in the state 0 with the probability lambda divided by lambda plus mu. And the system will be in the state 1 with the probability mu divided by lambda plus mu.

Similarly, if you go for i is equal to 0 you will get the same derivation and you can fill up what is the element here.

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$$\begin{bmatrix} \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \\ \frac{\lambda}{\lambda+\mu} & \frac{\mu}{\lambda+\mu} \end{bmatrix}$$

So, this is the limiting distribution probability matrix, and if you see that the rows are going to be identical. So, you will have the same identical rows in this row also so; that means, you will get the limiting distribution. I will discuss the limiting distribution in the after giving the one more example I will explain in detail. So, this is the transition probability system starting from the state 1 and being in the state 1 or 0 at a time t.