**Introduction to Probability Theory and Stochastic Processes**
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**Lecture – 76**

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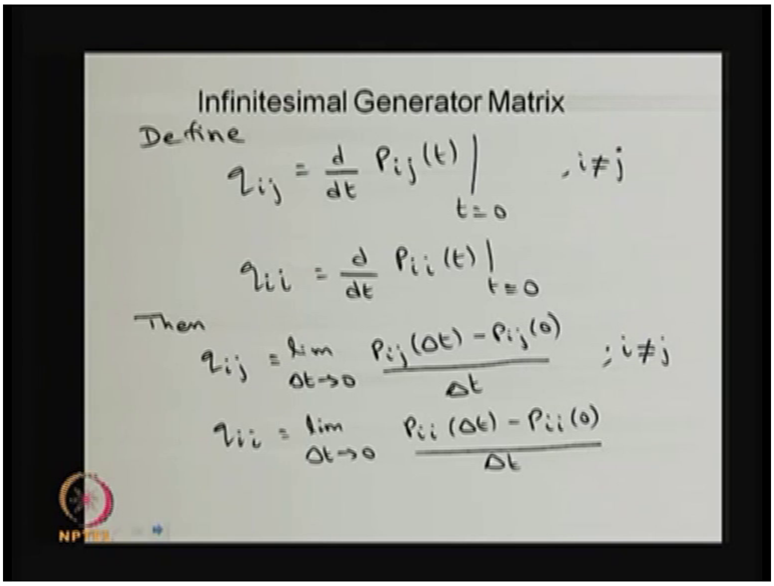


I am going to define the quantity called the qij, and later this is going to form a matrix that is going to be call it as a Infinitesimal Generator Matrix. So, let me start with the definition, qij that is nothing but take a derivative of Pij of t. That is a function of t you can find out the derivative, it is a it is differentiable function only.

So, you take a derivative then substitute t equal to 0 for all i not equal to j. Then you define qii that is also in the same way separately, because the qii the diagonal element is going to be different from all other elements therefore, I am defining separately. You know how to find out the derivative, derivative of Pij of t with respect to t that is, nothing but the limit delta t tends to 0 the difference divided by the delta t, since Pij of t is a transition probability of system moving from i to j.

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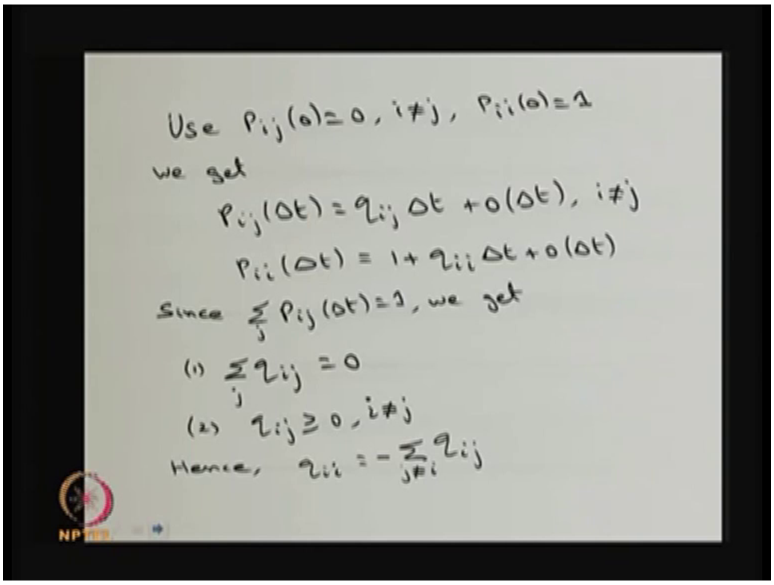


You can use Pij of 0 is equal to 0 for i is not equal to j. For j is equal to i that is Pii of 0 that is equal to 1; that means, the what is a transition probability of system moving from the state i to i in the interval 0 that is same as 1, that probability is 1. So, use this in the previous limit, in this Pij of 0 is equal to 0 and pi of 0 is equal to one you substitute.

Then the limit delta t tends to 0 therefore, the Pij of delta t this will go to this side. So, qij times delta t therefore, this is going to be Pij of delta t is nothing but the qij multiplied by delta t plus small o order of delta t ; that means, a as delta t tends to 0 this whole quantity will tends to 0.

Similarly, you substitute Pii a of 0 is equal to one here therefore, Pii of delta t that is same as a this will come to this side so 1 plus q ii delta t plus order of delta t. So, this order of delta t there is also tends to 0 has a delta t tends to 0. You know that the summation of Pij even at the time point delta t is small negligible time point to delta t.

At the time also over the i that is equal to 1 therefore, if you sum it up, you can conclude the left hand side is the probability. Right hand side for i is not equal to j you have qij whereas, that second expression you have 1 plus qii therefore, using the property of summation of Pij is equal to 1, you will get the summation of qij for all i sorry, for all j that is going to be 0.

When you add both the equations say for all j you will get the summation over j qij is equal to 0, as well as all the qij quantities are going to be great than or equal to 0 from the first one, because the left hand side is a probability and the this is multiplied by the delta t or delta t is always greater than 0.

Therefore, the qij is going to be greater than 0 for all i not equal to j whereas, if I add over all the j that is going to be 0 therefore, you will get the qii, that is nothing but; you make the summation for all qij for r for all j except i, then you make a sum minus sign. So, that is going to be the qii; that means, the diagonal element is nothing but, you make the row sum except that the diagonal term and put the minus sign that is going to be the diagonal term therefore, when you make a row sum that is going to be 0.

The details of the proof can be found in the reference books. So, the quantity qij that has the property the row sum is going to be 0. And other than the diagonal elements are greater than or equal to 0 therefore, the diagonally element is going to be summation of all the other terms with the minus sign.

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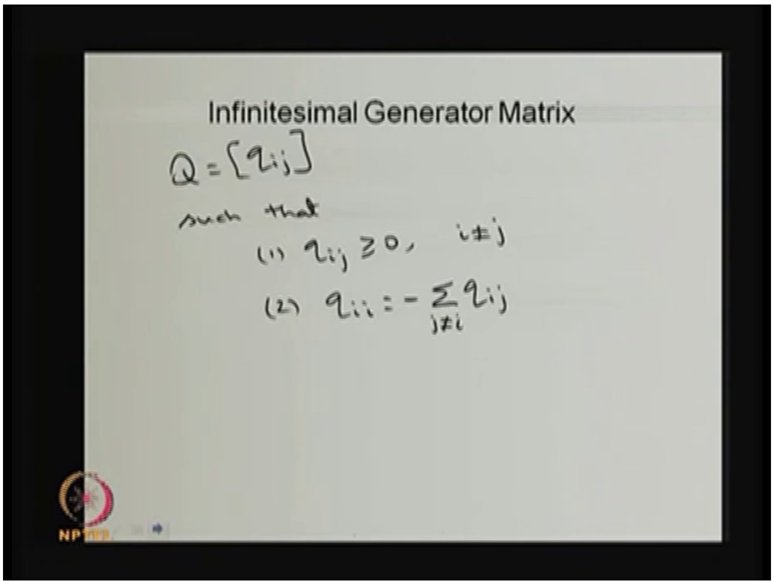


So, using this we can make a matrix that is going to be Q matrix, with the entities qij such that; satisfies the property, qij is always greater than or equal to 0 for i is not equal to j whereas, the diagonal element is minus of summation therefore, it has the property the row sum is going to be 0.

So, the difference between this matrix and the one step transition probability matrix in the DTMC, that is a probability matrix ; so the entries are probability values from 0 to 1, and the summation row sum is 1 whereas, here because qij is are obtained by differentiating the Pij, these are all the rates. And these rates are always great nor equal to 0, other than the diagonal elements and the diagonal elements are minus with the summation of all other row elements.

So, this matrix is called a infinitesimal generator matrix. Some books they use the word rate matrix also, and where as a here the rates are placed in the other than the diagonal elements. And the sum of the rates could be 0; that means, the probability of a system moving from that particular state to the that particular state is not possible; that probabilities 0 or there is a in a small interval of time there is the transition is not possible.

So, whenever the rates are greater than 0; that means, there is a positive probability that the system can have a transition of system moving from i to j.

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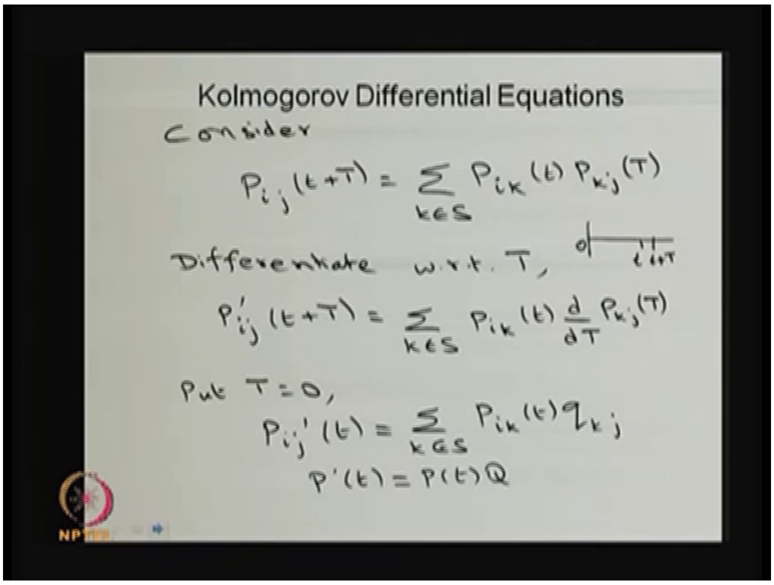


So, we have defined the Q matrix, now using the Q matrix we are going to find out the Pij of t. So, let me start with the Chapman Kolmogorov equation, now I am going to differentiate with respect to capital T; that means, I make the interval 0 to small t plus capital T as a 0 to t then I make a t to t plus capital T. Differentiate with respect to capital

T therefore, the left hand side is going to be I have written with a dash, so, the derivative comes inside the pkj of T then I am substituting t equal to 0.

So, basically I am making a system to move from state 0 to small t, then there is a small interval of time from t to t plus capital T that is the interpretation of this. Then substituting t equal to 0, I will get the left hand side is going to be Pij of dash t that is same as the summation over this whereas, this is nothing but the way we have defined the infinitesimal generator matrix entities. So, this is nothing but the q kj, that is a rate in which the system is moving from the state k to j.

In a matrix form I can make it as a Pij of t is going to form a matrix. So, the P dash of t that is same as a P of t times Q. So, this is a matrix and the P of t is also matrix, and this is the P dash of t meanseach entities are differentiated with respect to time t.

So, this is in the matrix form and this equation is called the forward Kolmogorov differential equation, because the derivation goes from 0 to t, then t to is t plus T, where considering as a very small interval of time. Therefore, this equation is called a forward Kolmogorov differential equation.

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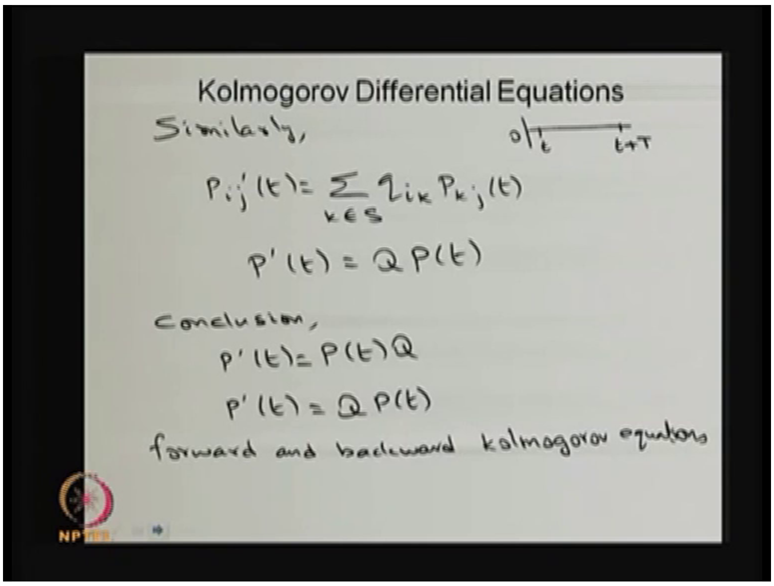


The same way if you do 0 to small t that has a small interval of time, and t to t plus capital T then you will get the P dash of t is equal to q times P of t, that is called the backward Kolmogorov differential equation. Whether you frame forward equation or a backward Kolmogorov equation, if you solve that equation you will get the Pij of t if you solve P dash of t is equal to P of t into q that is, a forward equation. P dash of t is equal to q times P of t that is a backward equation.

If you solve the equation with the initial condition because it is a differential equation. So, you need an initial condition, what is the probability, what is the transition probability of system moving from i to j a time 0. If you know the initial condition by supplying that solving this equation you will get the Pij of t. Once you know the Pij of t then you can get the distribution of x of t.

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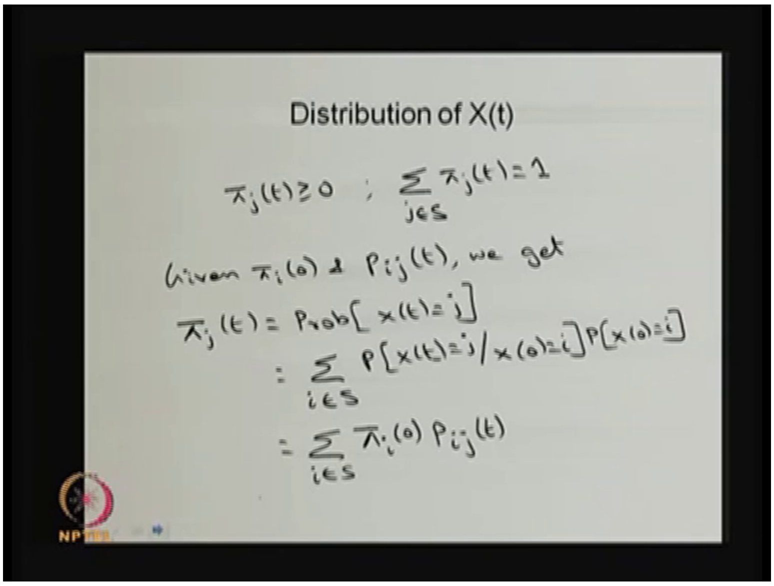


So, once you know the Pij of t, the given is pi of 0 and by solving that forward or backward Kolmogorov differential equation you will get the pi j Pij of t using the these 2 you can get the Pij of t. So, for given pi i of 0 and P ij of t; that means, the transition probability and the initial state probability vector, one can find out the distribution of x of t.

So, in this lecture I have started with the Markov process, then I have discussed the definition of a continuous time Markov chain. And also I have given what is the distribution of time spending in any state before moving into any other state, and also I explained the infinitesimal generator matrix, and using that how to find out the transition probability of Pijt from the Chapman Kolmogorov equation.

And we got a forward as well as the backward Kolmogorov differential equations by solving a forward or backward Kolmogorov differential equation, one can get the pi j of sorry one can get the Pij of t that is a transition probability, using this equation you can get the pi j of t that is nothing but the distribution of x of t. With this let me stop the this lecture and the next lecture, I will go for a simple example of a continuous time Markov chain as well as the stationary limiting distribution and the steady state distribution in the next lecture.