Introduction to Probability Theory and Stochastic Processes
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Lecture – 73

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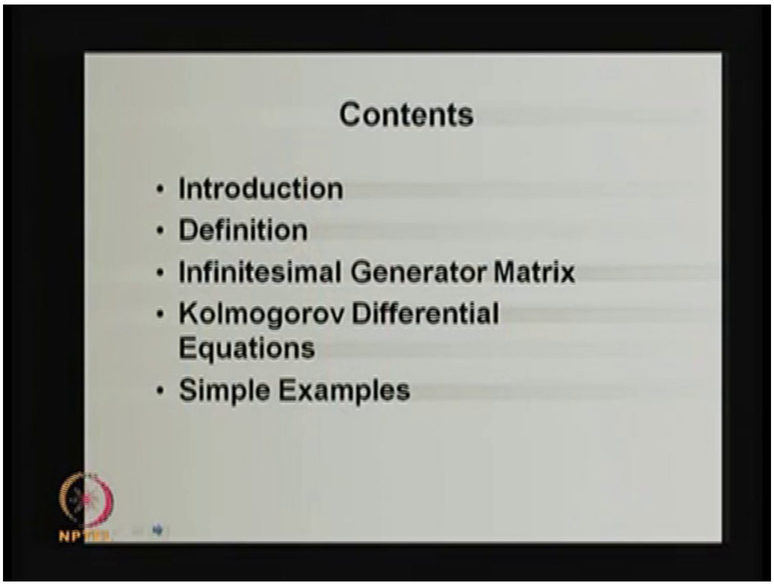

This is continuous time Markov chain. I am planning for 6 to 8 lectures in this model and I am going to start the lecture 1 with the definition of continuous time Markov chain then the derivation of Kolmogorov differential equations. And, I am going to give some simple examples for the continuous time Markov chain and also I am trying to give the stationary and the limiting distributions of continuous time Markov chain in this lecture.

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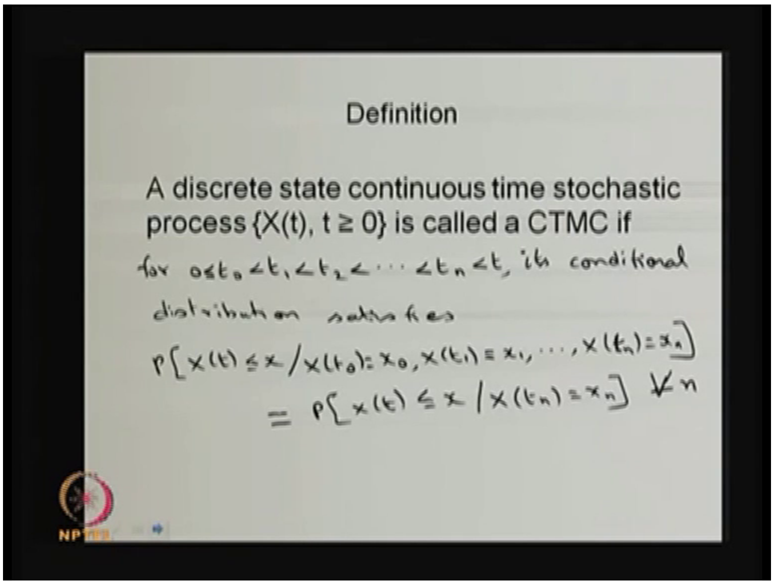

Let me start with the definition; definition of continuous time Markov chain. A discrete state continuous time; that means, the state space is discrete; that means, that the possible values of the random variable going to take the value for possible values of parameter space that is going to be finite or countably infinite. Therefore, the state space is going to be call it as a discrete.

Continuous time means, the parameter space or the possible values of the t, that collection is a uncountably infinite. Therefore, it is called a continuous time; that means a parameter space is continuous. So, a discrete state continuous time stochastic process X of t for t greater than or equal to 0 need not be t greater than or equal to 0 also, but here, I am making the very simplest one.

So, the X of t for fixed t, it is a random variable for every t that collection that is going to be a stochastic process. And the state space is discrete and parameter space is continuous and that stochastic process is going to be call it as a continuous time Markov chain if it satisfies the following condition.

If you take n time points, arbitrary time points n plus 1 time points, that is a t naught to t n, you can say if the t naught can be 0 also and with this inequality t naught less than t 1 less than t 2 and so on t n. And you take the any arbitrary t that is a t n less than t if this inequality.

For fixed t, that x of t is going to be a random variable. Therefore, now we are going to find out the conditional distribution for this n plus 1 random variable with the random variable x of t. That means, at t naught you have a x of t naught that is a random variable at t 1 x of t 1 is a random variable.

Similarly, at t n, x of t n is a random variable you have n plus 1 random variable with this n random variable given; that means, it takes already some values with the x naught x 1 x n. So, on respectively and you are finding the conditional CDF for the random variable x of t.

So; that means, you have n plus 2 random variables taken at the arbitrary time points a t naught t n as well as a small t and you are finding the conditional CDF of the random variable X of t given that already the other n plus 1 random variables taken at those arbitrary time points you taken the value x naught x 1 and so on t x.

And, it is taken already these values that conditional distribution conditional CDF if that is same as again, it is a conditional CDF of X of t given the last random variable X of t n is equal to x.

So, these a n plus 1 time points are arbitrary time points. So, if it satisfies for all n for every n; that means, the conditional distribution of n plus 1 random variable is same as the conditional distribution of the last random variable.

If this property is satisfied by the discrete state continuous time stochastic process for arbitrary time points, then that stochastic process is called a continuous time Markov chain. This is very important concept this is called the Markov property; that means, a the t is a sort of a future.

So, what is the probability that the random variable be in some state at the future time point t given that, you know the present state that is where this system is in time point t n that is small x n and I know the past information starting from X of t naught till X of t n minus 1. I know the information; that means, a what is the probability that a future random variable X of t will be in some state given that, it was in the states x naught at time point t naught, it was in the state x 1 at the time point to t 1 and so on.

Latest at the time point t n the system was in the state x n that is same as what is the probability that the future the random variable will be in some state at time point t given that, it is now in the state x n at that time point t n; that means, a future given present as well as the past information is same as future given only the present which and independent of the past information. That is called the memory less property or Markov property.

So, since this property is satisfied by the stochastic process, which has the state space is a discrete and the parameter space is continuous, then that stochastic process is called continuous time Markov chain so this is the definition. Now we are going to give some more properties over the continuous time Markov chain and some simple examples as well as the I am going to explain the limiting distribution and the stationary distribution for continuous time Markov chain in this lecture.

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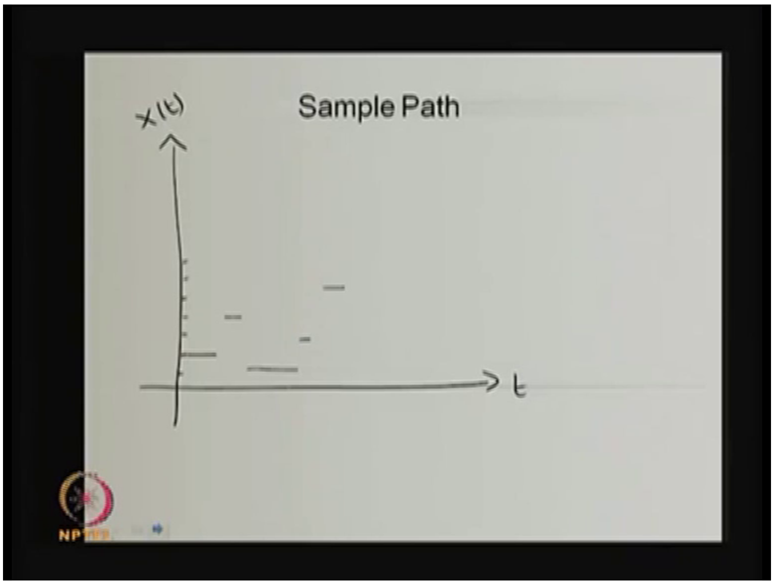

Let me show the sample path over the time t, that is, x axis the y axis is X of t. So, the system was in some state at time point 0. It was in the same state for some time then, it moved into the some other state. Then it was there in that state for some time then it moved into some other state and so on.

If you see the sample path, the following observation the system can stay in some state for some amount of time after that it will move to the some state.

So, there is no equal interval of a system going to be in some state also, it can be some positive amount of time the system can be in the some discrete states. So, here the observations are, the state space is discrete. Whereas, the parameter space is continuous and the time spent in each state that is going to be some positive amount of time before moving into any other states.

So, this is the observation in the sample path which I have drawn.

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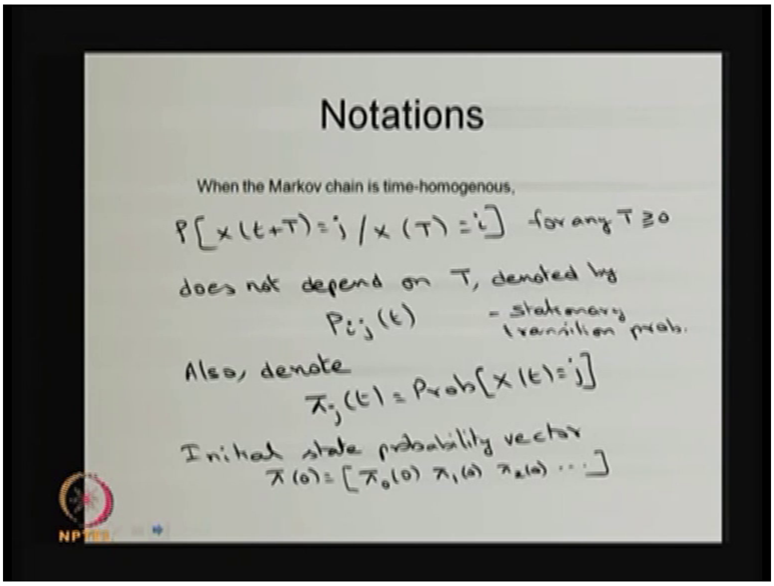

Now, I am going for few notations to study or to study the behavior of a continuous time Markov chain. Whenever the Markov chain that means, a here it is a continuous time Markov chain. It is at time homogeneous then, the conditional probability of system being in the state j at time point to t plus capital T, given that the capital T it was in the state i that does not depend on capital T.

Here we assume that the state changes from i to j at a future time point t plus capital T. This transition probability says the system was in the state i at the time point t. Let me draw the simple diagram.

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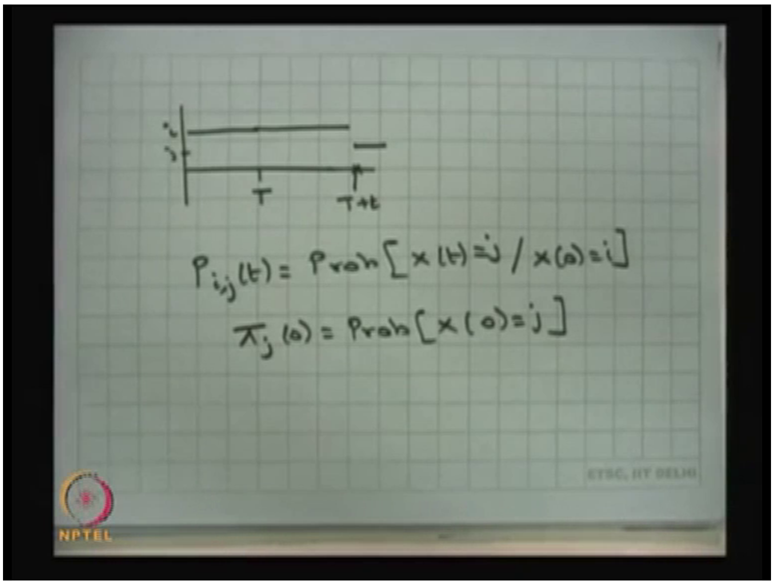

The system was in the state i at the capital T, then what is the probability that the system will be in the state j, what is the probability that the system will be in the state j at the time point T plus t. It is independent of capital T whenever, the Markov chain is going to be a time homogeneous, for any t greater than or equal to 0; that means, the actual time does not matter only the length matters.

The length of the transition time; that means, the small t is matters not the capital T. Whenever, it is at time homogeneous, that is, that we can denote it as a P i j of t because it depends on only the interval not the actual time. Therefore, it is a function of small t P i j of t; that means, that is the transition probability the system.

So, the same thing can be written as the P i j of t, this is a notation. What is the transition probability that the system was what is the probability that the system will be in the state j given that it was in the state i time 0.

Since it is valid for any interval of t to T plus t, it is independent of capital T. Therefore, I can represent this kind of transition probability as a probability that the system in the state j a time t given that it was in the state i at time 0. This denoted by P i j of t.

So, this notation you should remember. It is a transition probability with the suffix 2 let us i comma j of t this also call it as a stationary transition probability. Stationary means, it is at time invariant only the length of the interval is matters.

Similarly, I am denoting the next notation P i j of t. The P j of t is a conditional probability. Whereas, a the P i j of t that is unconditional one what is the probability that the system will be in the state j at time t. There is a possibility system would have been a coming to the state j before time t for time 0 itself or it would have come before just before t whatever it is. This probability will give the interpretation what is the probability that the system will be in the state j at time t only it gives the information at the time t, this is a unconditional probability.

I need a another notation for a initial state probability vector also that is, a pi naught. Pi naught is a vector which consists of entities what is the probability that the system was in the state 0 a time 0. Therefore, this I can write it as pi j of 0 that is nothing, but what is the probability that the system was in state j at time 0.

So, this is the meaning of pi j of 0. What is a probability that the system will be in the state sorry the system was in the state j at time 0 that is pi j of 0 like with these entities, you are framing the vector that is, pi naught. So, in these we are giving a 3 notations. One is the transition probability P i j of t; that is, a conditional probability. The other one is unconditional probability that is P i j of t and the initial state probability vector pi naught.

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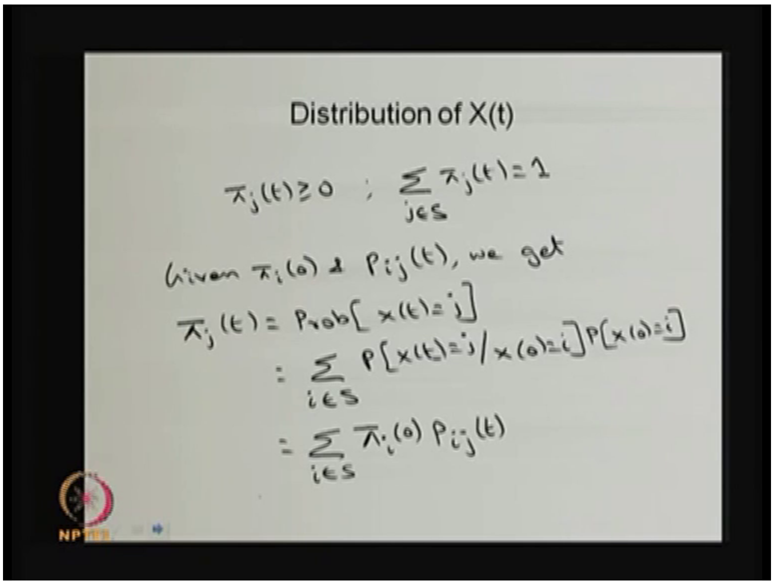

Using these, I am trying to find out what is the distribution of x of t for any time t for any time t x of t will form make a stochastic process. Here, it is a continuous time Markov chain. The default one is a time homogeneous a continuous time Markov chain and our interest is to find out what is the distribution of the random variable x of t.

It has the probability mass function that is pi j of t and if you make a summation over S, where S is the state space that summation is going to be 1.

If I know the initial probe initial state probability vector with the entities pi i of 0 as well as ah, if I know the transition probability of system moving from the state i to j from 0 to small t, I can able to find out what is the probability mass function of system being in the state j at time t.

That is and pi j of t that is same as probability that x of t is equal to j, that is same as I can make a summation I can make a conditional what is the probability that the system will be in the state j at time t given that it was in the state i multiplied by what is the probability that a system was in the state i at time 0.

For all possible values of i, where S is big, S is nothing but the state space. I know that a pi sorry I know that the probability of x of 0 is equal to i that is same as pi i of 0 and this transition probability since the Markov chain is at time homogeneous. So, 0 to t that is nothing, but 0 to 0 is the time point and it is any time point and i is the state in which the system was in the state in the at time 0 so P ij of t.

If I multiply pi i of 0 P i j of t for all possible values of i, I will get the probability that the system will be in the state j at time t. That means, if you want to find out the distribution of x of t for any time t, I need a initial state probability vector as well as the transition probability of system moving from one state to other states. This is given usually the initial state probability vector is given. So, what do we want to find out this P i j of t. So, how to find the P i j of t? That derivation I am going to do it in the another 2 3 slides.