Introduction to Probability Theory and Stochastic Processes
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Lecture - 71

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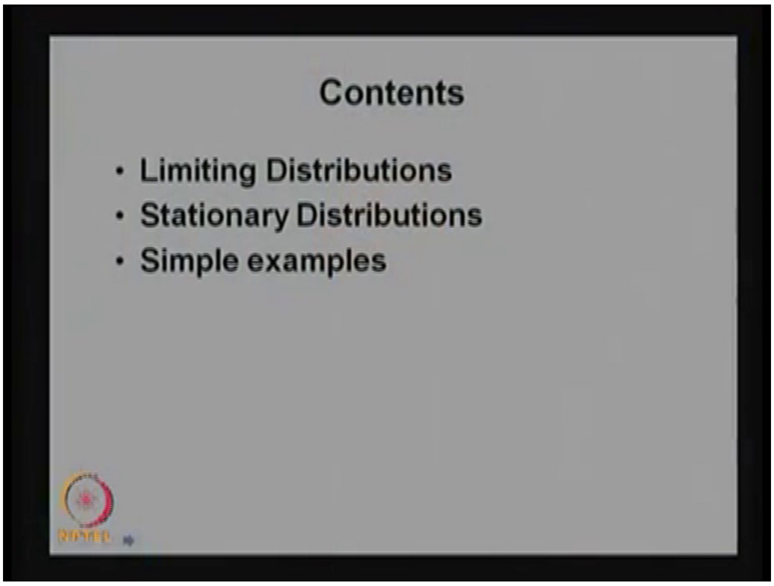

I am planning to explain a few examples for the classification of a states. Then I am going to give the definition of a limiting distributions, then followed by stationary distributions, then the same examples I am going to explain how to get the stationary distribution if it exists.

So, if you recall our earlier lecture, that is a lecture 3, we have given the lot of concepts through those concepts we can classify the states the state as a transient state or recurrent state. Then the recurrent state can be classified into the positive recurrent state and the null recurrent state and you can find out the periodicity of the states and if the period is going to be 1 then we say that state is going to be the a periodic state. And if any state is going to be a positive recurrent and aperiodic then we say that state is the ergodic state. And if one step transition probability if pia is equal to 1, then that state is going to be call it as a absorbing state.

And also we have discussed irreducible Markov chain; that means the whole state space is not able to partition into more than one closed communicating classes. Then that is going to be closed that is going to be call it as a irreducible Markov chain, otherwise it is a reducible Markov chain.

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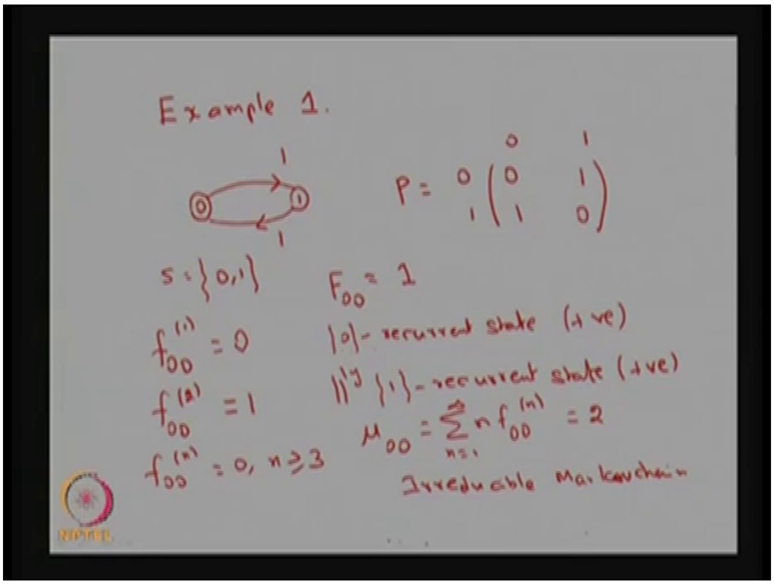

Now, I am going to give simple examples through that we are going to explain the classification of the states. The first example the simplest one, in this first simple example we have a only 2 states. So, the state space contains only 2 elements 0 and 1. The transition the one step transition probability from the system is moving from state is 0 to 1, that probability is 1 and the system is moving from the state 1 to 0 that probably is also 1.

So, the one step transition probability matrix can be obtained from the state transition diagram, both are one and the same. So, this is the one step transition probability matrix and this is a state transition diagram both are one and the same. So, 0 to 0 that probability 0 0 to 1 that probability is 1 1 to 0, 1 to 0 that probability is 1 and 1 to 1 is 0.

Now, we can find out whether these states are going to be a recurrent state or transient state. If you recall to find out the recurrent state or transient state, you have to find out what is the f ii. So, we start with the state 0 so, if you try to find out f 00 of 1, what is the probability that if the system start from the state 0, and reaching the state 0 in exactly first step for the first time. Then that probability is not possible, that is equal to 0. If you try to find out f 0 0 of 2 first visit to the state 0 given that started in the state 0, exactly in the second step it reaches the state 0. That is a possible because by seeing the state transition diagram, you can make out the first step the system is moving from state 0 to 1 and 1 to 0 it is possible coming back to the same state taking exactly 2 steps for the first time therefore, f 00 of 2 that probability is 1.

And by seeing the state transition diagram, you can visualize since it comes to the same state exactly second step therefore, all the further steps for the first time that is not possible. Therefore, all the f 00 of n that is going to be 0 for n is greater than or equal to 3. For n is greater than or equal to 3 the f 00 of n is equal to 0.

Now, if you try to find out what is a capital F 00, that is the probability of ever visiting to the state 0 starting from the state 0, that is going to be the summation of a F 00 superscript within bracket n, for all n vary from 1 to infinity. If you sum it up, then that is going to be 1.

Since F 00 is equal to 1, you can conclude the state 0 is the recurrent state. You can conclude this state 0 is the recurrent state. Similarly, if you do the same exercise for the state 1 by starting with the F 11 of a step one what is the probability F 11 of a step 2, what is the probability and f 11 of all the ns, and find out the summation. So, you land up a f 11 is also going to be 1. You can conclude similarly the state 1 that is also recurrent state.

Here after finding the recurrent state, now we can come find out whether in this is going to be a positive recurrent state or null recurrent state for that you have to find out what is the mean recurrence time or mean passage time. So, try to find out what is mu 00, that is nothing but summation n f ii of n, n varies from 1 to infinity. So, here the i is nothing but 00 n times f 00 of n, because this takes the value 1 for f 00 of 2. Therefore, you will get a 2 times one and all other quantities are 0 therefore, this is going to be 2.

And this is going to be a finite quantity therefore, you can conclude the 0 is the state 0 is the positive recurrent state. The same exercise you can do it for mu 11, that is also you may land up getting the value is equal to 2, therefore, you can come to the conclusion the state 1 that is also positive recurrent state.

So, in a in this a finite discrete time Markov chain, you have 2 states and both the states are going to be a positive recurrent state. And both are the communicating states therefore, you have a class that has the 2 states and the state space is also 0 and 1 and the closed communicating class is also 0 and 1. Therefore, you are not able to partition the state space into more than one communicating class and so on.

Therefore, we land up this Markov chain is going to be this Markov chain is the irreducible Markov chain. This Markov chain is a irreducible Markov chain because the state space has only 2 elements, and the both the elements are both the states are communicating each other. And we land up only one closed communicating class, therefore, this is going to be irreducible Markov chain.

We can find out what is the periodicity of these states also.

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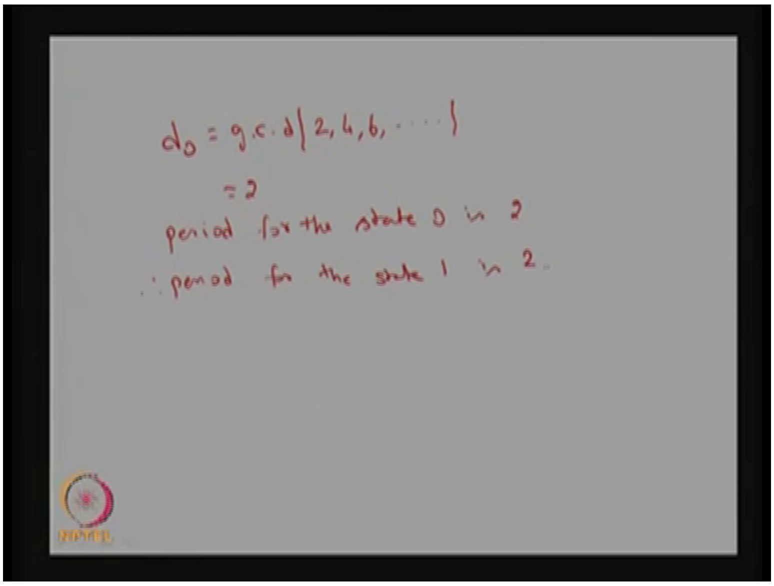

You can find out the periodicity for the state 0 by evaluating d 0, that is nothing but what is the greatest common divisor of all possible steps in which the system is coming back to the same state. So, if you find out, the system can come to the same state, if you see the state transition diagram if the system start from the state 0, coming back to the same state either by 2 steps or 4 steps or 6 steps and so on. You should remember that when you are trying when you are finding the periodicity, you are finding the number of steps coming back to the same state not necessarily the first visitor. Whereas, the f 00 f 00 of n to conclude it is a recurrent state you are find using the first time reaching that state in the exactly nth step so, there is a difference.

So, the gcd of all the possible steps in which the system is coming back to the same state. So, it can come back to the same state 0 in 2 steps or 4 steps or 6 steps and so on. So, the gcd is going to be 2; that means, the period for the state 2 sorry the state 1, the state 0 period for the state 0 is 2. Similarly, you can find out what is the period for the state 1 also, if you do the same exercise, but seeing this diagram you can make out the state 1 also going to have the gcd of a 2 6 2 4 6 8 and so on.

Therefore, the period for the state 1 also going to be 2, other ways also we can conclude both are communicating states since the period for the state 0 is 2, and since the state 1 is a communicating in the states 0; that means, it is accessible in both ways. Therefore, the state 1 is also having the same state same period.

In conclusion, you can make out if we have a one class with more than one states then all the states are going to have the same period. Therefore, the state 1 is also have the period for the state 1. That is also 2, that means this example you have only 2 states and this is a irreducible Markov chain and both the states are positive recurrent with the period 2. So, that is the way using the classification of the states we come to the conclusion of this particular example.

Later we are going to find out the limiting distribution and the stationary distribution and so on, but for that we need the classification. Here also you can visualize where the system will be for a longer run, if the system start from the state 0 or 1. You can visualize because it is only to state by seeing the state transition diagram, you can make out suppose the system start initially in the state 0 at every even number of steps, it will be come back to the state 0. In a longer run, based on the number is going to be even or odd accordingly, the system will be in any one of these states.

Similarly, in a longer run, you can make out if the system start from the state 1 initially all the even number of steps it will be come back to the same state 1, and all the odd number of the steps it will be in the state 0. In the longer run also it is going to be happen in the same way for a even n and the odd n accordingly the system will be in any one of these states.

In a longer run also the system will be any one of these 2 states only because it is a irreducible Markov chain because these 2 states are communicating each other. Therefore, in a longer run the probability that the system will be in any one of these states will be a sum value, and only the system will be in any one of these 2 states only. Later I am going to give the definition of the limiting distribution through that I am going to explain the same example again.