Introduction to Probability Theory and Stochastic Processes
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Lecture - 70
Classification of States

Now, we are going for the actual classification of a state. Using the concept of accessible communicate, closed set, then communicating class, then we have defined the first visit, then we have defined the mean passage time or mean recurrence time or mean first passage time so using these concepts, we are going to classify the states. The first definition is the recurrent state here, state j is said to be recurrent or the other word called persistent.

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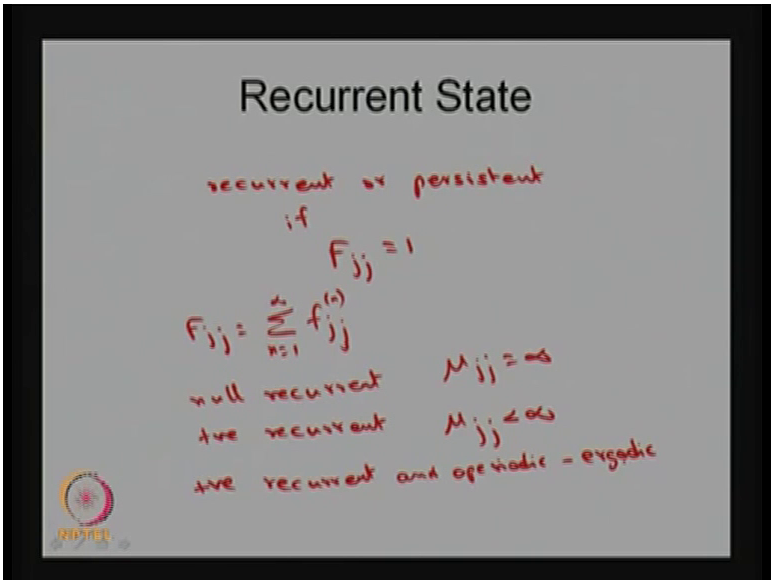

If the F suffix jj equal to 1, if you recall, what is F jj? F jj is the probability of ever entered to the state j, given that it was in the state j.

So, the F jj; I have given in the summation form of small F jj of n using the first visit. So, if you recall the F jj is nothing but what are all the possible ways the system can reach the state j as a first visit, you add all the combination all the probabilities that is going to be the F capital F jj. So, if capital F jj; that means, the probability of returning to the same state j if that probability is certain; that means, if the probability is 1, then that state is going to be the recurrent state.

We can classify the recurrent state into 2 form; one is called the null recurrent, the other one is called a positive recurrent based on the mean passage time value.

So, based on the capital F ijj, that is a probability, we classify the state is going to be a recurrent state. Now, based on the first passage time distribution, the mean first passage time, we are going to classify that recurrent state is going to be a null recurrent or positive recurrent accordingly, the mu jj. If it is a finite value, then we say that recurrent state is going to be the positive recurrent state. If mu suffix jj is going to be a infinite value; that means, on average the first passage time is going to be infinite, then that corresponding recurrent state is going to be call it as a null recurrent state.

So, whenever any state is going to be call it as a recurrent state, if the probability of ever entering into the state j starting from the state j, it is certain or the probability is 1. Then, that is a recurrent state and the recurrent state is going to be call it as a null recurrent if the mean first passage time or mean recurrence time or mean returned time is infinity. If that is going to be a finite quantity, then the recurrent state is going to call it as a positive recurrent state. If any state is going to be a positive recurrent as well as a periodic then that state is going to be call it as a ergodic state.

Any state is going to be call it as a ergodic whenever that state is a positive recurrent as well as a periodic. A periodic means, when they periodicity of that recurrent state is 1; that means, the greatest common divisor of all possible steps in which the system coming to the same state that values is 1. If the period is 1 and as well as the positive recurrent, it should be recurrent as well as positive recurrent; that means, if the mean recurrence time is going to be a finite quantity and then it is going to be call it as a ergodic state.

In a Markov chain, if all the states are going to be a ergodic one; that means, all the states are going to be a positive recurrent as well as a periodic, then we call that Markov chain itself ergodic Markov chain. That means, there is a possibility the Markov chain, may be a irreducible; that means, you will land up with only one class in which all the all the sets are going to be from a one close communicating class.

Suppose, each one state is going to be a positive recurrent and a periodic, then all other states are also going to be of the same time and same period. Therefore, all the states are going to be the ergodic states. Then, that Markov chain is going to be call it as a ergodic Markov chain.

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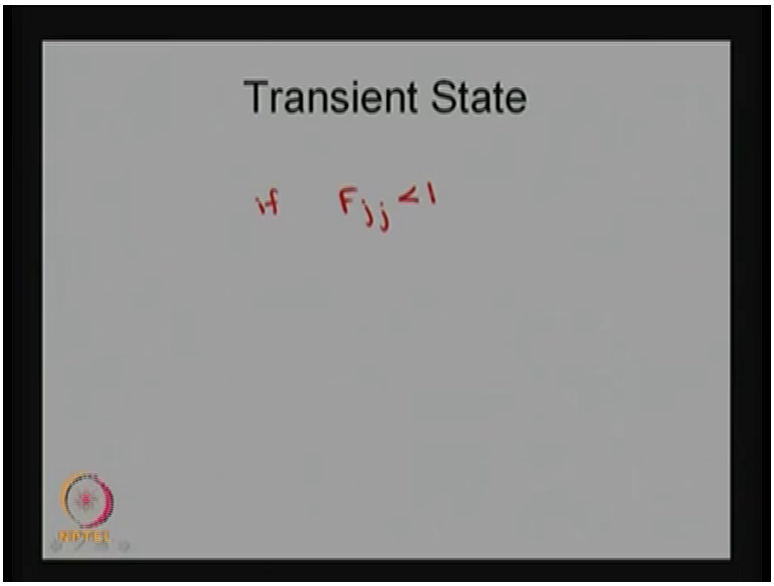

Now, I am going to classify the state has a transient state whenever the F jj value is less than 1, if you recall, we have consider only 2 cases. Whether the F jj is less than 1 or F jj is equal to 1; equal to 1 land up recurrent state and F jj is less than 1 that gives the transient state. That means, the probability of returning to the state j starting from the state j is not certain; that means, 1 minus of this probability with that much probability.

The system may not return to the same state j. if the system start from the state j; that means, with the some positive probability because, 1 minus this value is less than 1. Therefore, 1 minus of F jj is going to be greater than 0.

So, with some positive probability, the system may not return to the same state if it start from the state j, then that corresponding state is going to be call it as a transient state. By seeing the one step transition probability matrix or by seeing the state transition diagram of a discrete time Markov chain, you can easily come to the conclusion the state is going to be a recurrent state or transient state.

Whenever it is going to be a finite number of states, it is easy to come to the conclusion. If it is a infinite number of states, then we need some work to be needed to come to the conclusion whether it is a positive recurrent or null recurrent. So, but easily you can make out the given state is going to be a transient state that you can make out of from the state transition diagram or one step transition probability matrix.

The same the conclusion of the state is going to be the transient state that can be given via the random variable T j also. So, the state j is a transient if and only if the probability of the T suffix j is equal to infinity and that probability 0. Sorry, made a mistake, if this probability is strictly greater than 0, the probability of the mean the probability of the system returned to the first passage.

The first passage returned time that is infinity, if that probability is greater than 0; that means, a there is a certainty over the system returned to the state j with the infinite amount of time going to take. If that event is going to be the with the positive probability, then that state is going to be the transient state.

So, there are through 2 ways you can conclude the given state is going to be the transient state either F j j is less than 1 or the probability of T j equal to infinity which is greater than 0.

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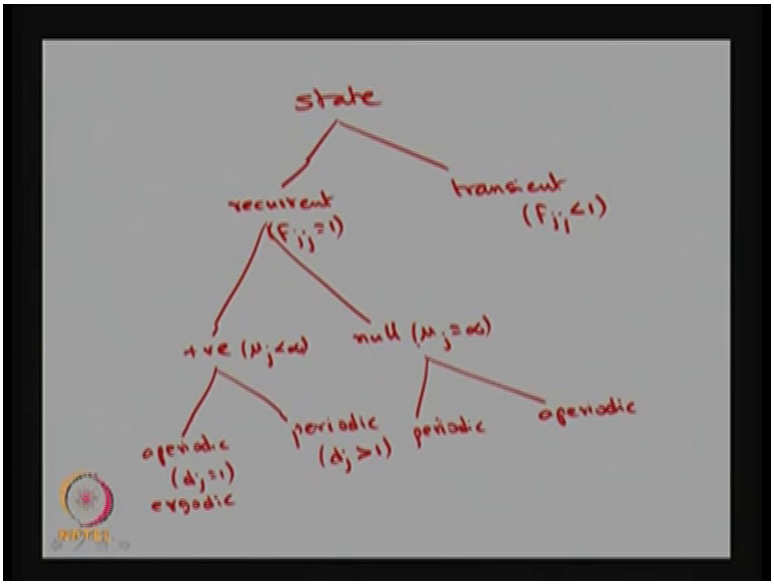

So, based on this I can come to the conclusion any state could be recurrent or transient; that means, this is corresponding to F jj is less than one and this is corresponding to F jj is equal to 1. I can classify the recurrent state into 2 form either it could be a positive recurrent or null recurrent.

Positive recurrent corresponding to the mu j or mu jj both are one and the same. That is going to be finite value or null recurrent is corresponding to mu j is equal to infinity; that

means, based on the mean recurrence time you can conclude, whether it is a positive recurrent or null recurrent.

Again, I can classify the positive recurrent into 2; one is a periodic and the other one is a periodic. Periodic means that, corresponding positive recurrent state that period is greater than 1. A periodic means the d j is 1. So the a periodic a positive recurrent state that is going to be call it as a ergodic state.

Similarly, I can classify the null recurrent state into 2. One is a periodic and the other one is a periodic. The observing state is a special case of positive recurrent state where, the transition probability from a state to itself is 1. So, this is the way you can classify the state is a recurrent state or transient state, positive recurrent state, null recurrent state again, each one could be a periodic or periodic state.

So, in this lecture, we started with the few concepts of a accessible, then communicate, then closed set, then we have discuss communicating class, then we have discussed what is the meaning of first visit, then we have given the first passage time, then we have given the mean first passage time distribution or mean recurrence time distribution. So, based on those concepts, we have classified the state as a recurrent state or transient state.

So, this is related to the probability, whereas, the conclusion of the positive recurrent or null recurrent is related to the average time. So, here, it is only it involves the probability that whether in certain probability, the system will come to the same state with the probability 1. Whereas, here there is a uncertainty the system may not come to the state j. If the system start from the state j, if there is a uncertainty of a returning; that means, with a some positive probability, the system would not be back, then that state is going to be call it as a transient state.

So, this you can easily visualize in the state transition diagram of any discrete time Markov chain, you can see it whether just from by seeing the state transition diagram you can come to the conclusion whether the state is going to be the transient or recurrent. But through these diagrams you cannot come to the conclusion whether it is going to be a positive recurrent or null recurrent. Unless otherwise, you evaluate this quantity mu j is going to be n times F jj of n.

So, you find out that summation. So, based on the summation, values is going to be a finite one or infinite one. Accordingly; that means, whether the mean recurrence time or mean return time or mean first passage time is going to be a finite quantity or infinite quantity accordingly you can conclude whether that recurrent state is going to be a positive recurrent or null recurrent.

So, here you need a computation whereas, by seeing the state transition diagram, sometime you can come to the conclusion it is a positive recurrent or sorry sometime you can come to the conclusion whether it is a transient state or recurrent state. Now, the issue of a periodicity the periodicity is important to conclude whether the limiting distribution exists or not whether that is going to be unique. So, you need to find out the a periodic or periodic.

So, if the period is going to be 1, then that state is going to be call it as a periodic. If the period is greater than 1, then it is a period with that not integer. When it is going to be a null recurrent then also we can come to the conclusion whether it is a period periodic or a periodic. Whenever you have a Markov chain with a finite number of states, then it is easy to find out whether it is going to be a positive recurrent or transient.

So, you need a quite good exercise is needed whenever the Markov chain is have a infinite number of states, then you need some work to be done for come to the conclusion it is a null recurrent and so on. In today's lecture, with this classification, I stop here and all the simple examples and the limiting distribution that I will explained in the fourth lecture.

Thanks.