Introduction to Probability Theory and Stochastic Processes
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Lecture - 07

Now, we will move into the next topic that is distribution function.

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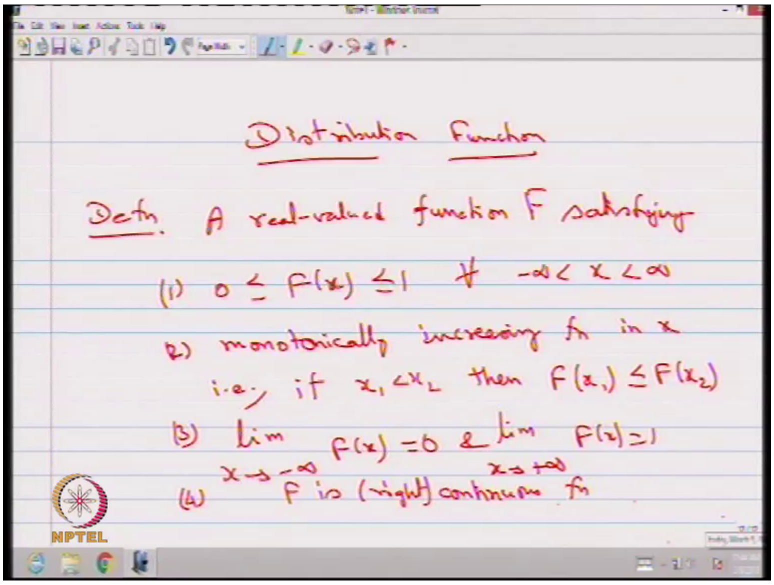

That is distribution function this is nothing to do with probability. So, first let me define the distribution function, then we will connect the distribution function with the probability. Therefore, the same distribution function will be call it as a cumulative distribution function of a random variable x. So, when I define now the distribution function, there is no probability at all ok.

Let me define distribution function definition. A real valued function capital F, because later we are going to use this capital F as a CDF. Therefore, I am going to use now itself capital F. A real valued function capital F satisfying 4 conditions the first condition; it is always lies between 0 to 1 for all x lies between minus infinity to infinity; this is the first condition. Second condition it is a monotonically increasing function in x; that means, if two values you take x 1 x 2, x 1 is less than x 2, then the F of x 1 is always less than or equal to F of x 2. It is same as a non decreasing function whether you say it is a monotonically increasing function for two different values of x 1 less than x 2, then your

F of x 1 is less than or equal to F of x 2 or non decreasing function both are one of the same.

Third condition limit x tends to minus infinity of F of x that is 0 as well as limit x tends to plus infinity of F of x is 1. It is not substituting the F at minus infinity is 0, F at plus infinity is 1, it may holds in some examples, but the definition is limit of x tends to minus infinity that has to be 0, limit of plus infinity that has to be 1 it is not the subscription it is a limit.

The fourth condition that is a the capital F is right continuous function, I can use the word I can make right in the bracket. That means, the function can be continuous, if not it is a right continuous. The function is continuous function if not it has to be a right continuous function. If these 4 conditions are satisfied then you can conclude the F is distribution function.

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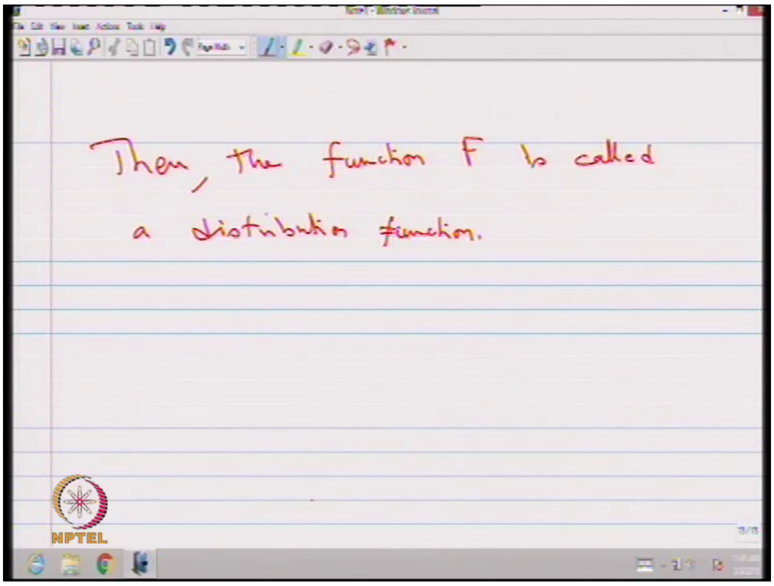

Then the function the real valued function capital F that is called a distribution function. As I said this is nothing to do with the probability, a any real valued function satisfying these 4 conditions is going to be call it as a distribution function if given any one of the properties are not satisfied then you cannot conclude. It has to be real valued function value lies between 0 to 1 it should be a monotonically increasing limit value has to start from 0 at minus infinity and it had end up 1 as a limit x tends plus infinity and either it should be continuous or right continuous.

You can draw some diagram for the function capital F and you can get the field. After that we can introduce the probability, then we land up cumulative distribution function of the random variable x. So, let me start what are all the graph of capital F so, that that is going to be the distribution function. You can also try you can also try to draw the graph x axis x, y axis is capital F, so that satisfying this 4 conditions. Therefore, that is a distribution function you try I will also going to give some two 3 diagrams so that you can conclude which one is going to be distribution function which is not a distribution function and so.

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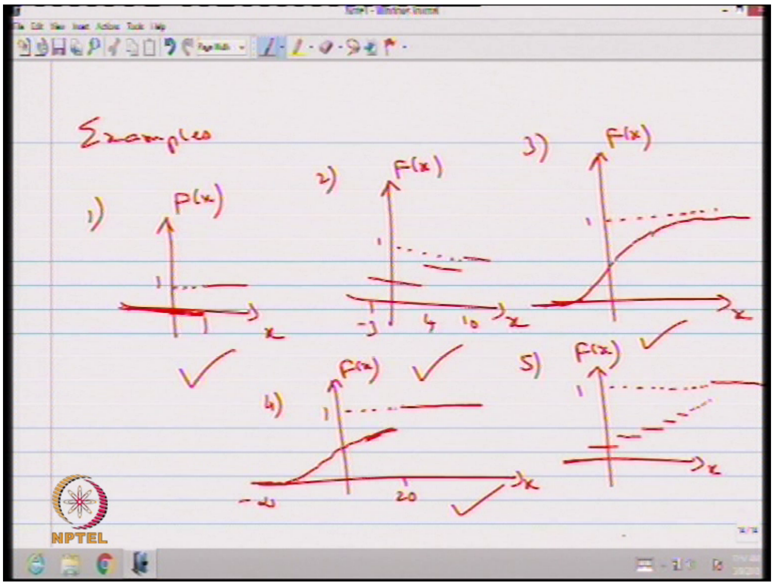

Examples: first example when I asked the students to draw the diagram, they can make it different type whether it is going to satisfy the 4 condition or not that is important. Since, it is values from 0 to 1 the easiest diagram the values is 0 till this point, and this point it increases becomes 1. Therefore, this satisfies all the 4 conditions therefore, it is distribution function I can put the tick mark.

I draw another diagram, at some point it has a increment. So, this point you treat it as minus 3, at some other point it is not scaled at some point it increased, at some other point again it increased and this value suppose that is going to be 1 suppose this value is going to be 1; that means, till minus 3 it is 0, minus 3 to 4 it has some value constant. There is a increment at the point 4 and between interval 4 to 10 it has a same value, at the point 10 it becomes 1 and it is 1 till the end therefore, this also satisfies.

And go for another diagram as a third example it is 0 and it is increasing and it is a touches it becomes 1 at infinity; that means, asymptotically it goes and it becomes 1 at infinity. So, you see the difference of the 3 different distribution function, the first one has the only one jump at the point 1, the second one has 1 2 3 three jumps and the fourth third example which does not have any discontinuity or no jumps it starts from 0 and it goes to 1 and it is a continuous function, whereas the first and second example are the right continuous function. That means, the way I define the value at the point wherever there is a jump and where what the value at the right limit are same, which is different from the left limit in the first 2 examples, whereas in the third example it is a continuous function or I can go for one more diagram.

It take a value 0 and it is increased to some value at some point, there is a jump at some point there is a jump, then it becomes 1. That means, it is continuous between the interval minus infinity to suppose this point is a 20. This function is a continuous between the interval minus infinity to 20, 20 there is a jump and it becomes one from 22 infinity. Therefore, it has a continuous in some interval jumps at one point or there is a possibility it may have a jumps at many points, it may have a combination of continuous in some interval as well as jumps.

Whereas, if you see the first example and second example, the distribution function increases the values only by jumps I am repeating the word the first example and the second example the distribution function start from 0, it incremented at some value and it becomes a same value till the next jump, then the next jump it increases till the next jump then it again increase then it goes to the same value.

Therefore, the value increases only by jumps in the first example and the second example whereas in the third example it increases continuously from one point till the end whereas, in the fourth example it increases in some interval as well as it is retain the same value with some jumps. Therefore, all 4 types are going to be the distribution function, I have not given what is the function which is not a distribution function.

So, you can think of a wrong example of increase decreasing or not values are lies between 0 to 1; it more than one or less than 0 or it is discontinuities. So, violating these 4 conditions you can always create n number of examples, which are going to be a non distribution function. But what we want is a real valued function satisfying the 4 conditions therefore, this is going to be a distribution function.

So, this 4 example makes the first example has a only one jump, the second example has a 3 jumps like that you can have a finite number of jumps. The third example does not have a jump at all, it is increasing in the whole interval and the fourth example is the combination of increasing as well as having a jumps. We can go for one more example which has the countably infinite jumps, that makes a completeness of the distribution function.

So, the fifth example is having a countably infinite jumps countably infinite jumps. So, it start from 0 and it has a jump, again it has a jump, again it has a jump, again it has a jump like this it is keep going it reaches one at some point.

The first example has only one jump, second example has a 3 jumps, fifth example has the countably infinite number of jumps, third example does not have a jump, fifth example has a one jump as well as the increasing between the some interval; this shows the distribution function can be of many form. So, this will lead has to create types of random variable in the later stage, but before that let me explain what is cumulative distribution function, through the distribution function that is going to be the next definition is the cumulative.

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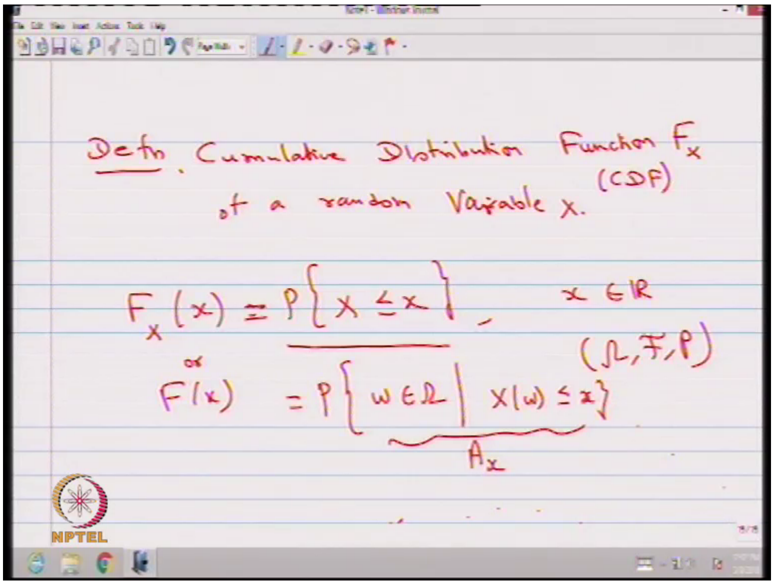

Next definition is cumulative distribution function of a random variable x. So, this is a

CDF is denoted by capital F suffix x, that is denoted by capital F suffix x, that is a cumulative distribution function and how it is defined for all x that is same as P of x is less than or equal to small x, where small x is in the real line.

The capital F which we define that is a distribution function, extra we are using a suffix capital X that is to show that the same distribution function capital F is call it as the cumulative distribution function of a random variable x. So, whether I write capital F suffix x or capital F of x both are on in the same.

So, I can use a word or here to distant gives or to tell that both are distribution function and the way you define the distribution function through the P, that P comes from the probability space omega F p the same capital P is used. In the probability space you have a capital P that is a probability measure set function. So, if you find out P of x is less than or equal to x for all x belonging to r, this way if you define the distribution function and that distribution function is going to be call it as a cumulative distribution function of a random variable x.

Now, the way we write capital P x is less than or equal to x, the way you define this way the distribution function is going to be call it as a cumulative distribution function of a random variable x. In short they use CDF; CDF means cumulative distribution function. I can expand what is the meaning of capital P less than or equal to x; that means, that is same as capital P of the capital X less than or equal to x is nothing but for possible outcomes w belonging to omega such that, under the operation x the w gives the values x of w gives the values less than or equal to x, that is a meaning of a capital P x less than or equal to x.

Means you are collecting few possible outcomes which satisfies the condition x of w less than or equal to x. This is nothing but a set this is nothing but event, the way I explained in the earlier this can be denoted by the later called A suffix x; that means, it is a P of event A suffix x. A suffix x as a event, this event is belonging to the F by using a kolmogorov asymmetric definition, the P of a suffix x gives the values greater or equal to 0 for all events and the P of omega is equal to 1 and P of union of mutual it is joint events that is same as sum of P of A x.

The using the same kolmogorov asymmetric condition, the capital F is defined for all x belonging to real line; that means, when x between minus infinity to plus infinity you will get the values of P of x less than or equal to x, that will satisfies the 4 condition which we said it earlier that is it is a real valued function lies between 0 to 1 monotonically increasing, limit is a minus infinity 0 plus infinity is 1 and F is a right continuous function. So, all these 4 conditions will be satisfied. Therefore, this distribution function the capital F that is going to be call it as a cumulative distribution function of the random variable x, that is denoted by capital F or capital F suffix x. When we have more than one random variable the suffix is inverter now we have only one random variable. Therefore, whether you right F of x or F suffix x both are one and the same

Now, let us prove how these function capital F x is going to be the distribution function. Once you prove that it is a distribution function through the P x is less than or equal to x, then you can conclude the same distribution function is going to be call it as a cumulative distribution function of the random variable x. Even though there are 4 points to be proved the first one is lies between 0 to 1, and second one is monotonically increasing or non decreasing function and the limit is minus infinity 0 and plus infinity is 1 and right continuous. So, we will prove only few things the remaining proof can be done it in the similar way.

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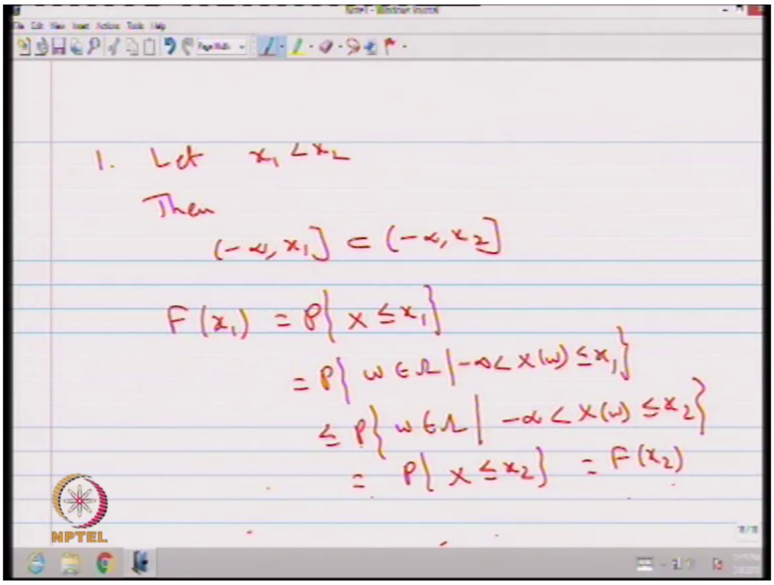

The first one to prove that it is non decreasing or monotonically increasing, you take two points with x 1 is less than x 2, and you can always conclude minus infinity to x 1 which is contained in minus infinity to x 2 and this is a one set and is another set and once it is contained in the other set.

Suppose you go for finding what is the F of x 1, that is nothing but the P of x is less than or equal to x 1 that is same as P of set of all ws belonging to omega such that x of w is lies between minus infinity to x 1. Since once it is contained in the other set or equivalent of one event is contained in the other event, then the probability is going to be less than or equal therefore, that is less than or equal to the P of collection of ws such that minus infinity to x of w is less than or equal to x 2 this is nothing but P of x is less than or equal to x 2 and that is same as F of x 2. Therefore, we can conclude F is the non decreasing in x or monotonically increasing.

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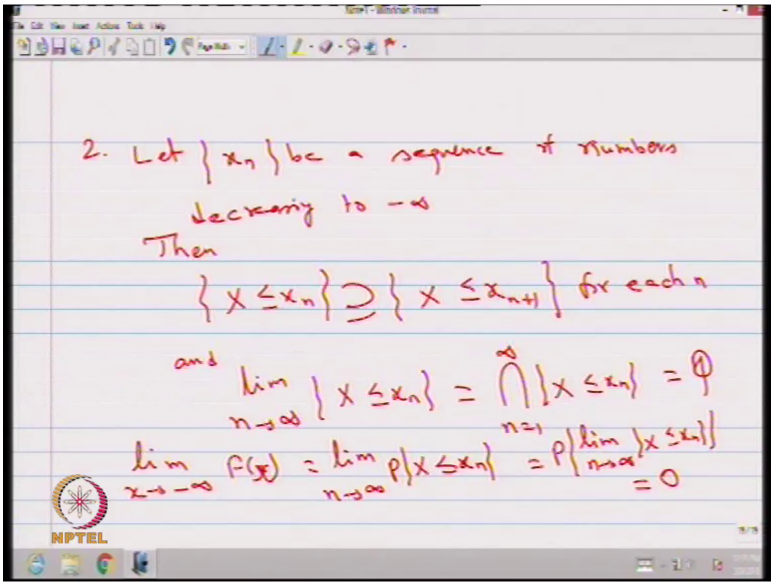

The second one we can prove limit x tends to minus infinity of F of x is 0, for that we take sequence x n be a sequence of real numbers such that it is decreasing to minus infinity, then you can conclude x less than or equal to small x n that is satisfies this condition. For each n not only that the limit n tends to infinity the event x less than or equal to x n that is nothing but intersection of x less than or equal to x n where n goes from one to infinity.

Since xns are the sequence of a numbers decreasing to minus infinity satisfying this condition therefore, a limit n tends to infinity x is less than or equal to xn is same as intersection of x is less than or equal to xn, where n running from one to infinity that is nothing but the empty set that is nothing but empty set.

Therefore limit x tends to minus infinity of F of x is same as limit n tends to infinity of P of x is less than or equal to x n. Just now we concluded limit n tends to infinity x is less than or equal to x n is empty set. Therefore, this is nothing but P of I can interchange the role of a limit and probability, because P is a continuous function P, limit intensity infinity of x is less than or equal to x n and that is empty set. So, the P of empty set is going to be 0.

So, we have proved limit x tends to minus infinity F of x that is going to be 0.

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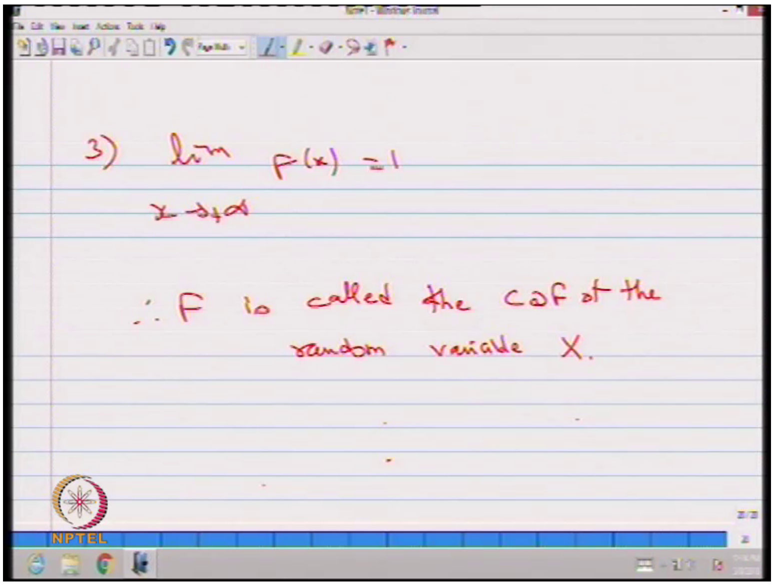

Similarly, one can prove limit x tends to plus infinity of F of x that is going to be 1 and forth one, one can prove the F of x is the right continuous in x. Since, all the 4 conditions are satisfied for the distribution function. Therefore, this distribution function capital F is called a, is called the CDF of the random variable x. So, the distribution function satisfying P of x is less than or equal to x form that distribution function will be the cumulative distribution function for the random variable x.