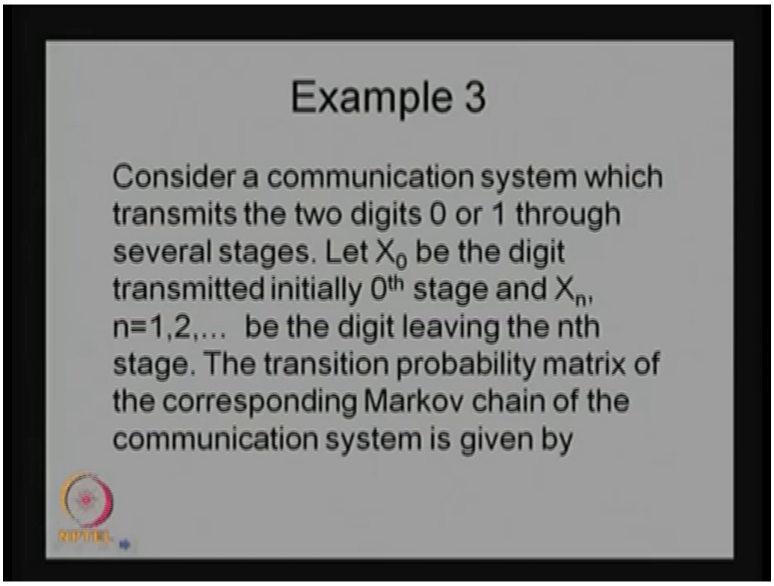
Introduction to Probability Theory and Stochastic Processes

Prof. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture – 66

(Refer Slide Time: 00:01)

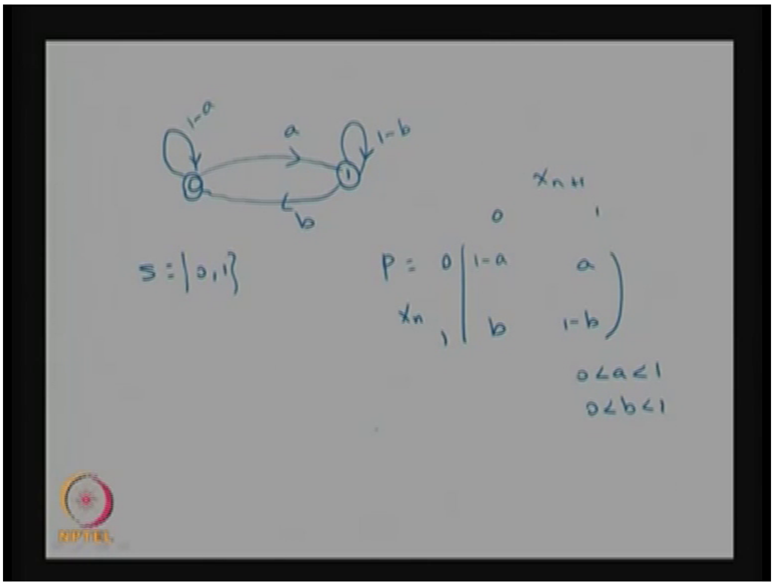


This example talks about the communication system in which whenever the transmission takes place with the digits 0 and 1 in the several stages. Now, we are going to define the random variable X naught be the digit transmitted initially that is a 0th step. Either the transmission digit will be 0 or 1 therefore, only two possibilities can be takes place at the any nth step transmission either 0 or 1 like that we are making the transmission over the different stages. Therefore, this X n over the n will form a stochastic process. Because you never know which digit is a transmitted in the nth stage.

So, each stage is going to be a one random variable, and you have a collection of random variable over the stages therefore, it is a sequence of random variables. So, this is going to form a stochastic process, and this stochastic process is nothing but the discrete time, discrete state stochastic process because the possible values of X n is going to be 0 or 1 therefore, the state space is 0 or 1. And it is a discrete time discrete state stochastic process.

The way the subsequent transmission takes place depends only on the last transmission not the previous stages. Therefore, you can assume that this follows a Markov property. Therefore, this stochastic process is going to be call it as a discrete time Markov chain. Now, our interest is to find out so now, I will provide what is the one step transition probability for the Markov chain, or let me give the transition diagram for that.

(Refer Slide Time: 01:55)



So, state transition diagram the possible states are 0 or 1 the, because the state space is 0 and 1; and the probability that in the next step also the transmission is 0 with the probability 1 minus a. This is a conditional probability, this is the conditional probability of the nth stage the transmission was 0, the n plus 1th stage is also the transmission is 0 with the probability 1 minus a.

The one step transition probability of system is moving from 0 to 1, that probability is a; that means, the nth stage the transmission was the digit 0, the n plus 1th stage also the trans the n plus 1th stage transmission will be the digit one with the probability a. Similarly, I am going to supply the one step transition probability of 1 to 1 that is 1 minus b, and this is b; that means, this 1 to 0 that probability is 1 to 0 is a b and 1 to 1 is 1 minus b; obviously, these a is lies between 0 to 1 and b is also lies between 1.

(Refer Slide Time: 03:30)

$$P^{(n)} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \qquad\qquad 1 \end{array} \\ \begin{bmatrix} \dfrac{b+a(1-a-b)^n}{a+b} & \dfrac{a-a(1-a-b)^n}{a+b} \\ \dfrac{b-b(1-a-b)^n}{a+b} & \dfrac{a+b(1-a-b)^n}{a+b} \end{bmatrix} \end{array}$$

$$for\ |1-a-b| < 1$$

So, this is the a is the probability that the system is transmitting from the 0th step, 0th sorry nth stage with the digit 0 and the n plus 1th stage with the probable with the stage with the digit 1 that probability is a therefore, the negation is 1 minus a because the system can transmit either 0 or 1. So, once you say that the one step transition probability of 0 to 1 is a then 0 to 0 will be 1 minus a.

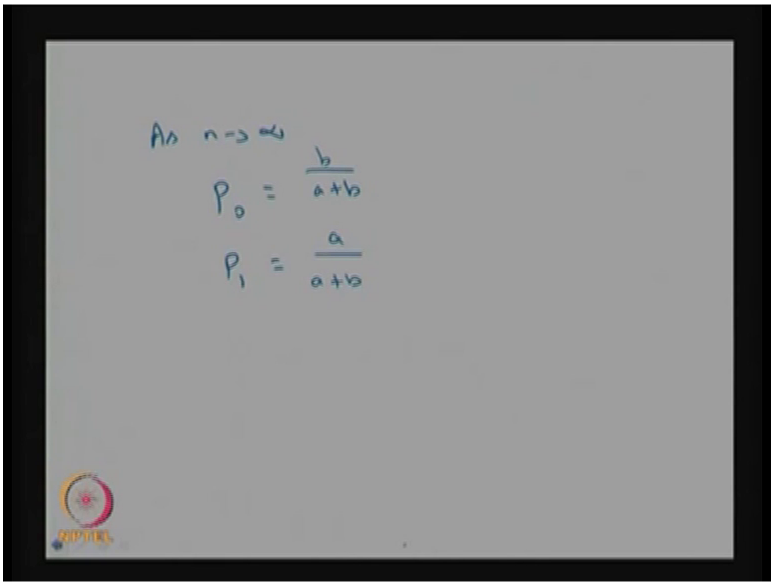
Similarly, 1 to 0 is given as the probability b, and the other digit transmission will be one therefore, it is going to be 1 to 1 will be 1 minus b. So, this is the state transition diagram and this is a one-step transition probability for a given time homogeneous discrete time Markov chain.

Our interest is to find out what is the distribution of X n for n. For that you need what is the n step transition probability matrix. Since the one step transition probability matrix is given you can find out the P square P power 3 and so on by induction method you can find out the P power m. But using the P power m you can find out the P m plus n. Therefore, you can come to the conclusion what is the n step transition probability of system is moving from 0 to 1 and do 0 to 0 and so on. So, this is nothing but I am just giving the only the result b plus a times 1 minus a minus b power n divided by a plus b. And this is nothing but a minus a times 1 minus a minus b power n divided by a plus b.

Similarly, if you find out the n step transition probability of system moving from 1 to 0, that is b minus b times 1 minus a minus b power n divided by a plus b. This is nothing but a plus b times 1 minus a minus b power n divided by a plus b. So, here I am just giving the n step transition probability matrix form by given P you should find out the P square P power q by induction you can find out the P power n.

And this is valid provided 1 minus a minus b which is less than 1 because, you are finding the P power n matrix. So, here it needs some determent also unless otherwise the absolute of 1 minus a minus b which is less than 1 this result is not valid. So, provided this condition the P, P of n that is the matrix so, that is same as P power n also, P of n is same as P power n.

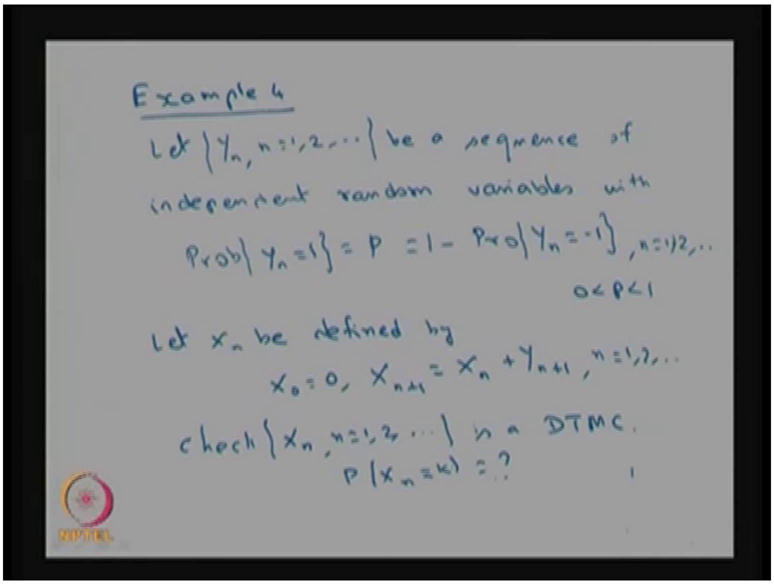
(Refer Slide Time: 06:50)



So, as n tends to infinity, you can come to the conclusion what is the probability that the system will be in the state 0 that is same as b divided by a plus b. And similarly what is the probability that the system will be in the state one as n tends to infinity that will be a divided by a plus b. This can be visualized from the state transition diagram easily whenever the system is keep moving into the state 0 or 1 with the probability ab and with the self-loop 1 minus a and 1 minus b. The subsequent stages the system will be in any one of these 2 states.

So, the with the proportion of b divided by a plus b the system will be in the state one. Similarly, with the proportion a divided by sorry a divided by a plus b the system will be in the state 1, with the proportion b divided by a plus b the system will be in the state 0 in a longer run. The interpretation of as n tends to infinity, this probability is nothing but in a long run with this proportion the system will be in the state 0 or 1. So, this state transition diagram will be useful to study the long run distribution, or where the system will be as n tends to infinity, to study those things the state transition diagram will be useful

(Refer Slide Time: 08:42)



Now, we will move into the next problem that is example 4. Let it is a sequence of random variable be a sequence of independent random variables with condition the probability of Y n takes a value 1, that probability is P, that is same as 1 minus the probability of Y n takes a value minus 1. We have a stochastic process, and each random variable is a independent random variable. And the probability mass function is provided with this situation, the probability of Y n takes a value one is P you can assume that the P takes a value 0 to 1. That is same as 1 minus of probability of Y n takes a value minus 1 for all a.

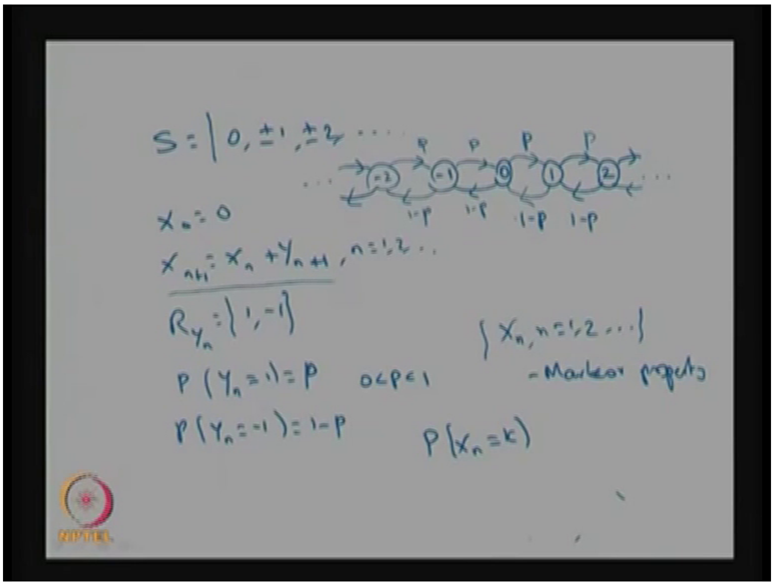
Now, I am going to define another random variable let X n be defined by X naught is equal to 0. Whereas, a X n plus 1 onwards that is going to be X n plus Y n plus 1 for n is equal to 1 2 and so on. So, we are defining a another random variable X n with the X naught is equal to 0 and the X n plus 1 is equal to X n plus Y n plus 1.

Now, the question is check X n that stochastic process is the DTMC's. If it is a DTMC also find out what is the probability of X n takes a value k. We started with one stochastic process and we define a another stochastic process with the earlier stochastic process.

And check whether the given the new stochastic process is a discrete time Markov chain, that is a default one that is a time homogeneous discrete time Markov chain. If so, then what is the probability of X n takes a value k? That is nothing but find out the distribution of X n.

So, how to find out this? The given or the X n is going to be the DTMC. Since Y n takes a value one with the probability P, and Y n takes a value minus 1 with the probability 1 minus P, you can make out the possible values of Y n is going to be 0 or plus or minus 1 plus or minus 2 and so on.

(Refer Slide Time: 12:18)



Because the relation is a X naught is equal to 0, and X n is X n plus Y n plus 1, and the range of Y n the range of Y n is 1 comma minus 1. Therefore, the range of X n that in form of state space and the X naught is equal to 0 therefore, X, X n that relation is a X n plus 1, sorry. Xn plus 1 is equal to X n plus Y n plus 1 so, n takes a value one and so on. Therefore, the possible values of X n will be 0 plus or minus 1 or plus or minus 2 and so on therefore, that mean from a state space.

Now, the given clue is a probability of y n takes a value 1 is probability P, and probability of Y n takes a value minus 1, that is 1 minus P and the probability P is lies between 0 to 1; So, using this information, you can make a state space of the Y n that is going to be, sorry, 1, 2 and so on, minus 1, minus 2 and so on.

Now, we can fill up what is the one step transition of system is moving from 0 to 1; that means, the X 1, 0 to 1 suppose you substitute 0 here, then suppose it takes a value 1, then the system can move from the state 0 to 1 in one step. Suppose you put the value X naught X n is equal to 0, suppose you put X n is equal to 0, and Y n plus 1 takes a value one with the probability P, then the X n plus 1 value will be 1 with the probability P, correct?

Now, we can go for what is the state transition probability of 1 to 0. Suppose X n value was 1, suppose the Y n plus 1 value was minus 1, then the X n plus 1 value will be 0. So, the one step transition of a system moving from a 1 to 0 because of happening probability of Y n plus 1 is equal to minus 1, that probability is 1 minus p so, this is 1 to P.

So, whenever the system is moving from one step forward, that probability will be the probability P and one step backward that probability will be 1 minus P. So, this is the way it goes forward step, and this is the way it goes to the backward step so, you can fill up all other probabilities forward probability with the probability P and the backward probability with the 1 minus P.

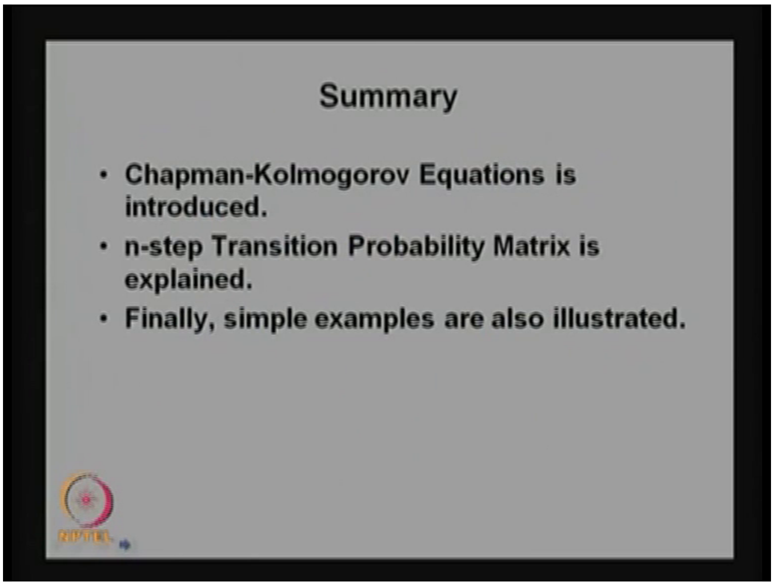
Also we can come to the conclusion the way we have written X n plus 1 is equal to X n plus Y n plus 1, and all the y is are independent random variable the X n plus 1 going to take the value depends only on X n not the previous X n minus 1 or X n minus 2 and so on therefore, the conditional distribution of X n plus 1 given that X n X n minus 1 till X naught that is same as the conditional distribution of X n plus 1 given Xn; that means, the X n is going to satisfy the Markov property because of this relation. Because of X n plus 1 is equal to X n plus independent random variable. Therefore, the X n n is equal to 1 2 3 and so on this stochastic process is going to satisfies the Markov property therefore, this discrete time discrete state stochastic process is going to be the discrete time Markov chain because of the Markov properties satisfied.

Once it is Markov property satisfied by using the Chapman-Kolmogorov equation, you can find out what is the distribution of X n takes a value k. That is nothing but where it started at time 0, and the what is the conditional distribution of the n step transition probability and n step transition probability is nothing but the element from the P power n, and from here you can find out the one step transition probability matrix from the one step transition probability matrix you can find out the P p square P cube and so on and you can find out the P power n, and that the element is going to be the n step transition probability using that you can find out the distribution.

And since we do not know the value of P where P is lies between 0 to 1, it is a I am not going to discuss the computational aspect of a finding out the distribution. This is left as an exercise and the final answer is provided. The difference between the earlier example and this example and this example the state space is going to be a countably infinite. Therefore, the P is not going to be a easy matrix it is going to be a matrix with the many elements in it. Therefore, finding out the P square and the P power n it is going to be little complicated than the usual square matrix.

So, hence the conclusion is by knowing the initial probability vector and the one step transition probability matrix or the state transition diagram, we can get the distribution of X n for any n. There is a small mistake the running index for X n plus 1 value is equal to X n plus Y n plus 1, that is starting from 0 comma 1 comma 2 and the similarly the previous slide X n plus 1 is equal to X n plus Y n plus 1 and the n is running from 0 comma 1 comma 2 and so on.

(Refer Slide Time: 19:36)



So, in this lecture we have discussed Chapman-Kolmogorov equation. And also we have discussed the n step transition probability matrix. So, the n step transition probability matrix can be computed from the one step transition probability matrix with the power of that n. And also we have discussed 4 simple examples for explaining the Chapman-Kolmogorov equation and the n step transition probability matrix.

Thanks.