Introduction to Probability Theory and Stochastic Processes
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Lecture – 54

(Refer Slide Time: 00:01)

Here the lambda is a constant, and there is another name for the default Poisson process is called the homogeneous Poisson process, because there is a another one called the non homogeneous Poisson process in which the lambda need not be a constant it can be a function of a time t also. Therefore, the one we have derived now it is a homogeneous Poisson process, in which the lambda is a constant which is greater than 0. When lambda is going to be a function of t the corresponding a Poisson process is called a non-homogeneous Poisson process.

So, this is the one particular and very important continuous time or continuous parameter discrete state stochastic process, that is a Poisson process or this is also we can say this is going to be a very important continuous time arrival process that is the Poisson process. The way we are counting N of t is going to be a number of arrivals over the interval 0 to t, or number of occurrence of the event over the t the way you are counting over the time.

Poisson process is an example of counting process so, the N of t is also called a counting process. So, the Poisson process is also called it as the counting process.
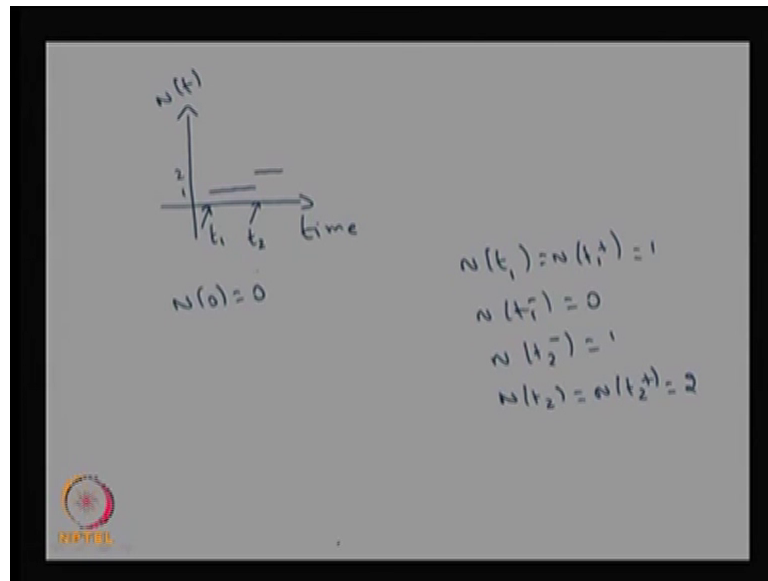
I can go for giving the sample path of N of t over the time what is the different values of N of t is going to take; obviously, N of 0 is equal to 0. Whenever some arrival occurs in some time, then the arrival is going to occur therefore, suppose the arrival occurs at this time, I make it as the up arrow.

Then the value of N of t is going to be incremented by 1, till the next arrival comes. Suppose the next arrival takes place at this time point, then the N of t values is going to be 1 till the time and it is going to be a right continuous function; that means, the time point in which the first arrival occurs suppose you make it as a T 1.

So, the N of T 1 minus is going to be 0, and the T 1 and N of T 1 plus T 1 as well as a N of T 1 plus that is going to be 1. Whereas, the left limit N of T 1 minus that is going to be 0. Suppose the second arrival occurs at the sometime point T 2, then the N of T 2 minus that is a left limit at the time point T 2 that is going to be 1, and N of T 2 that is same as a N of T 2 plus that is going to be 2. So, therefore, it is incremented by 1 so, the values is going to be 2.

So, this is the random random time in which the arrival is going to occur, and the way we have made the assumption in a very small interval only one maximum only one arrival can occur. Therefore, the N of t is going to be a non-decreasing right continuous and increased by jump of size 1 at the time you poke of arrival.

So, whenever you see the sample path of the Poisson process, it is always going to be a non-decreasing right continuous, and increase by a jumps of size 1, at the time of at the time you poke of arrivals. Now I am going to relate the; another random variable which involves in the Poisson process or I am going to discuss the another stochastic process which involved in the Poisson process.
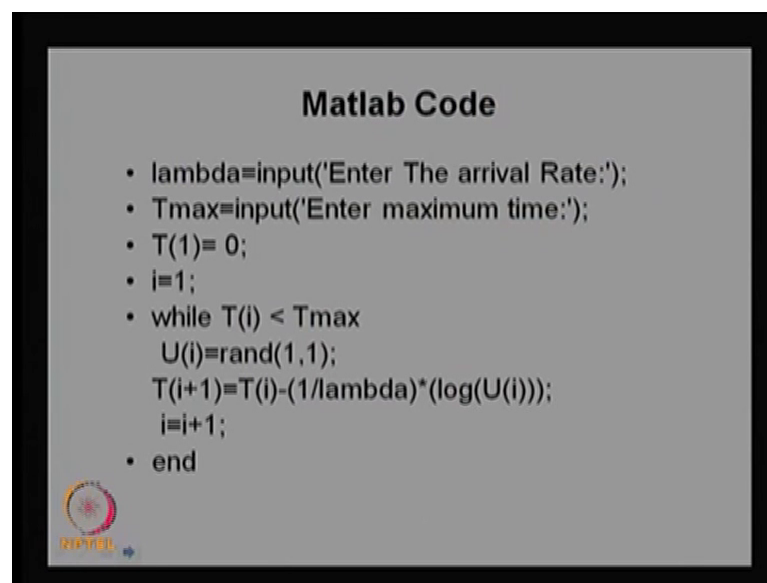
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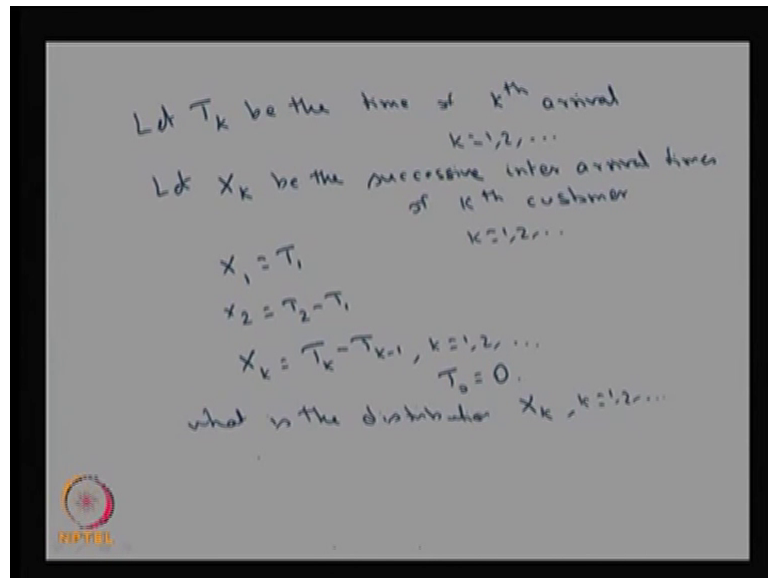
So, for that I am going to define the new random variable as let T suffix k be the time of k th arrival. So, k can take the value 1 or 2 and so on so, therefore, the t be the random t be the random variable takes what is a time point in which the k th arrival occurs; that means, the way I have given the sample path in the previous slide, the T 1 and T 2 the small T 1 and T 2 are the different values of the capital Tk. I am going to define another random variable x of Xk be the successive inter arrival times of k th customer.

So now the k can takes a value 1 2 and so on. So, the Tk be the time point whereas, the Xk be the inter arrival time; that means, the X 1 is nothing but T 1 minus a t 0 and; obviously, t 0 is 0 therefore, X 1 is same as the T 1, and X 2 is nothing but T 2 minus T 1; that means, what is the inter arrival time for the second arrival, that inter arrival time is what time the first arrival occurs that is a T 1, and what time the second arrival occurs that difference is going to be the inter arrival of the second customer. So, this is the way I can define xk is going to be T suffix k minus t suffix k minus 1.

So now the running index for k can take the value one and so on; obviously, t 0 is going to be 0. Our interest is to find out what is the distribution of, what is the distribution of Xk for all k 1 2 and so on. Is it feasible to find out the distribution of xk? It is possible first we can start with a k equal to 1, what could be the distribution of X 1.

Then once we get the X 1 distribution, the same analysis can be repeated to get the distribution of X 2 and x 3 and so on. Because the scenario which you are going to take it for finding out the distribution of X 1 that is the same as for the X 2 and so on.

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So now our interest is to find out what is the distribution of X 1. First we will try to find out that X 1. Only now we will find out the distribution of X 1.

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$$P(x_1 > t) = P(N(t) = 0)$$
$$= \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$P(x_1 > t) = e^{-\lambda t}$$
$$P(x_1 \leq t) = 1 - e^{-\lambda t}$$

$$x_1 \sim \text{Exponential dist} (\lambda)$$
$$x_2 \sim \text{Exp}(\lambda) \qquad x_i \sim \text{Exp}(\lambda)$$
$$i = 1, 2, 3, \ldots$$

Since X 1 is a continuous random variable, we can go for finding out what is the complement CDF of X 1. So, this is a complement CDF of X 1, that is nothing but what is the probability that the first arrival occurs after time t, that is same as what is the probability that till time t no customer enter into the system. The left hand side is the unknown whereas, the right hand side is the known one so, we are relating 2 different random variable.

So, here this is the what is the probability that the first arrival occurs after time t, that is same as what is the probability that no arrival takes place during the interval 0 to small t. But we know what is the probability of N t is equal to 0 because just now we have made it for each t this is going to be a Poisson distribution with the parameter lambda times t. Therefore, the probability of N t equal to 0, that is same as e power minus lambda t and lambda t power 0 by 0 factorial. And this is same as e power minus lambda t.

So, the left hand side is the unknown, the unknown is what is the probability that X 1 takes a value greater than t, that is same as e power minus lambda t, therefore, you can get what is the probability of X 1 less than or equal to t, that is same as 1 minus e power minus lambda t. So, this is going to be a what is the CDF for the random variable X 1, and the CDF of X 1 is the same as the CDF of exponential distribution with the parameter lambda times t. Therefore, we can come to the conclusion X 1 is going to be a

exponentially distributed. The X 1 is exponentially distributed with the parameter lambda.

So, the unknown distribution X 1 first we are trying to find out what is a complement CDF of X 1, and that land up to be e power minus lambda t, therefore, the CDF of X 1 is going to be 1 minus e power minus lambda t. The from this we conclude the X 1 is going to be exponential distribution with the parameter lambda where lambda is greater than 0.

The way we have compute the way we get the distribution of X 1, similarly one can show X 2, that is the inter arrival time of the second customer enter into the system. That is also can be proved it is exponential distribution with the parameter lambda. Not only X 2 we can go for the further all the Xi's so, we can able to prove all the Xi's are going to be exponential distribution with the parameter lambda for i takes a value 1 2 and so on. Not only that we can able to prove all the Xi's are independent random variable, also and identical with the each one is exponential distribution with the parameter lambda.

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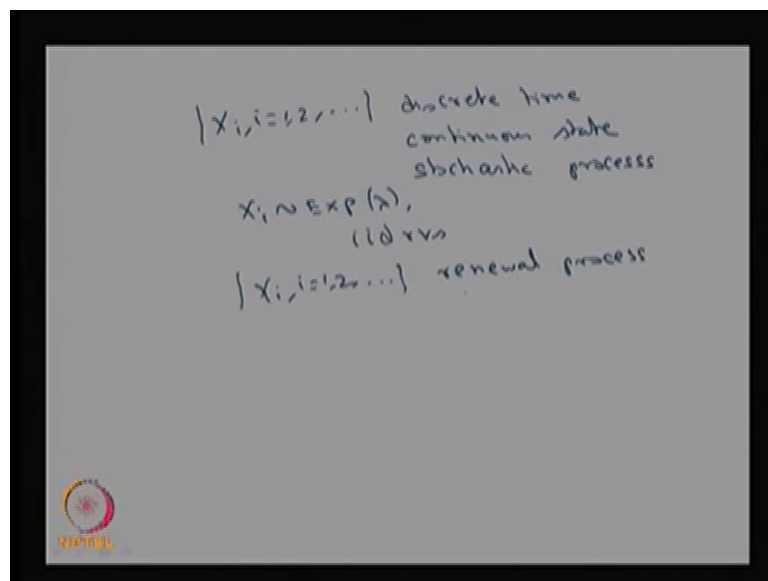
Therefore, the way we land up relating Poisson process with the inter arrival time. So, this Xi's will form a discrete time or discrete parameter continuous state stochastic process in which each random variable Xi is going to be a exponential distribution with the parameter lambda, and all the Xi's are iid random variable also.

And this each Xi's are nothing but inter renewal time therefore, this is going to be call it as renewal process. We are going to discuss the renewal process in detailed later of this course, but here I am just explaining how will you create the renewal process of from the Poisson process and the N of t is a Poisson process for different values of t.

Whereas the inter arrival time that is the time in which the renewal takes place or the arrival takes place. Therefore, the renewals will form a stochastic process and that corresponding process is called a renewal process. Therefore, this is going to be a one particular type of renewal process in which the renewal takes place of exponentially distributed time intervals and all the times are iid random variables also.