Introduction to Probability Theory and Stochastic Processes
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Lecture – 53

So till now, we have discussed what is the discrete time arrival process? Now we are going to discuss the continuous time arrival process that is a Poisson process.

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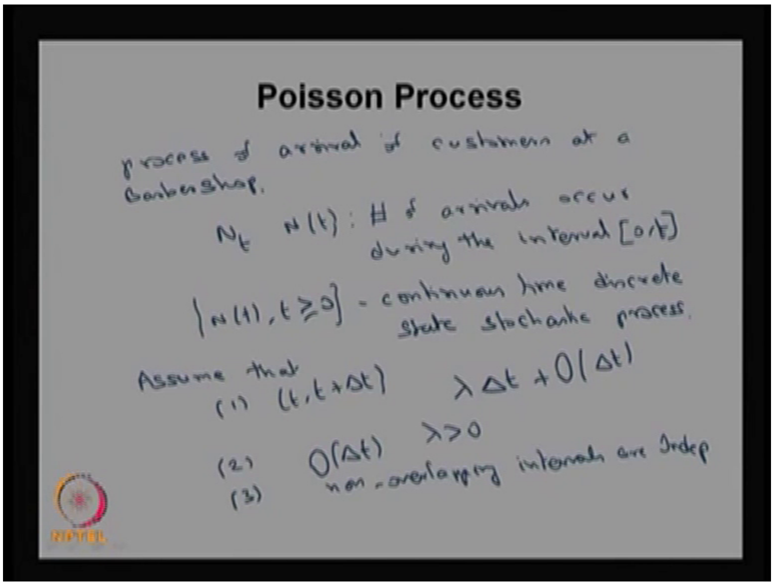

So, in this lecture, I am going to develop what is the Poisson process and how we can get the Poisson process from the scratch. Suppose you consider the process of arrival of customers; consider the process of arrival of customers at a barber shop.

So, this is the same example we have discussed in the beginning of this course also. So, over the time how many arrivals is going to take place that is going to be a random variable. So, let N t, N suffix t or some books they use as a N of t. So, the N of t denotes number of arrivals occur during the interval, during the interval 0 to the closed interval 0 to t; that means, we are defining a random variable N of t that denotes a number of arrival occurs a during the interval 0 to t.

For fixed t; N of t is going to be a random variable therefore, N of t over the time because t is greater than or equal to 0. This is going to be a. Since the Poisson values of a capital T that is the parameter space is going to 0 to infinity. Therefore, this is going to under the classification of a continuous parameter or continuous time.

And the possible values of N of t for different values of t, that is going to be takes the value 0 or 1 or 2 therefore, it is going to be a countably infinite. Therefore, this is going to be a continuous time or continuous parameter discrete state stochastic process. So, this is the N of t over the t greater than or equal to 0; that is going to be a continuous time discrete state stochastic process.

Now, we are going to develop the theory behind the Poisson process. To create the Poisson process, you need a few assumptions so that you can able to develop the Poisson process. The first assumption in a small negligible interval, if the interval is a t to t plus delta t, if the small negligible interval is a t to t plus delta t.

Then the probability of one arrival is going to be lambda times delta t plus capital O of delta t. The probability of one arrival occurs during the interval t to t plus delta t is going to be lambda times that smaller interval delta t plus capital Order of capital O delta t. Here the lambda is going to be strictly greater than 0, and we are going to discuss what is lambda and so on the in the later after this explaining the Poisson process. So, here the lambda is going to be a constant and which takes the value greater than 0. And the capital O delta t means as a delta t tends to 0, the order of a delta t that is going to be tends to 0 as delta t tends to 0. So, this is the first assumption.

The second assumption the probability of more than one arrival is going to be a order of delta t. In the same interval t to t plus delta t, more than one arrival in this small negligible interval that probability is order of delta t; that means, as a delta t tends to 0 this values is going to tends to 0.

Then the third assumption occurrence of arrivals in a non-overlapping intervals are mutually independent, non-overlapping intervals are independent. So, this is a very important assumption; that means, what is a probability that the arrival occurs in a non-overlapping intervals that probability is the same as the product of a probability of arrival occurs in the in each interval, therefore, it is going to satisfies the independent property occurrence of evens in non-overlapping intervals are mutually independent. Therefore, the probability is going to be probability of intersection of all those things is same as the probability of individual probability and their product.

So, with these 3 assumptions we are going to develop the Poisson process.

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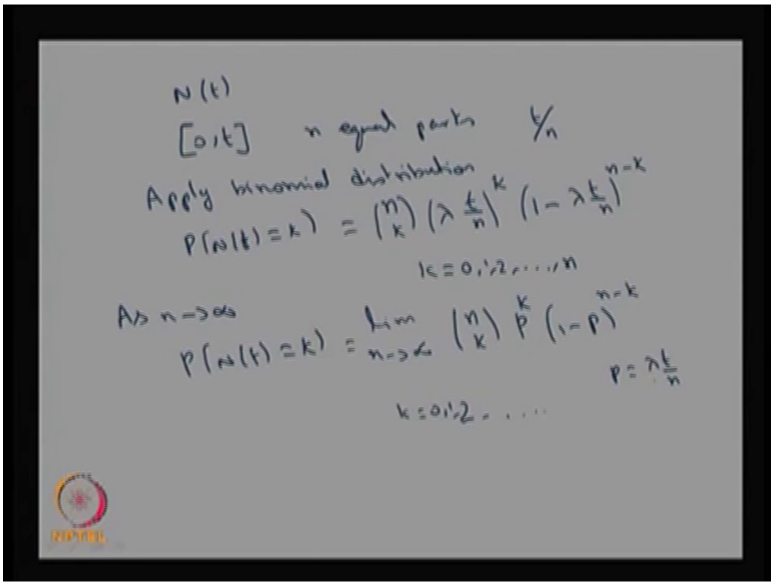

So, what I am going to do? Since I started with the random variable N of t is the number of arrivals in the interval 0 to t, I am going to partition the interval 0 to t into n equal parts. I am going to I am going to partition the interval 0 to t into n equal parts, since I made it the interval 0 to t into n equal parts. Then each will be of the length t by n.

And since I made the assumption the non-overlapping intervals are independent, and the probability of one arrival is lambda times delta t, and the probability of more than one arrival is order of delta t and so on. Therefore, I can apply binomial distribution the way I have partition the interval 0 to t into n pieces.

Therefore, this is going to be of n intervals of interval length t by n, therefore, I can say what is the probability that I can able to find out, what is the probability that k arrivals takes place in the interval n intervals of each length t by n, what is the probability that k arrivals takes place therefore, the possible values of k is going to be 0 to n. And I can able to find out by using the binomial distribution, what is the probability that N of t takes a value k.

Since non overlapping intervals are independent and each probability of what sorry, probability of one arrival is lambda times delta t, where delta t is a t by n. So, each interval behave as a Bernoulli trial whether the arrival occurs or there is no arrival. And like that you have n such independent trials, therefore, the sum of n independent Bernoulli trials land up binomial trials, therefore, by using the binomial distribution, I can able to get what is a probability that N of t takes a value k, that is a what is the possible n c k ways.

And what is the probability of arrival takes place in one interval that is lambda times this interval length is t by n, lambda times a t by n power k. And what is the probability of no arrival takes place in each interval that is 1 minus lambda times t by n power n minus k. So, this is the way I can able to get what is the probability that a k arrival takes place in the interval 0 to t by partitioning np n intervals. So, this is a probability.

But the way I made a partition n equal parts so now, I had to go for what is the result as n tends to infinity; that means, my interest is what could be the result if n tends to infinity of k of, what is the probability that N t takes a value k as n tends to infinity, therefore, the running index for k is going to be 0, 1, 2 and so on what is the probability of N t takes a value k; that means, in the right hand side I have to go for finding out as n tends to infinity, what is the result for the right hand side, what is the probability of N t takes a value k.

We take a n tends to infinity, because we need to study the limiting behavior of the stochastic process. So, that is same as limit n tends to infinity of n c k, I can make it as a P power k where P is going to be lambda times t by n. And 1 minus P power n minus k. Now I have to find out what is the result for limit n tends to infinity of this expression n c k P power k 1 minus P power n minus k, where P is going to be lambda times t by n.

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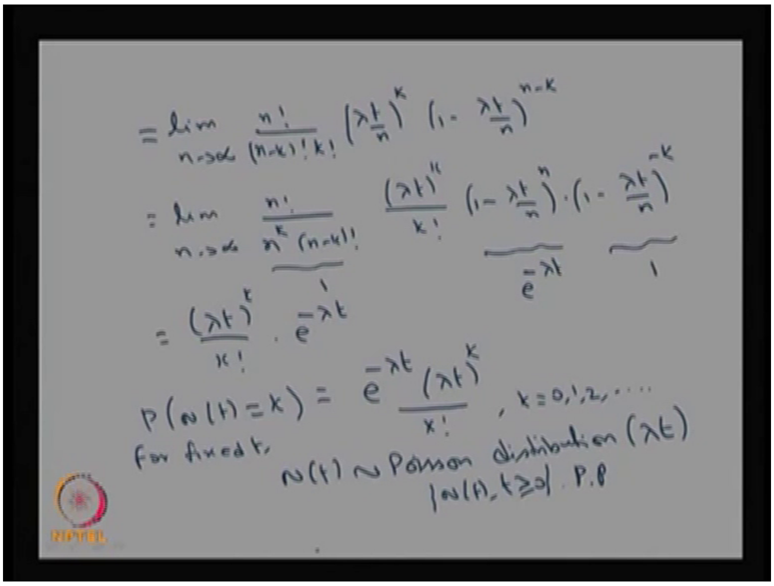

If I do the simple calculation, let me explain the limit n tends to infinity that is same as limit n tends to infinity of n c k, I can make it as a n factorial n minus k factorial and the k factorial. And that is a lambda t by n power k, and that is 1 minus lambda t by n power n minus k.

And that is same as the limit n tends to infinity of n factorial, and here this n power k, I can take it outside, and n minus k factorial. And lambda t power k and divided by k factorial so, this k factorial I take it inside. And the power 1 minus lambda t by n power n minus k I split it into 1 minus lambda t by n power n into 1 minus lambda t by n power minus k n.

So now I can look as n tends to infinity, this is nothing to do with the n, therefore, lambda t power k by k factorial will come out. So, this result is going to be lambda t power k by k factorial. And this will land up as n tends to infinity, this is going to be e power minus lambda t, and this will land up 1. And this is also land up 1 as n tends to infinity therefore, I may land up it is e power minus lambda t.

Hence the final answer of what is the probability that k arrival takes place in the interval 0 to t, that is going to be e power minus lambda t, and the lambda t over k by k factorial and the possible values of k can be 0, 1, 2 and so on.

For fixed t, if you see this is same as, for fixed t it is going to be a random variable; for all possible values of t it is going be a stochastic process. So, for fixed t the N of t is a random variable. And that probability mass function is e power minus lambda times t, lambda t power k by k factorial. So, lambda is a constant for fixed t lambda into t that is going to be a constant.

Therefore, the right hand side look like the probability mass function of the Poisson distribution, therefore, for fixed t the N of t is a Poisson distribution, the random variable N of t for fixed t, it is going to be a Poisson distribution with the parameter lambda times t, lambda is a constant and for fixed t, t is a constant.

So, lambda multiplied by the t, again this is going to be a constant. Therefore, for fixed t it is going to be a Poisson distribution with the parameter lambda multiplied t. Therefore, for possible values of t the N of t is going to from a stochastic process. And the since a for fixed t it is going to be a Poisson distribution, the collection of a random variable and each random variable is a Poisson distribution. Therefore, this is going to be call it as the Poisson process.

The way I have we have explained earlier, each random variable is a Bernoulli distributed random variable the collection of random variable is a Bernoulli process. Similarly, each s n is going to be a binomial distribution therefore, the collection is going to be a binomial process. The same way for fixed t it is going to be a Poisson distribution therefore, that collection is going to be call it as Poisson process. So now, we have developed N of t is going to be a Poisson process.