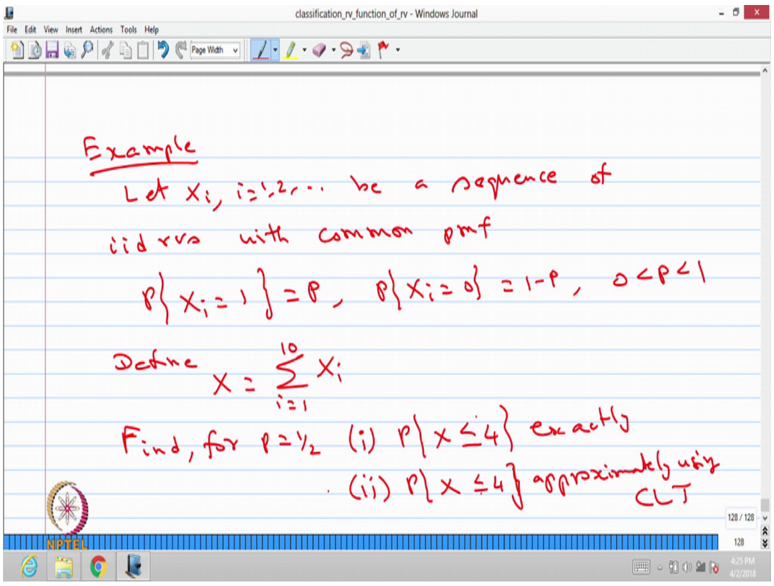
Introduction to Probability Theory and Stochastic Processes
Prof. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 46

(Refer Slide Time: 00:03)

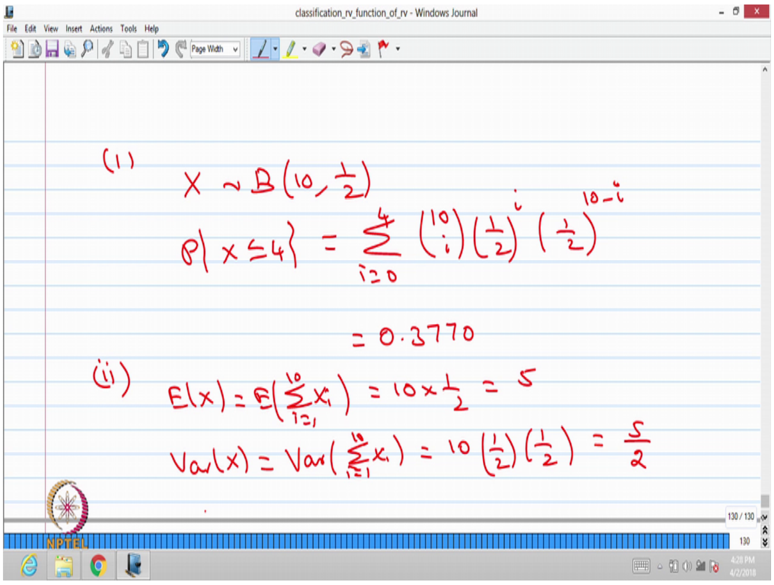


As a first example let us consider let Xi be a sequence of iid random variables with common probability mass function that is probability of Xi take the value 1 that is P and probability of Xi takes a value 0 that is 1 minus, where P is lies between 0 to 1. It is basically a Bernoulli trials you have a sequence of random variable with the common probability mass function.

We define a random variable X which is sum of a 10 random variables. The question is find for P is equal to 1 by 2; find for P is equal to 1 by 2, find probability of X is less than or equal to 4 exactly. The second one probability of a X is less than or equal to 4 approximately using central limit theorem CLT.

Let us go for finding this probability exactly the first one, X is sum of first 10 random variables.

(Refer Slide Time: 02:15)

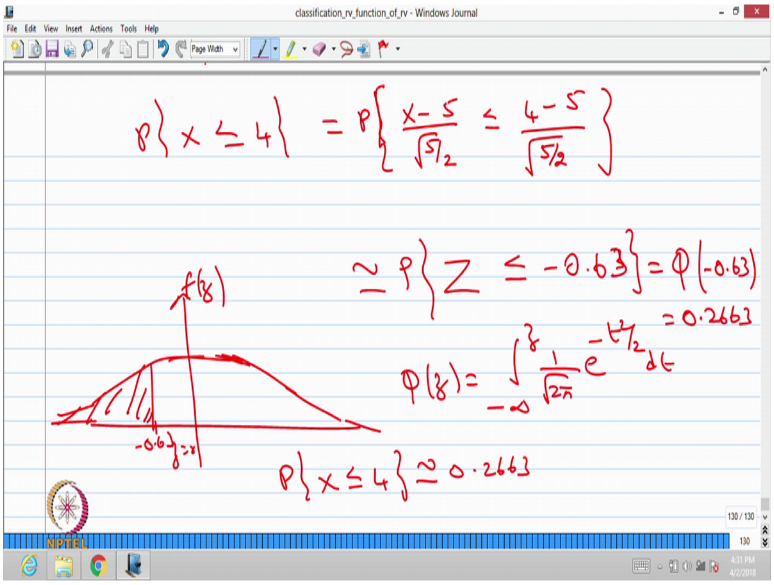


Each random variable is Bernoulli distributed. Therefore, the X is going to be Binomial distributed with the parameters 10 comma 1 by 2. Each one is the Bernoulli distributed with the parameter 1 by 2. Therefore, all are n such independent Bernoulli distributed random variable. Therefore, X is binomial distributed random variable with the parameters 10 comma 1 by 2.

So, now you can find out the probability of X is less than or equal to 4; that is summation i is equal to 0 to 4 10 c i P power i 1 minus P power 10 minus i. If you do the simplification, you will get the answer 03770. This is by finding the probability exactly because you know the distribution of X. The second part finding the probability approximately using CLT; for that we can use the mean and variance of X without using the distribution of X.

So, if you want to find out the probability of X is less than or equal to 4 approximately, first we will find out what is the expectation of X. That is nothing but expectation of sum of 10 random variables. So, the mean is going to be P. So, 10 times P. So, 10 into 1 by 2, that is equal to 5. Similarly, you can find the variance of X, that is going to be variance of sum of random variables that is going to be 10 into P into 1 minus P; that is equal to 5 by 2. So, to apply the central limit theorem, we need those random variable to be iid’s and mean and variance should be known, then you can find out the probability approximately using a central limit theorem.

(Refer Slide Time: 05:00)



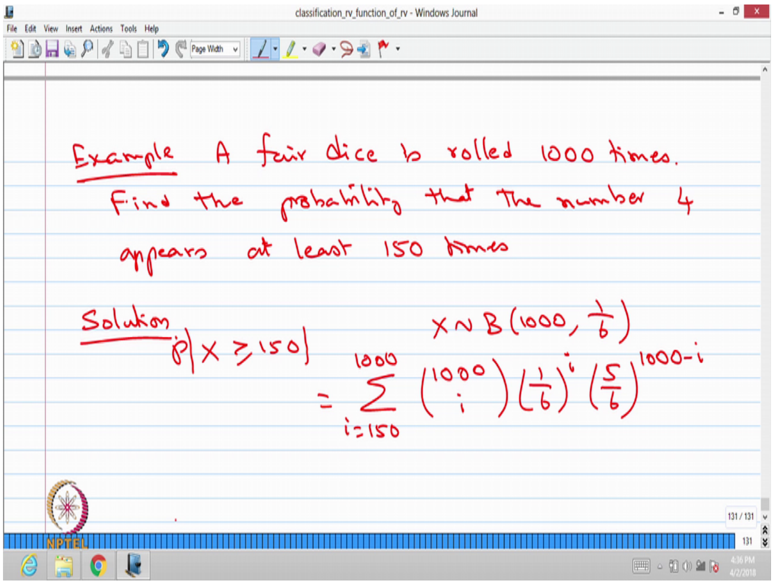
So, the probability of X less than or equal to 4 that is same as the probability of X minus mean; mean is the 5 divided by square root of variance that is 5 by 2 that is less than or equal to 4 minus 5 divided by square root of 5 by 2. That is approximately probability of I am replacing X minus 5 divided by square root of 5 by 2 approximately with the standard normal distribution. Less than or equal to this is by the central limit theorem. The first step probability of X is less than or equal to 4 that is same as probability of X minus 5 divided by square root of 5 by 2 less than or equal to 4 minus 5 divided by square root of 5 by 2.

After subtracting the mean and the standard deviation that is approximately a standard normal. I am using this approximation symbol, that is less than or equal to 4 minus 5 divided by square root of 5 by 2 that is going to be minus 0.63. That is nothing but when you have a standard normal distribution probability density function z is equal to 0 and this is the probability density function of z. So, minus 0.63 somewhere suppose this is a point minus 0.63. So, this probability that is nothing but we usually we make the notation psi of z is minus infinity to z 1 divided by square root of 2 pi e power minus t square by 2 d t.

So, this is nothing but psi of minus 0.63. So, from the table 1 can find what is the psi of minus 0.63 when psi is defined in the form of minus infinity to z 1 divided by square root 2 pi e power minus t square by 2. This value is going to be 0.2663. So, when you find out approximately probability of X is less than or equal to 4 that is approximately 0.2663 by using a central limit theorem; whereas, by getting the exact value you are getting 0.3770. You are getting a exact answer because you know the distribution of this random variable X; many times when you had the random variables you may not be able to easily get this distribution of the random variable. In that case in that case one can always apply the central limit theorem to find out the approximate probability, not the exact probability.

So, exact probability is possible only if you know the distribution; whereas, if you know mean and variance and all the random variables are independent, need not be identical still you can go for finding the approximate probability using the central limit theorem.

(Refer Slide Time: 08:44)



We will discuss one more problem. Example 2, A fair dice a fair dice is rolled 1000 times. The question is find the probability, find the probability that the number 4 appears at least 150 times. Find the probability that the number 4 appears at least 150 times.

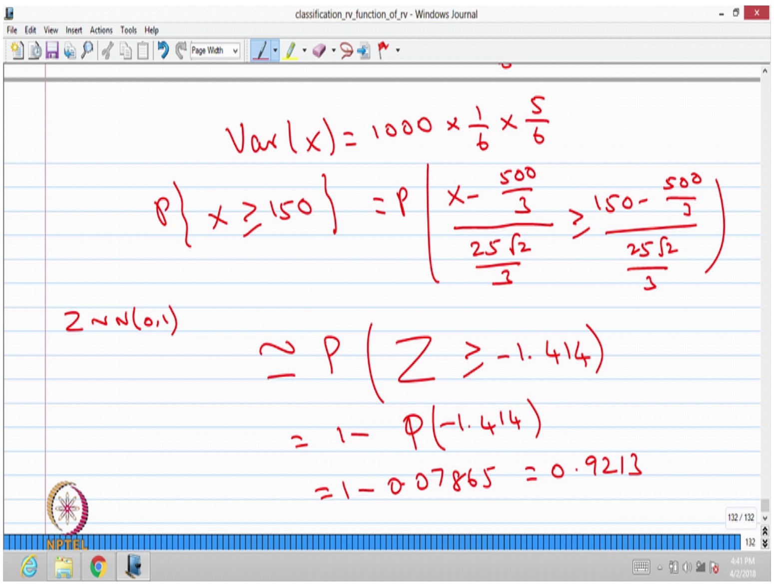
Here, we dint say that use the central limit theorem, but we can apply the central limit theorem because of the difficulty while getting the answer for this or the computation is very difficult; therefore, we go for the central limit theorem.

Let X denotes the number of times the number 4 is obtained and the question is; what is the probability that X is going to be greater than or equal to 150? X is the number of times the number 4 is obtained and the question is; what is the probability that X is greater than or equal to 150?

Since, a fair dice is rolled 1000 times independently; therefore, you can conclude X follows sum of 1000 Bernoulli distributed random variable with the parameter of getting a number 4 probability is a 1 by 6. Therefore, the parameter is 1000 comma 1 by 6. The n is 1000 and the probability of success P is 1 by 6. By computing probability of X is greater than or equal to 150 is nothing but by using this formula that is i is equal to 150 to 1000 of 1000 c i 1 by 6 power i 5 by 6 power 1000 minus i. To get this result, you need factorials which involve which comes from the 1000 c i which is very difficult or the negation is 1 minus summation i is equal to 0 to 140 and both are tedious calculation.

Therefore, we are going for getting the approximate probability instead of exact probability. That means we would not use the binomial distribution for the random variable X; whereas, we will use only the mean and variance of the random variable X, not the distribution. Because if we use the distribution, you can get the exact probability; but computationally it is not feasible. So, we apply the central limit theorem, apply central limit theorem.

(Refer Slide Time: 12:51)



For that you need a mean of that random variable. Mean since it is a binomial distribution, the mean is going to be n into P or even you can use the mean of each Bernoulli distributed random variable then you can just multiply by 1000. So, that is going to be 1000 multiplied by 1 by 6. This is going to be the mean and you can get the variance of X that is going to be 1000 multiplied by P into 1 minus P. So, this is going to be the variance of X.

Now, I can apply the CLT that is probability of X is greater than or equal to 150; that is same as probability of X minus mean. If you simplify that is 500 divided by 3 divided by ; if I simplify that it is 25 times square root of 2 divided by 3 greater than or equal to 150 minus 500 divided by 3 divided by 25 times square root of 2 by 3. This is same as approximately probability of Z; where Z is where Z is a standard normal distribution greater than or equal to. So, this quantity is minus 1.414; that is same as 1 minus psi of minus 1.414; where the psi is defined in the same way in the earlier form.

So, if you see the table for the psi of 1.414, you will get the answer that is 1 minus 0.07864. So, this is 0.9213. In this problem, we are not able to get the exact probability even though we know the distribution. Even though we know the distribution of X is binomial distribution computationally it is not possible, but by applying the central limit theorem we are able to find out the probability of X is greater than or equal to 150. That is the advantage with the central limit theorem.

As long as at least a second order moment exist and the random variables are independent, need not be identical and sum of random variable for larger n approximately normal distribution. In other words, sum of random variables their mean divided by the standard deviation will be approximately standard normal distribution by using central limit theorem.

So, with this we are completing this model of limiting distributions with the 3 lectures; first lecture is modes of convergence. Second one is law of large numbers; in that we discuss the weak law of as well as the strong law of large numbers. Then finally, we discuss the central limit theorem with the proof and two examples.