Introduction to Probability Theory and Stochastic Processes
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Lecture – 43

Now, we will discuss the third mode of convergence that is convergence in rth moment.

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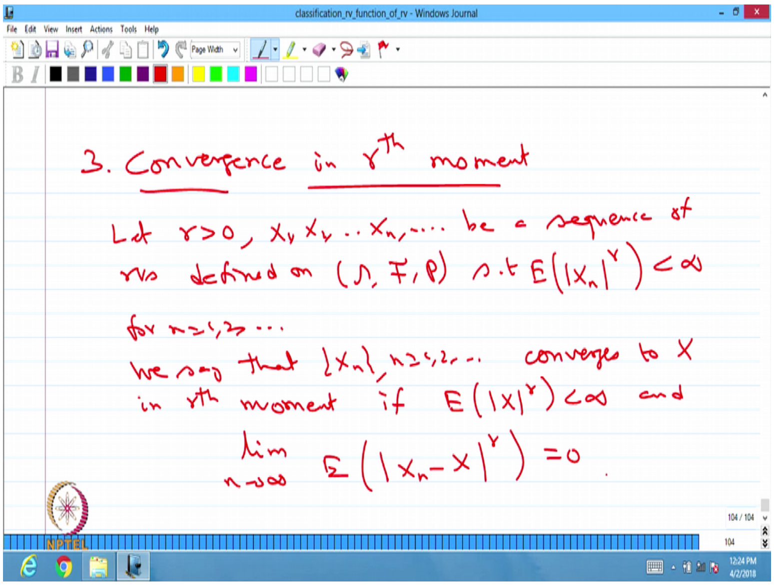

Let, r greater than 0 x1, x2 and so on xn and so on be a sequence of random variables defined on omega F capital P such that the rth moment for the sequence of random variable is exist; the absolute sense expectation is going to be finite for n is equal to 1, 2 and so on.

We say that, we say that this sequence of random variable converges to the random variable X in rth moment, if the expectation of the random variable X the rth moment that is finite and X also defined in the same probability space of course.

And limit n tends to infinity; limit n tends to infinity expectation of in absolute sense X n minus X to the power r that expected quantity is going to be 0. We are keeping r greater than 0 and we have a sequence of random variable defined in the probability space such that rth moment exist for those sequence of random variable.

We say that the sequence of random variable converges to X in rth moment provided the rth moment exist for the random variables X, which is defined in the same probability space and the limit n tends to infinity expectation that difference of this random variable, the rth moment is going to 0.

If these conditions are satisfied, then we can conclude the sequence of random variable converges to the random variable in the rth moment.

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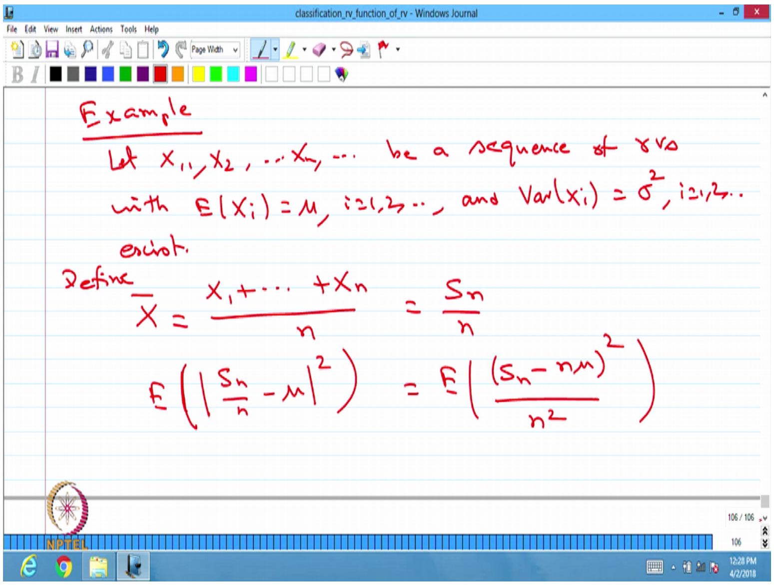

We can go for the example to explain this mode of convergence also. That is let X1, X2, Xn and so on be a sequence of random variables with mean mu ok, identical mu and variance of each random variable that is sigma square or exist both exist and is denoted by mu for the mean and variance is sigma square. I am going to define the new random variable that is sum of n random variables divided by n; sum of random variable we usually use the notation S n.

But now, I am going to divide that sum divided by n. So, I am going to use a notation called X bar. X bar is the random variable; it is a function of n. I am not writing n in the left side, X bar is the sum of random variables divided by n fine or we can another notation we can use as a function of S. S n divided as a function of n. S n divided by n. Either I can use X bar or S n divided by n, both are one and the same that is sum of random variable. So, this is the.

So, I am going to define this random variable for different n will go for finding first expectation of expectation of absolute of S n by n minus mu the whole square. In absolute sense, we will find what is the value of S n by n minus mu square in absolute sense that is same as expectation of S n is the square S n minus n mu divided by S n minus n mu whole square divided by n square.

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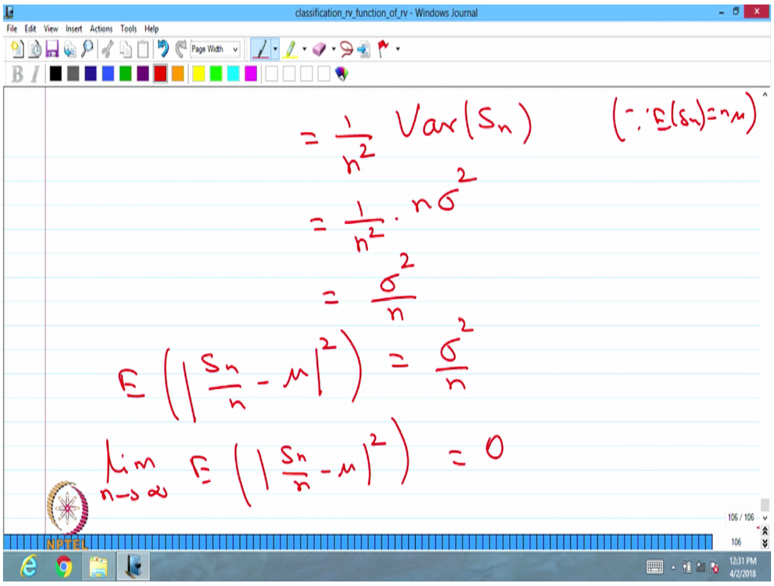

That is same as, I can take 1 by n square outside, if you see S n minus n mu whole square expectation. If you find out the expectation of S n expectation of S n that is the expectation of X1 plus X2 plus Xn all the expectation of Xi's are same that is mu. Therefore, it is n mu.

So, since expectation of S n is n mu. So, the expectation of S n minus n mu whole square is nothing but variance of S n; that is same as 1 divided by n square. I will make a additional condition X is are sequence of independent random variable with the mean is mu and the variance is sigma square.

Therefore, the variance of S n is nothing, but variance of X1 plus X2 plus so on plus Xn and variance of X is or sigma square. Therefore, it is going to be n sigma square. Therefore, this is going to be sigma square by n. We got the expectation of absolute of S n by n minus mu whole square that is going to be sigma square by n.

Now, let me apply limit n tends to infinity of expectation of absolute of S n by n minus mu the whole square. Since, n is in the denominator this becomes 0.

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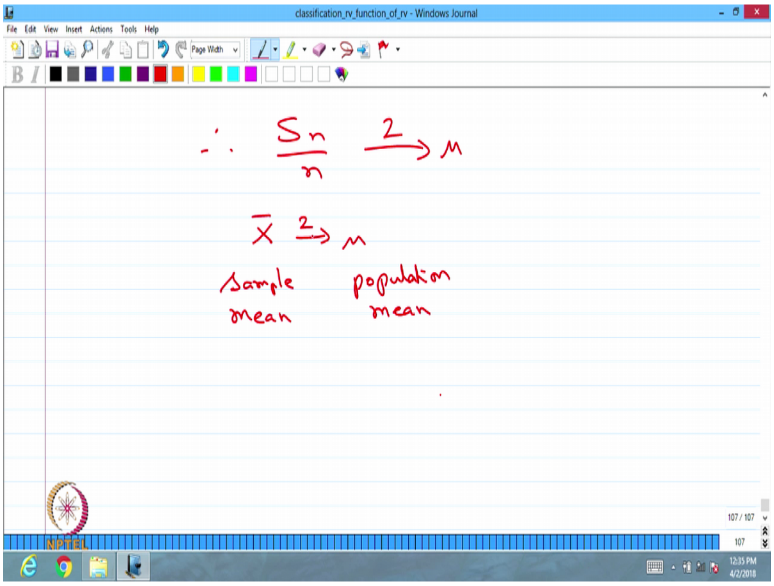

So, this is the definition we wanted to convergence in rth moments; that means, I can conclude, S n by n converges to mu in second order moment. The left right hand side it is not a random variable, it takes a value mu; that means it is a random variable takes a value mu with the probability 1.

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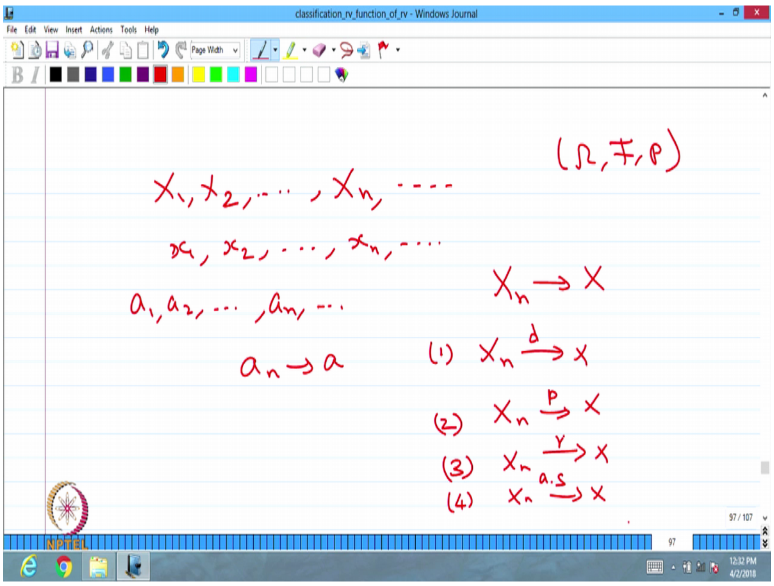

So, Xn converges to rth moment with number r. In notation, we write S n by n converges to mu in the second order moment. In the right hand side mu means it is a random variable takes a value mu with the probability 1. I have already written X n by n as the X bar. So, X bar converges to mu in second order moment.

In statistics we use the X bar that is sum of random variable divided by n that is from the n random variable we call it as a Sample mean the sequence of random variable having the mean mu, variance sigma square. So, the mu is called Population mean. So, the conclusion is if you have a population with the mean mu and if you get a sample of size n, then the sample mean will converge to the population mean as n tends to infinity.

That means, if you for a large sample or a large sample the sample mean will converge to the population mean and this converges takes place in the convergence in rth moment, where r is equal to 2 here. Because we are finding the second order moment convergence is used. Therefore, sample mean converges to the population mean in second order moment.

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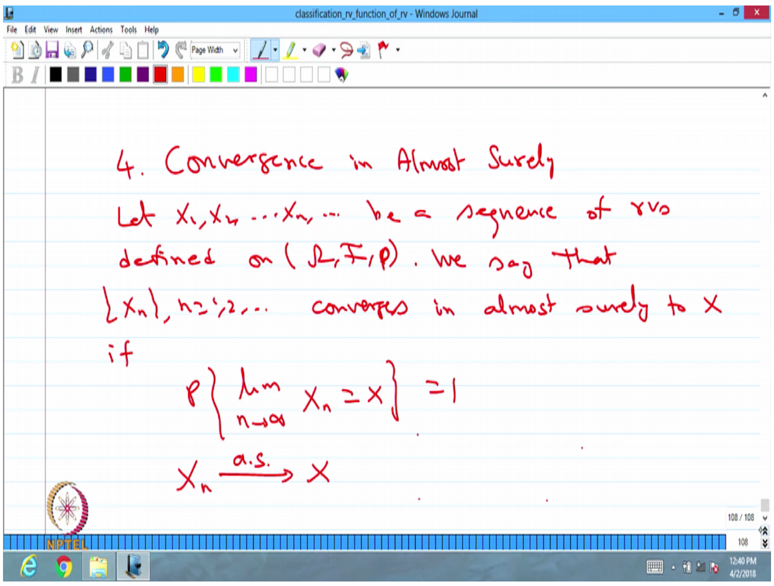

We move to the fourth mode of convergence; convergence in Almost Surely. Let X 1 X 2 X n and so on be a sequence of random variables defined on the probability space omega F P. We say that the sequence converges; the sequence converges in almost surely to the random variable X, if the probability of limit n tends to infinity. X n is equal to X that is 1.

So, here the random variable X is defined in the same probability space. The probability of limit n tends to infinity X n is equal to X that is going to be 1, then one can conclude the sequence of random variable convergence to the random variable X in almost surely. So, we write it X n converges to X above the arrow, you write a dot S; that means, this convergence in almost surely.

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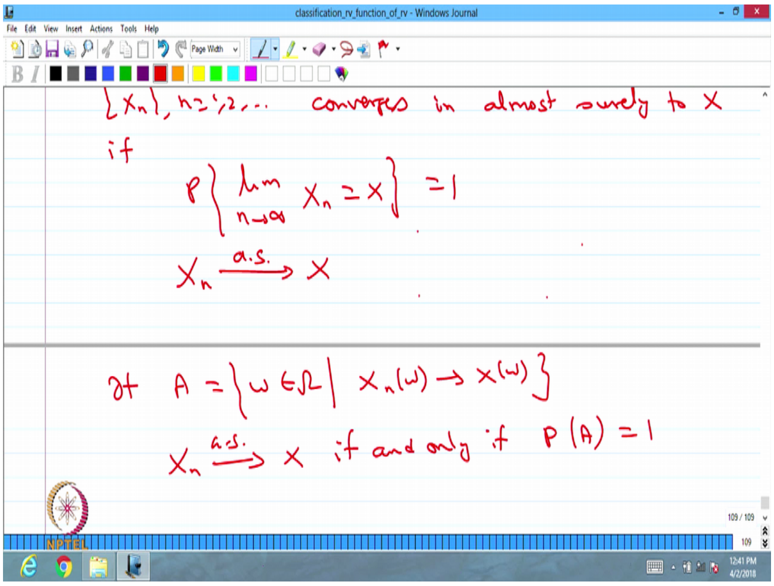

Suppose, we make a suppose we make a event A is nothing but collection of w's belonging to omega such that X n of w tends to X of w. In that case when I say X n converges to X almost surely, if and only if the probability of the event A that is going to be 1. If I make a event A that is nothing but as n tends to infinity X n of w will tends to X of w and collect those w's that is going to be the event A. If the probability of that event A is going to be 1; then, we can conclude the X n converges to X almost surely.

In other words, negation of the event A that probability is going to be 0; that means, you collect all possible outcomes from omega which does not converge X n of w to the X of w, you collect those possible outcomes whose measure is 0. Then, we also we can conclude X n converges to X almost surely; that means, it is the whole unit mass is attached with the collection of possible outcomes; those outcomes or the outcomes of X n of w converges to X of w.

Those outcome satisfies X n of w converges to r tends to X of w as n tends to infinity. So, if you include those possible outcomes or w's those possible outcomes whose probability mass is 1 or whichever is not satisfy in this condition, those possible outcomes whose probability mass is 0. Then we can conclude X n converges to X in almost surely. We will go for the example for this.

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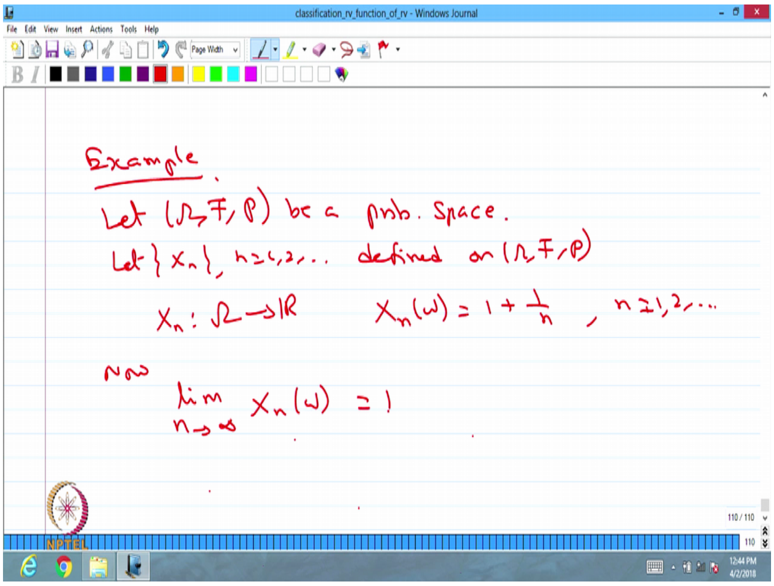

Let omega F and P be a probability space. Let sequence of random variable defined on the probability space omega [FL] P as X n of X n is mapping from omega to real line such that X n of w is equal to 1 plus 1 by n for n is equal to 1, 2 and so on.

Now, will go for computing limit n tends to infinity X n of w where w is belonging to make a limit; n tends to infinity of X n of w that is going to be 1.

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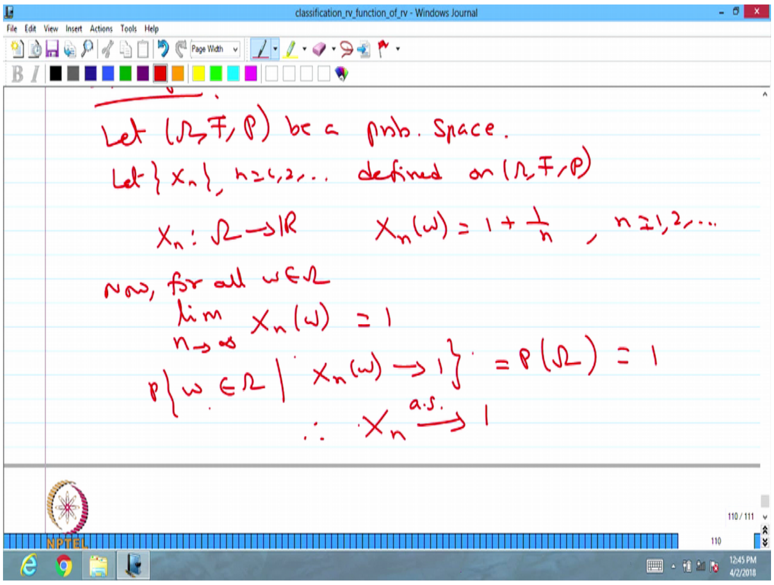

That means, the set of all w's belonging to omega such that X n of w will tends to 1. You collect those possible outcomes find out the probability of that that is nothing but P of omega that is same as 1. In that case, one can conclude X n converges to 1 almost surely.

Because for all because for all w belonging to omega the limit n tends to infinity X n of w is equal to 1. Therefore, finding the probability of w belong into make a satisfying this condition that is X n of w tends to 1 as n tends to infinity, that is going to be the whole possible outcomes that is nothing but the p of omega that is equal to 1. Therefore, it satisfies the condition for almost surely. Therefore, X n converges to 1, almost surely. So, the same example can be considered for other mode of convergence also.

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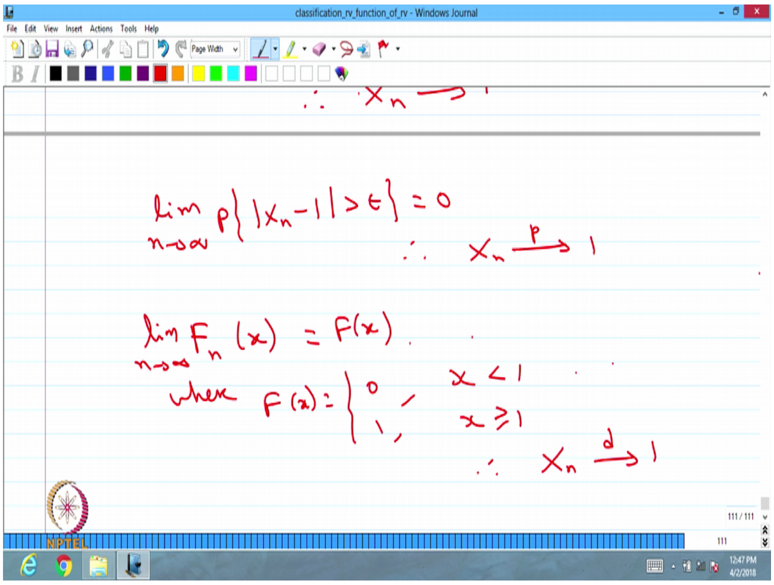

That means, one can prove limit n tends to infinity. The probability of absolute of X n minus 1 greater than epsilon, that also we can prove this is going to be 0. This implies X n converges to 1 in probability.

Similarly, for the same example, one can prove the sequence of random variable CDF's has a limit n tends to infinity will converges to F of X, where F of X is where F of X is takes a value 0 till 1 and 1 onwards, it is going to be 1. Therefore, we can conclude the same example X n converges to 1 in distribution.

That means, there are few a sequence of random variable may converge to more than 1 mode of convergence. So, that can be connected in 1 nice way that is suppose you have a sequence of random variable and the random variable X defined on the probability space omega F P convergence in distribution and we have a convergence in probability.

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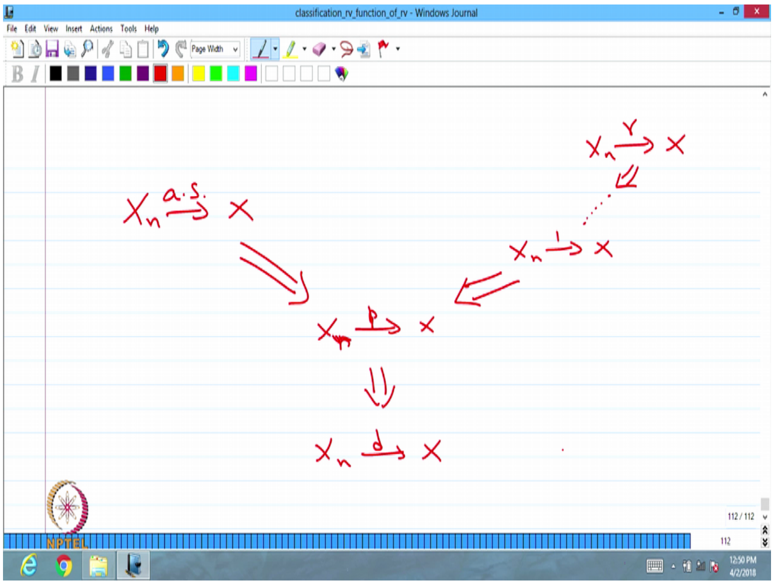

Suppose, some sequence of random variable convergence in probability this implies convergence in distribution. The converse is not true it may be true for some few problems, in generalities not true.

Therefore, I am not writing the up adder converges not true in general. Similarly, if you have a X n converges to X in the first order moment, this implies the converges in probability; Converse is also not true in general here. Suppose, you have a convergence in rth moment that implies the r minus 1 and so on till the first order moment.

But here, again converse is not true because of 2 reasons. The first order moment exist that does not being that the second order moment is going to exist; even if it exist it does not imply the convergence takes place. Therefore, the converse is not true. Convergence in a rth moment exist, then you can go for tilde first order moment convergence that implies the convergence in probability that implies convergence in distribution.

Now, I am giving the connection with the convergence in almost surely that is X n converges to X almost surely. This implies convergence in probability, you see the different direction. Almost surely convergence implies convergence in probability convergence in probability implies convergence in distribution; no where converse is true in general.

There are some additional condition we can mention for some sequence of random variable, then we can conclude the X n converges in X in probability that convergence in almost surely also. But for that you have to make a some additional condition. In general, the converse is not true for all the cases.

So, with this relation I am going to stop the convergence or modes of convergence. In the next class we will discuss about the law of large numbers.