Introduction to Probability Theory and Stochastic Processes
Prof. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Module – 08
Lecture – 42
Limiting Distributions

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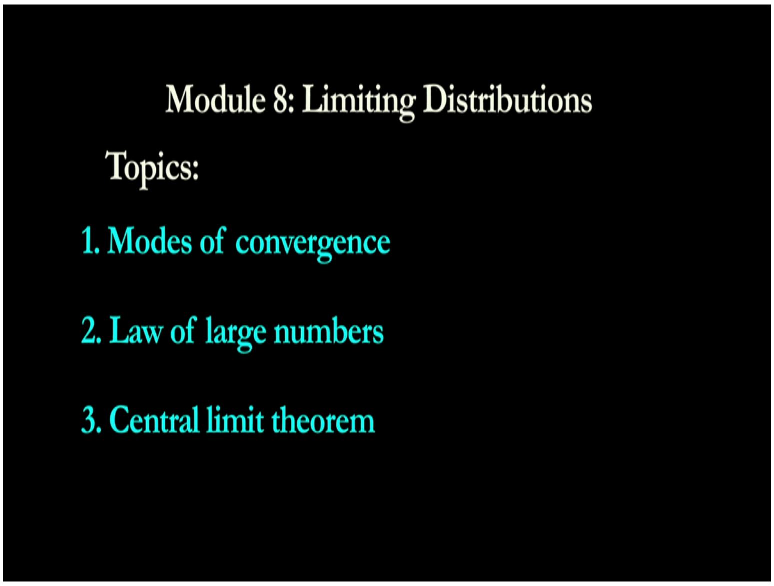

In this model, we are going to discuss a limiting distributions.

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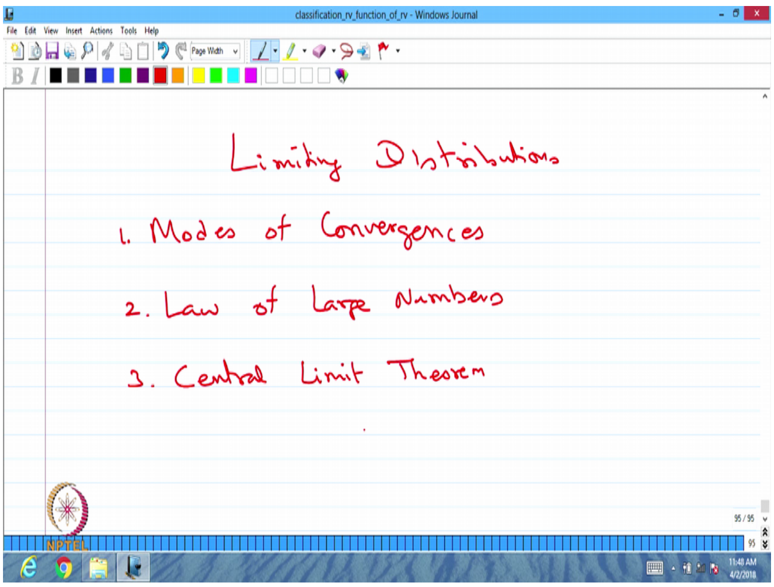

So, we cover three important topics; first topic is called modes of convergences and the second topic we are going to discussed we are going to discuss law of large numbers and the third topic which we are going to discuss, that is a very important topic, that is central limit theorem. In this model, we are going to discuss these three topics has a three different lectures. In the first lecture, we are going to discuss modes of convergences; second lecture, we are going to discuss law of large numbers; third lecture, we are going to discuss central limit theorem.

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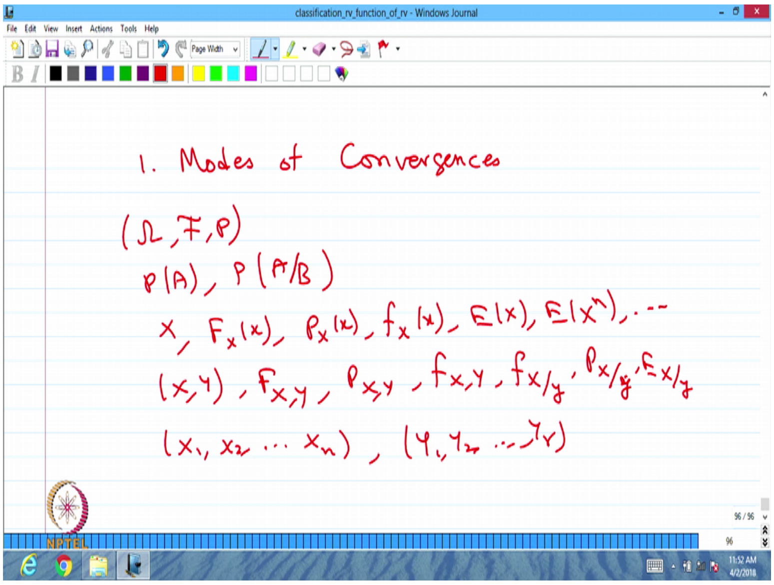

The first lecture is on modes of convergences. Till now what we discussed the probability, space probability of a event, then we discussed a conditional probability of events, then we have introduce a random variable, then we discussed the CDF of the random variable. Then, based on the discrete type or continuous type random variable we discussed the probability mass function, probability density function.

After we introduced one random variable in a probability space we have discussed the distribution in the form of a CDF density function or mass function. Later, we said one random variable is not enough to solve some particular problems; you may need more than one random variables, to be defined in the same probability space to solve the given problem.

Then we introduce two random variables then we discuss the CDF, we discussed the joint probability mass function, we discussed joint probability density function, then later we is discuss the conditional probability density function, conditional probability mass function for the first random variable, we discuss the mean variance and so on, that I missed earlier second order moment in a third other moment and so on and here we discuss the conditional distribution, conditional probability conditional expectation and so on.

Not only two random variables, the later we introduce n random variables in the same probability space, then we discuss the joint distribution of n dimensional random variable, after that we introduce the transformation from one dimensional to another n dimensional random variable or r dimensional random variable and so on.

Now, we are going to discuss not a 1 random variable, not 2 random variables not n finite random variables we are going to discuss sequence of random variables. This is also possible when you solve a given problem you may need to know the sequence of random variable and what is their distribution as n tends to infinity?

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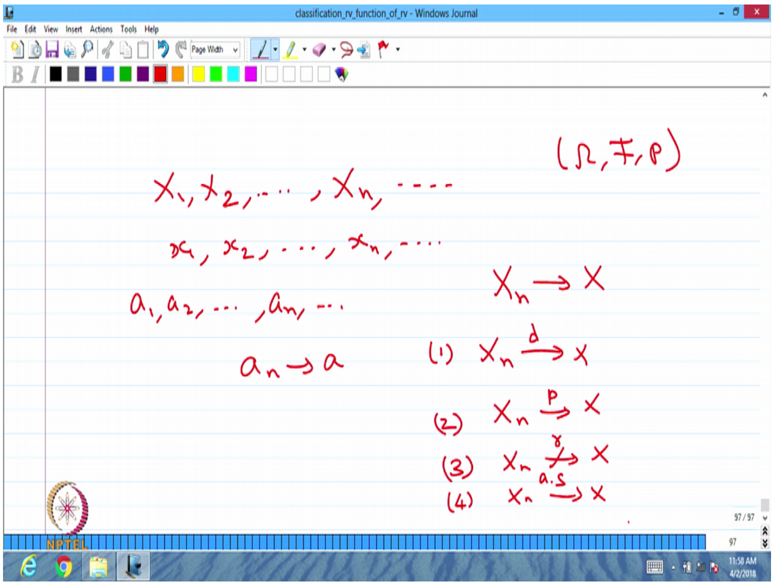

The first question is comes whether if you have a sequence of random variables whether if you have a sequence of random variable whether this sequence of random variable converge or not converges to one random variable or not if it converge what is the distribution of that sequence of random variable that is going to be the topic of modes of convergence for the random variables.

For these first we can understand how the sequence of a real number real numbers converges suppose you have real numbers a 1 a 2 and so on a n and so on whether this sequence of a real numbers converges or not if it is converges; that means, a n converges to say a what is the value of a, that we have discussed in the any real analysis courses of a sequence and series of real numbers and so on.

The same concept the we are planning to introduce for the sequence of random variables , but the only difference is the random variable takes a real numbers with some distribution, that means; the X 1 random variable can take the real values x 1 with some distribution. Similarly, the random variable X 2 may takes a random may takes a values a real values x 2 with some distribution and so on.

Similarly, the random variables X n can take the real number x n with some distribution. If you know the distribution of a each random variable X 1, X 2 and so on you have a sequence of random variable all are defined in the same probability space, that is very important all are defined in the same probability space and if you know the distribution of this sequence of random variables whether this sequence of random variable converges to one random variable or not.

If it converges what is the distribution of that that means, if I write X n converges to X this is notation am I write X n converges to X; that means, I have a sequence of random variable X 1, X 2, X 3 and so on whether this sequence of random variable converges to the one random variable that is call it as X. If I know the distribution of X size what could be the distribution of x? That is the question.

In sequence of real numbers converges to one real number a that may be easy comparing to the sequence of random variable converges to the one random variable. Since, each random variable attached with some distribution for some random variables moments of a first order may exist further moment may not exist and so on. Therefore, you cannot make a only one way you can conclude this sequence of random variable converges to one random variable x. There may be more than one ways you can conclude that this sequence of random variable converges to one random variable that we call it as a modes of convergence.

So, we are going to discuss there are 4 modes of convergence the first mode of convergence we write it as X n converges to X in distribution by writing small letter d above the arrow I am going to give the definition one by one in detail with examples. Comparing to the sequence of real numbers how it converges to one random variable. Similarly, we are going to discuss how the sequence of random variable converges to one random variable in different ways the different ways we say it as a different modes.

The first mode of convergence that is called the convergence in distribution with the letter small d above their the second one that is convergence in probability the third one convergence in r-th moment, the forth one convergence in almost surely a dot s, above the arrow if you write d that means, convergence in distribution, above the arrow if I write small p that means, convergence in probability, above the arrow I put the slash with r; that means, it is a convergence in the r-th moment the forth one convergence in almost surely.

So, these are all the four ways the sequence of random variable different in a same probability space converges to one random variable which we denoted as a capital X that is also defined in the same probability space in four different modes of convergence.

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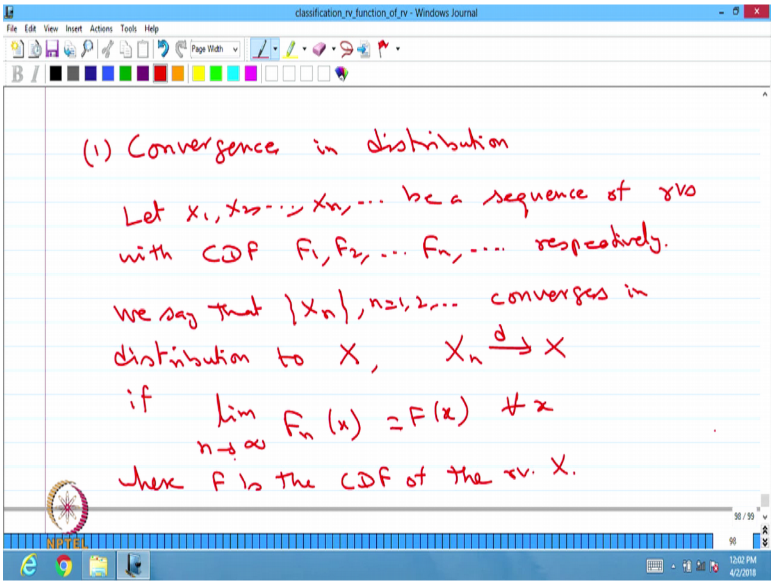

Let us start with the first mode of convergence that is convergence in distribution in distribution let X 1 X 2 and so on X n and so on be a sequence of random variables with CDF F 1, F 2, and so on F n respectively that means, the random variable X 1 has a CDF F 1 the random variable X 2 has a CDF F 2 and so on. All these random variable defined in the same probability space.

We say that we say that X n the sequence n take the value 1, 2 and so on converges in distribution to the random variable denoted by capital X can be written as X n converges to X in distribution if limit n tends to infinity F n of x that is same as F of x for all x where F is the CDF of the random variable.

X as long as the limit n tends to infinity F n of x is same as F of x where F of x is CDF of the random variable X that is valid for all X this condition valid for all X then we can conclude the sequence of random variables convergence in distribution to the random variable X note that a F 1, F 2, F n and so on that is the CDF of the sequence of random variables respectively and that converges to a function that is the CDF of the random variable X, then we can conclude this convergence in distribution to the random variable X.

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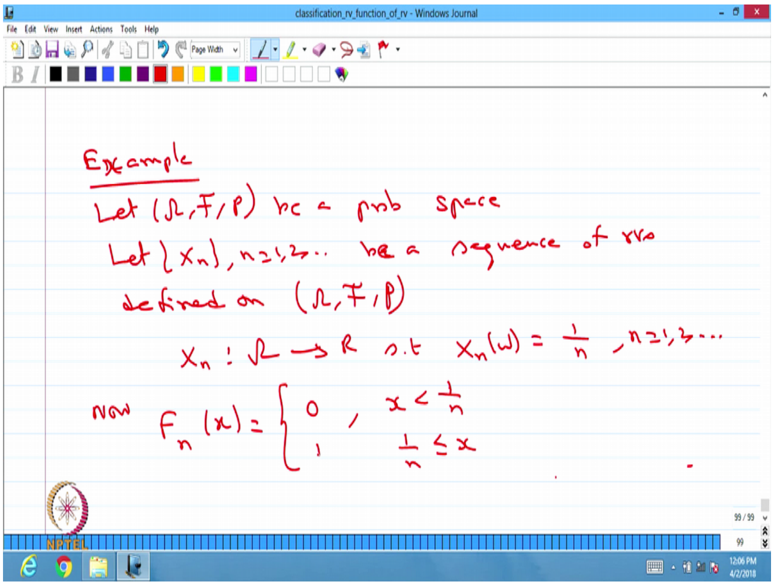

Let us give a one simple example through that we can understand the definition clearly. Example; let omega, F, capital P be a probability space. Let X n; n is equal to 1, 2, and so on be a sequence of random variables defined on the probability space omega, F, P that is defined as X n is defined from omega to R such that such that X n of W that is equal to 1 by n for n is equal to 1, 2 and so on. So, we are defining the sequence of random variable from omega to R such that X n of the W takes a value 1 by n for n is equal to 1, 2, and so on.

Now, we will find out what is CDF of a this sequence of random variable interview. So, if you find out the CDF of the n-th random variable as a function of x this is going to take the value 0, when x is going to be lesser than 1 by n from 1 by n onwards when x is going to be 1 by n onwards it is going to take the value 1. So, this is going to be the CDF of the sequence of random variable x size. So, here n takes a value 1, 2 and so on.

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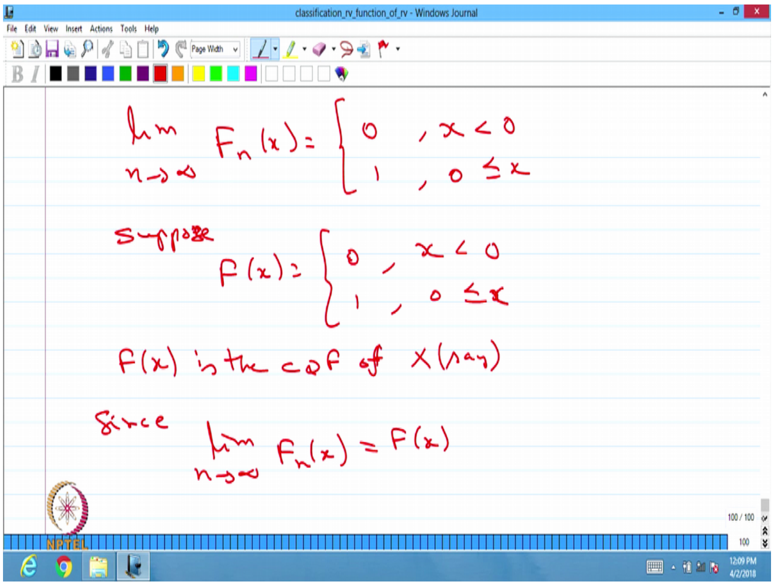

Now, let us go for finding the limit n tends to infinity of F n of x what could be the value as limit n tends to infinity for the F n of x this is going to be 0 when x is lesser than 0 because as n tends to infinity of 1 by n that becomes 0 and 1 from 0 onwards. This is a limit n tends to infinity of F n of x.

Suppose, I denote this as the F of x suppose I make F of x that takes a value 0, when x is lesser than 0, 1 from 0 onwards verify whether this is going to be the CDF of some random variable it start from 0 land up 1 and so on. It is satisfies all the properties of a CDF therefore, this is the F of x is the CDF of some random variables you denote it as a capital X say F of x is the CDF of the some random variable X by seeing limit n tends to infinity of F n of x that is same as F of x.

Since limit n tends to infinity F n of x that is same as F of x, where F of x is the CDF of some random variable x and the left hand side F n of x is the CDF of the sequence of random variable or F n of x is the CDF of the random variable X n as the limit n tends to infinity that is same as the CDF of the random variable x. Therefore, we can conclude this is the condition is satisfied by the convergence in distribution.

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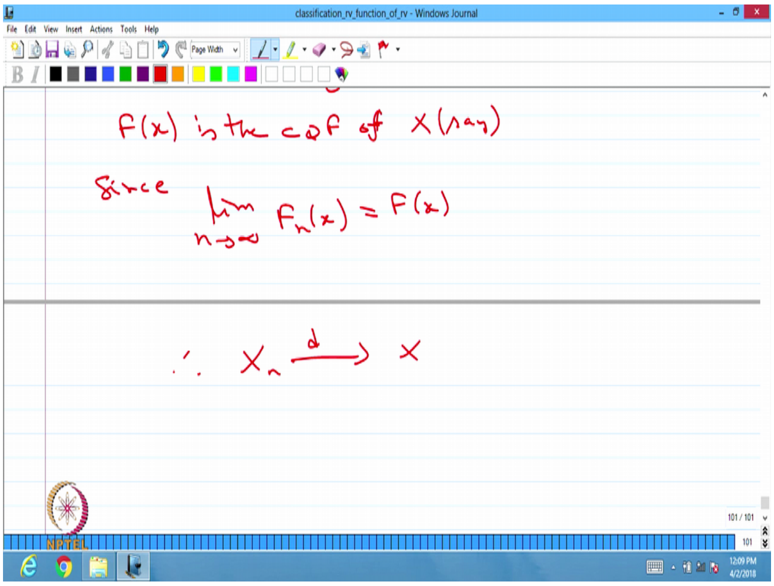

Therefore, you can conclude the X n converges to X in distribution. There is a possibility the sequence of random variable CDF’s may converge to some function that may not be the CDF. So, as long as this sequence of CDF converges to some function that is also the CDF of some random variable then you can conclude the X n converges to X in distribution. So, like that we have some more problem that we will discuss little later; that means, when we discuss other mode of convergence we can verify whether this satisfies a convergence in distribution also.

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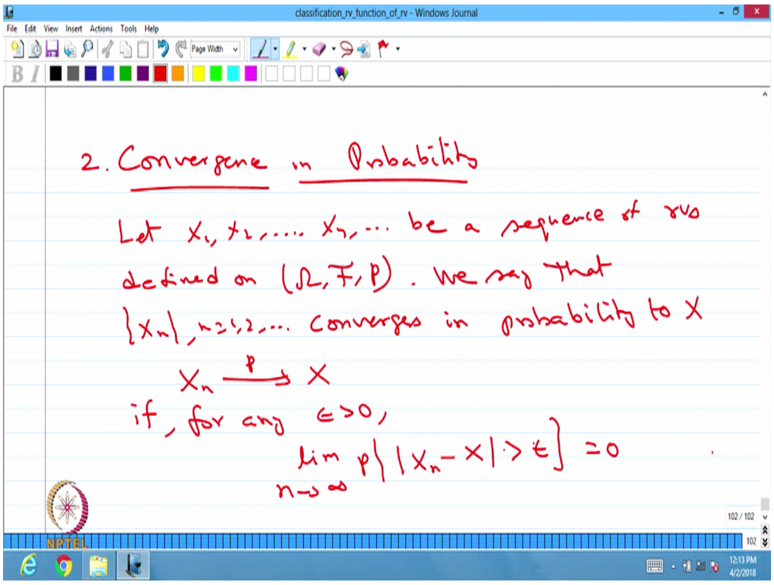

So, now will move into next mode of convergence that is a convergence in probability. Let X 1, X 2, X n and so on be a sequence of random variables defined on the probability space omega, F, P. We say that this sequence of random variable converges in probability to the random variable capital X and write it as X n converges to X in probability if for any epsilon, which is greater than 0. Limit n tends to infinity probability of absolute of X n minus X greater than epsilon that is equal to 0. If this condition is satisfied for any epsilon greater than 0 finding out the probability in absolute is X n minus X greater than epsilon limit n tends to infinity is going to be 0, then we conclude the this sequence of random variable converges in probability to the random variable X.

Note that to verify this sequence of random variable converges in probability you should know the random variable X, then finding of the probability after is verified then you can conclude this sequence of random variable convergences in probability that means, you should know about the distribution of the random variable X or at least you should know how to compute the probability of absolute of X n minus X greater than epsilon for any epsilon greater than 0 that means, beforehand you should have a the distribution of the random variable X along with the distribution of the sequence of random variable X n’s then only you can conclude whether this sequence of random variable convergence in probability.

So, for this mode of convergence will go for the example, through that we will understand.

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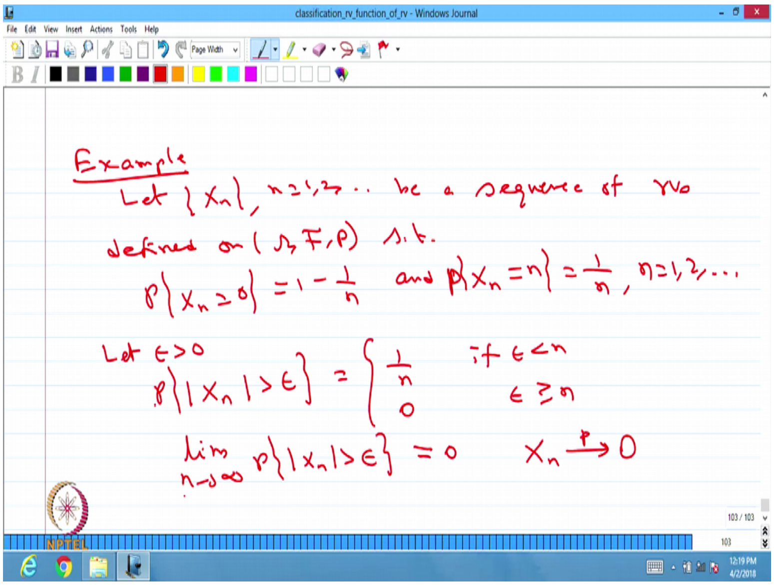

The example is a example let X n be a sequence of random variables defined on omega, F, capital P such that the probability of X n is equal to 0 that is 1 minus 1 divided by n and the probability of X n takes a value n it is 1 divided by n. So, this is true for all n, n is equal to 1, 2, and so on.

We have a sequence of random variable whose distribution is defined X n takes a value 0 with the probability 1 minus 1 by n or X n takes a value n with the probability 1 by n; that means, this sequence of random variables are of the discrete type which has the two points, 0 and n are the mass points either the mass is at 0 or n for the n-th random variable and you have sequence of random variable n is equal to 1, 2 and so on.

In this example we can go for taking let epsilon greater than 0 you can go for finding probability of absolute of X n which is greater than epsilon finding out the probability of absolute of X n greater than epsilon that is same as this is going to be 1 by n if a epsilon is going to be lesser than n. If epsilon is going to be greater than or equal to n the probability of absolute of X n greater than epsilon is 0 this is for fixed epsilon is greater than 0.

Now, you can go for taking a limit n tends to infinity of probability of absolute of X n greater than epsilon. As a limit n tends to infinity this quantity is going to be the right inside is going to be 0. Since the limit n tends to infinity probability of absolute of X n greater than epsilon is 0 therefore, you can treat the absolute of X n minus 0 that is equivalent of concluding X n converges to 0 in probability. 0 you can treat it as the a random variables X takes a value 0 with the probability 1. You can make a X n tends to X in probability, where X is a degenerated variable or constraint which takes a value 0 with the probability 1.

So, sometimes are the sequence of random variable converges to constraint also. So, this is example in which we have given sequence of random variable converges to constant 0 in probability. Since it is a converges to 0 therefore, we are directly going for probability of absolute of X n greater than epsilon sometimes if you have a random variable X and whose distribution is known then you can go for finding out the probability of absolute of X n minus X greater than epsilon, then whether the limit n tends to infinity.

This quantity is going to be 0 or not accordingly you can conclude sequence of random variable converges to random variable X in probability or not. So, here it is a very easiest example in which we are land up the sequence of random variable converges to 0 in probability.