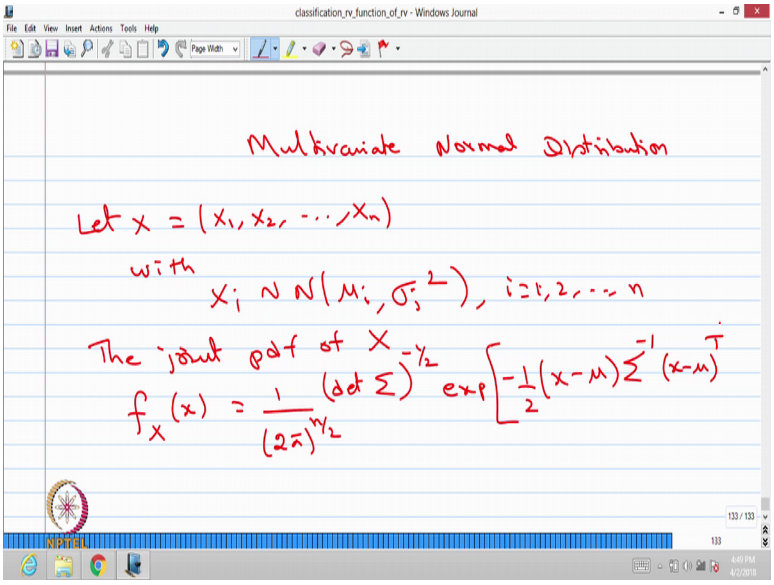
**Introduction to Probability Theory and Stochastic Processes**
**Prof. S. Dharmaraja**
**Department of Mathematics**
**Indian Institute of Technology, Delhi**

**Lecture – 37**

So, we have discussed covariance variance matrix then we have discussed, as a example we have discussed the discrete type random variable. Now we are going to discuss one continuous type random variables in which we can describe the covariance variance matrix in a nice way. That is one very important multidimensional random variables of continues type that is called multivariate normal distribution which has a lot of applications.

(Refer Slide Time: 00:35)

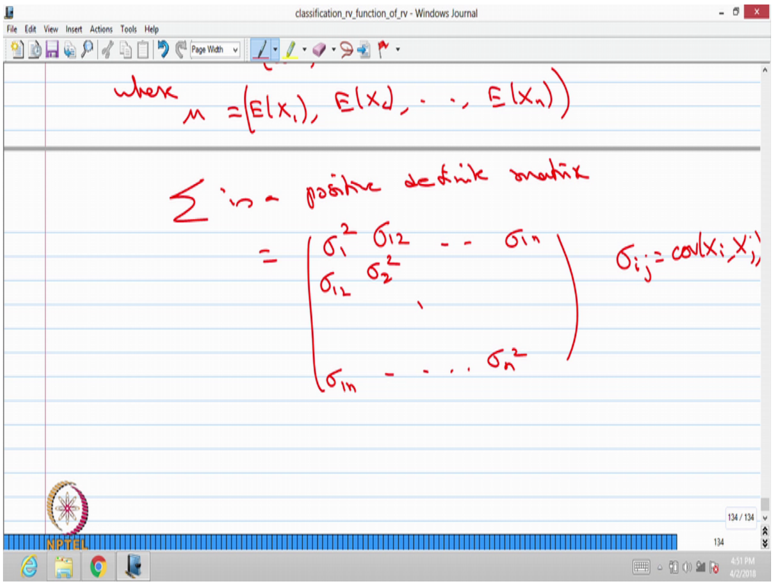



The way the central with there on which we are going to discuss later which has lot of applications in the real world problems he same way the multivariate normal distribution also going to play important roles in many complicated problems in probability.

Let X be a vector whose elements are X 1, X 2 and so, on it is a n dimensional random variables are of continuous type; with each random variable X i follows a normal distribution with the mean mu i and the variance in sigma i square; for i is equal to 1 to n. Then we call the random variable X as the multivariate normal distribution whenever each random variable is normal distributed at random variable, then the n dimensional random variable is going to be call it as a multivariate normal distributed normal variable.

Then we can define the joint probability density function, the joint probability density function of the random vector capital X whose elements are X 1, X 2, X n is given by X is the vector that is 1 divided by 2 pi power n by 2 multiplied by the determinant of the covariance variance matrix, the whole power minus 1 by 2 multiplied by exponential of a minus 1 by 2 x minus mu multiplied by the covariance variance matrix with the inverse multiplied by x minus mu the whole transpose.

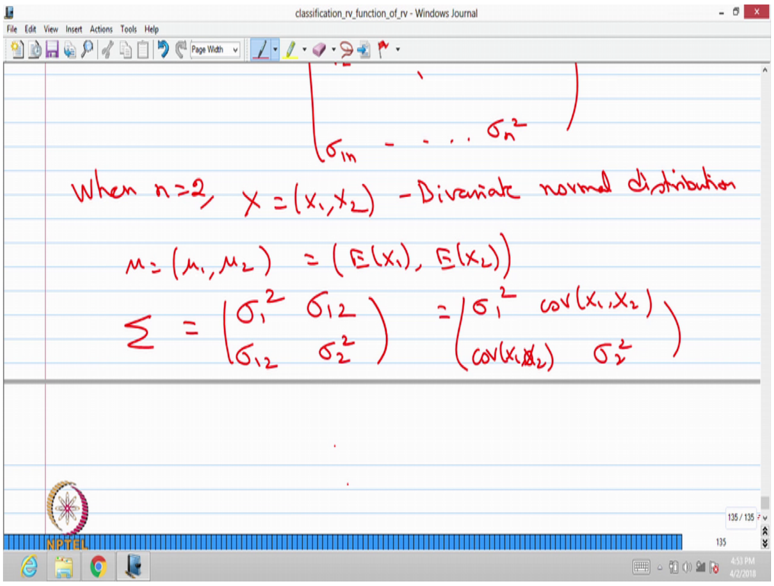
(Refer Slide Time: 03:09)



Where the mu is a vector that is a expectation of the vector; that means, whose element are expectation of X 1, expectation of X 2 and so, on the n th element is expectation of X n; mu is the vector whose elements are the expect individual expectations. And summation is a positive definite matrix. It is the covariance variance matrix of the n dimensional random vector X 1, X 2, X n; that means, whose elements are the diagonal elements are variance and off diagonal elements are covariance between any 2 random variables which is denoted by sigma 1 2. Since sigma 1 2 is same as sigma 2 1; so, both we write it as a sigma 1 2 and the last element is sigma 1 n.

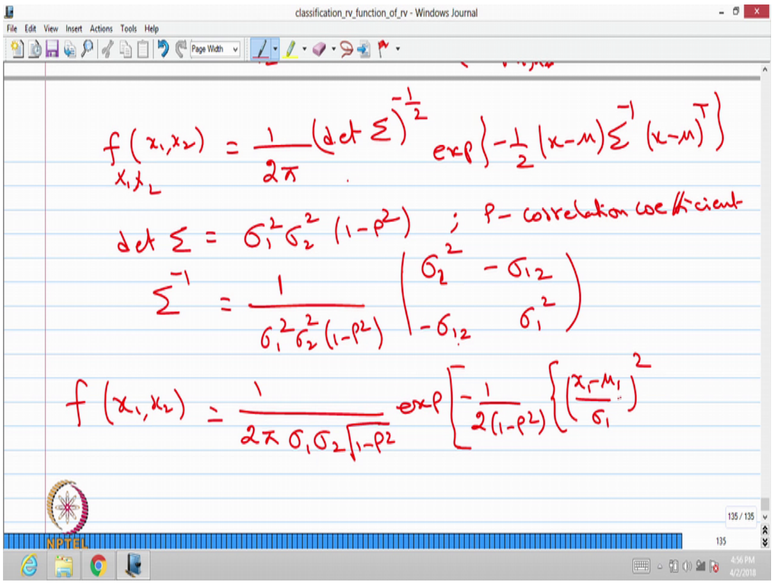
Here the first element is sigma 1 n where sigma i j is nothing, but the covariance of X i with X j. When i and j are same them it becomes a variance of i; variance of the random variable X i. So, the joint probability density function can be written in the form where am use the vector and sigma is the matrix and similarly small x is also vector.

(Refer Slide Time: 05:18)



For example, when n is equal to 2 we call it as a bivariate X as elements X 1, X 2 that is called a bivariate normal distributed random variable, we call it as a bivariate normal distribution. In that case the mu is going to be mu 1 comma mu 2 where mu 1 is nothing, but expectation of X 1 and mu 2 is nothing, but the expectation of X 2. And the summation matrix that is covariance variance matrix is nothing, but variance of X 1, covariance of X 1 X 2, covariance of X 1 X 2 and variance of X 2; that is sigma 1 square covariance of X 1 with X 2 covariance of X 1 with X 2 and variance of X 2.

(Refer Slide Time: 06:43)

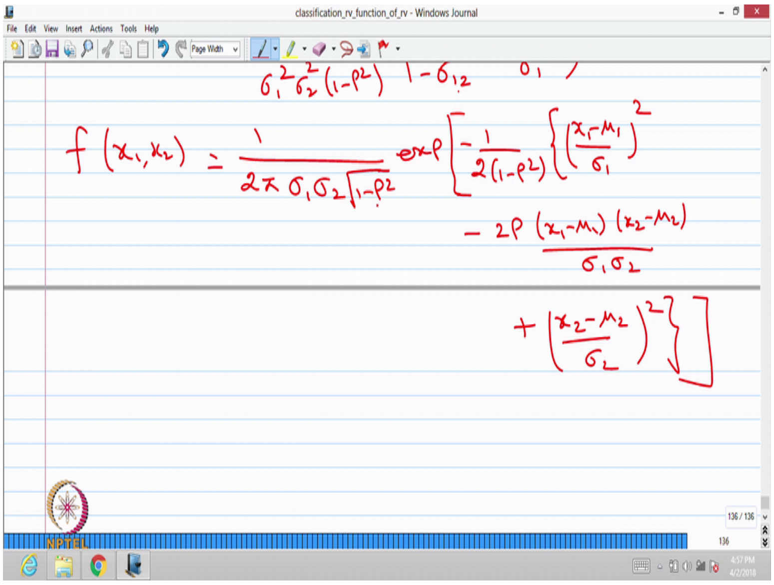



Now, the joint probability density function will be much simplified that is f of X 1 comma X 2 whose elements are x 1 and x 2 that is nothing, but 1 divided by 2, 2 pi power n by 2 n is 2 here. Therefore, 2 pi and determinant of some matrix raise 2 to the power minus 1 by 2 and exponential of multiplied by exponential of minus 1 by 2 x minus mu summation inverse x minus mu transpose.

So, we find out each quantity separately determinant of summation matrix; if you simplify you will get sigma 1 square sigma 2 square and covariance of X 1 and X 2; since we have only 2 elements we can make it as a rho therefore, it is going to be 1 minus rho square. Similarly, if you find out the inverse of covariance variance matrix that is going to be 1 divided by variance x 1 x2 multiplied by 1 minus rho square multiplied by 2 is the correlation coefficient correct; sigma 2 square minus sigma 1 2 that is covariance of x 1 with x 2 minus x 1 with x 2 sigma whole squared.

Therefore, you substitute in the joint probability density function therefore, f of x 1 with x 2 that is going to be 1 divided by 2 pi sigma 1, sigma 2 multiplied by square root of 1 minus rho square; exponential of minus 1 by 2 times 1 minus rho square multiplied by x 1 minus mu 1 divided by sigma 1 the whole square minus 2 times rho x 1 minus mu 1 x 2 minus mu 2 divided by sigma 1 sigma 2 plus x 2 minus mu 2 divided by sigma 2 the whole squared this is in the curly bracket this closed bracket.

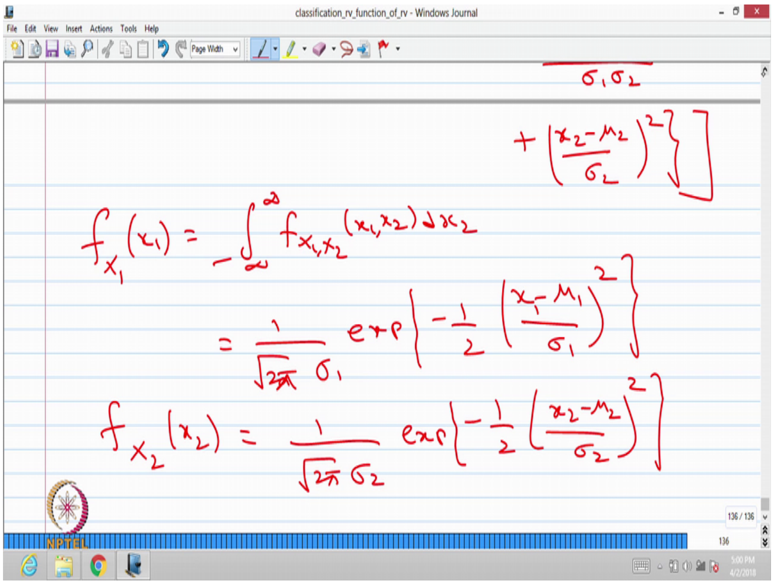
(Refer Slide Time: 09:33)




So, this is the joint probability density function of bivariate normal distribution in which each one is a normal distributed with the parameters mu 1 comma sigma whole squared. Here there is another observation, we are not making the assumption of both the random variables are independent. If they are independent then the correlation coefficient becomes 0; then we would have the middle term. So, this term will vanish; so, you will have first term as well as the third term. Similarly when the rho square is becomes 0 then square root of 1 minus rho square would not exist.

So, when they are independent random variable then easily you can write as the product of 2 probability density function of a normal distributed random variable. But immaterial of both the random variables are independent; we can get the marginal distribution of the random variable X 1.

(Refer Slide Time: 11:10)

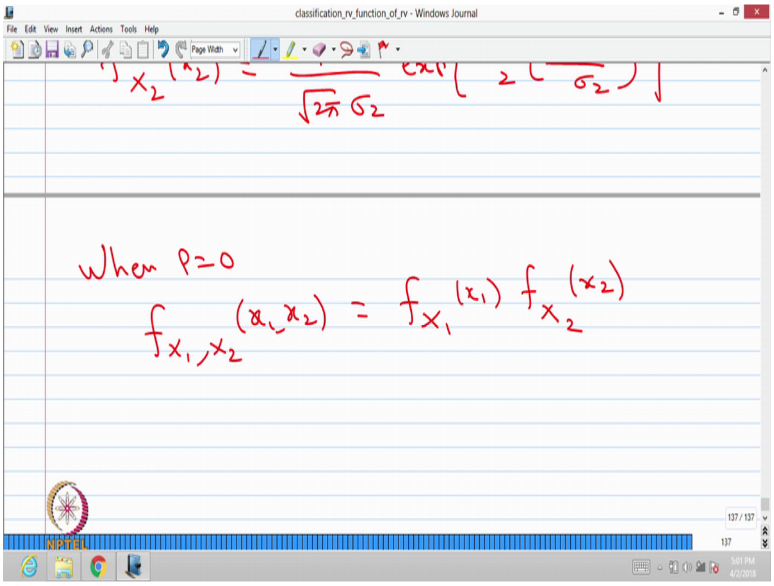


From the joint you can always get the marginal by integration with respect to the other variable; that is minus infinity to infinity the joint probability density function of x 1 x 2 with respect to x 2. If you simplify you will get the answer that is 1 divided by square root of 2 pi sigma 1 exponential of minus 1 by 2 x by x 1 minus mu 1 divided by sigma 1 square. So, this is the original distribution of the random variable x 1.

Similarly, you can find the marginal distribution of X 2 by integration with respect to X 1 of joint probability density function. So, that is going to be 1 divided by square root of 2 pi sigma 2 exponential of minus 1 by 2 x 2 minus mu 2 divided by sigma 2 the whole square. By seeing the probability density function, you can make out this is a normal distribution with the mean mu 1 and the variance sigma whole square for the random variable x 1. Similarly for the random variable x 2 it is also normal distributed with the mean mu 2 variance sigma 2 square. Whereas, the joint one is given as this is the joint probability density function of bivariate normal distribution. So, this is a very good example of how the covariance variance matrix play a role.

So, we have discussed earlier in the discrete type now we are describing the continuous type random variable. As a example for described in the covariance variance matrix, there is another important observation in the joint probability density function of normal distribution. If you substitute rho equal to 0 that is a correlation coefficient that is 0, you will get the first term and the third term in that we can come to the conclusion you will get joint probability density function of x 1 x 2 is same as the product of probability density function of x 1 x 2..

(Refer Slide Time: 13:55)




That is when rho is equal to 0; the joint probability density function of bivariate normal distribution is land up the probability density function of a normal distribution in the product.

This is a very important result in the sense when rho is equal to 0, that is a correlation coefficient is 0 we are getting a independent relation. If they are independent then the joint probability density function is going to be the product of probability density functions.

Usually or in general the correlation coefficient is 0 that does not imply they are independent random variable whereas, independent random variable implies the correlation coefficient or covariance between any 2 random variables going to be 0; the converse is now rho in general, but for the normal distribution the converse is also true. That means, the covariance between any 2 random variables or the correlation coefficient between those 2 random variables are 0 implies those random variables are independent.

So, this is a very important result the if and only if condition for the correlation coefficient 0 and independent are going to be satisfied only for normal distributed random variable.