Introduction to Probability Theory and Stochastic Processes
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Lecture – 36

We are moving into the next important result of finding variance of sum of random variables.

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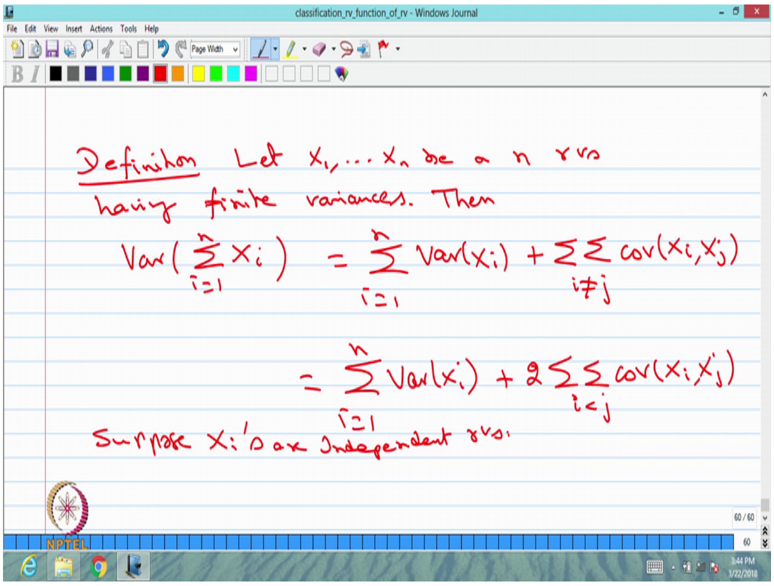

Next definition that is let X 1, X n be a n random variables having finite variances; then one can define variance of sum of random variables from i is equal to 1 to n let X 1, X 2, X and be a n random variables, having a finite variances; that means, first and second order moment exist and it is finite.

Then we are defining the sum of random variable that is going to be individual variance plus the double summation over covariance between any 2 random variables provided i and j is not equal. That is a variance of sum of random variable is the variance of individual random variable with summation plus covariance of any 2 distinct random variable that is going to be the variance of sum of random variable.

This can be rewritten, this can be rewritten that is summation of i is equal to 1 to n variance of each random variable.

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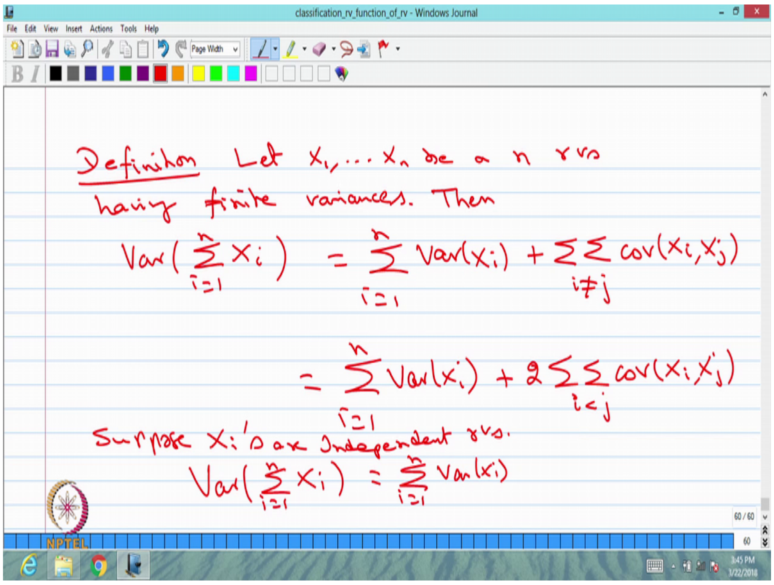

Since using the result number 2 covariance of X comma Y that is a means covariance of Y comma X; that means, the second summation can be rewritten in the form of a 2 times double summation. Covariance of X i with X j; now the condition is instead of i is not equal to j I can write i is less than j because we have put the 2 times. So, both the statements are one and the same.

Here also we can go for one special case, suppose X i's are independent random variables independent, when I say independent random variables; that means, they are mutually independent random variable. In that case we can use the previous result that is result number 5.

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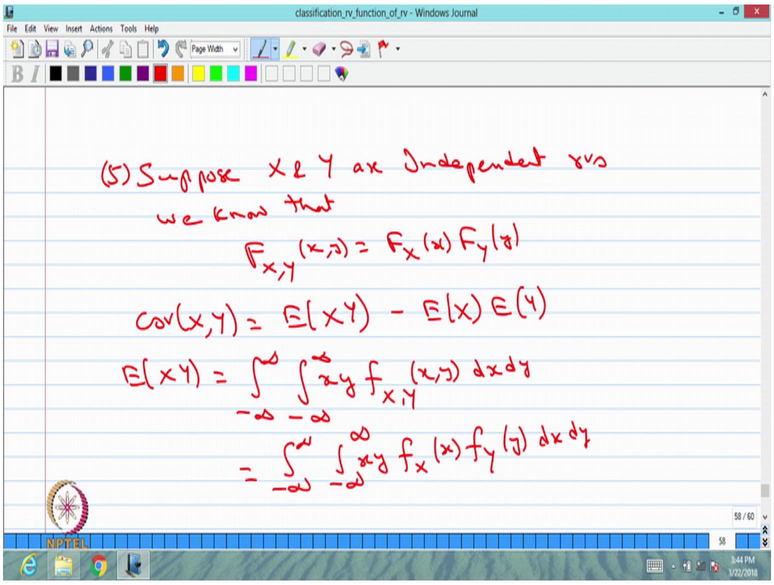

Well any 2 random variables are independent then the covariance is going to be 0. Therefore, the second the summation whole thing; it will not come therefore, variance of sum of a random variables is same as sum of variance of individual random variables when X i's are mutually independent random variables.

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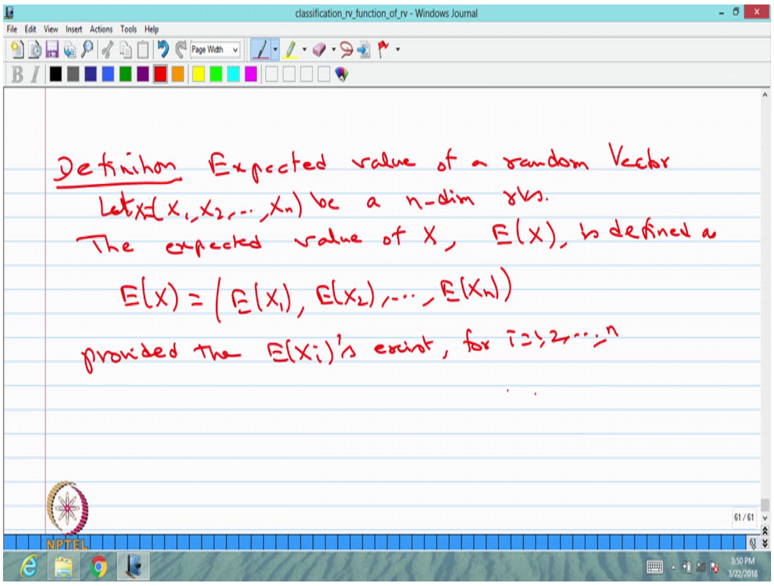

Now, we will give the definition of a expected value of a random vector; we have already given the definition of expected value of function of random vector. Now we are going to give the definition of expected value of a random vector; let X 1, X 2, X n be a n dimensional random variables.

The expected value of ; let me denote this random vector in the form of capital X, let me denote X 1 X 2 X n random vector with a notation capital X. So, I am going to define the expected value of the random vector capital X, which is denoted by expectation of X is defined as; it is defined as expectation of X is nothing, but since X 1, X 2, Xn's are vector the expected value of X is also going to be vector; whose elements are expectation of X 1, expectation of X 2 and so, on expectation of X n.

We are finding the expected value of a random vector therefore, that is also going to be a vector whose elements are expected value of individual random variables; provided the expectation of X i's exist for i is equal to 1, 2 n. So, as long as a individual expectation exists one can define expected value of a random vector with the elements is X 1, X 2, Xn. In the same way we are going to create a matrix whose elements are variance and the covariance between any 2 random variable that is a next definition.

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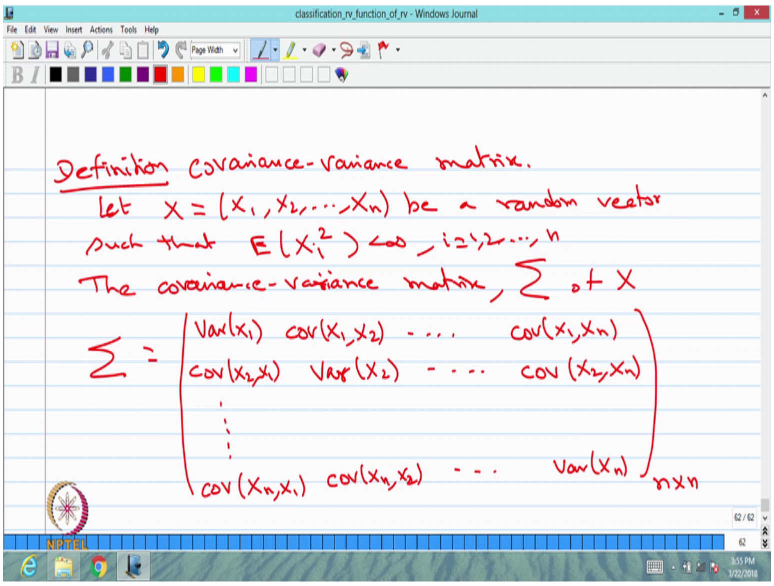

The next definition is covariance variance matrix; let capital X it is a random vector for the random variables X 1 X 2 and so, on; X n be a random vector either you can say random vector or n dimensional random variables or random vector with the n random variables. Such that the expectation of X i square that is a finite for the random variable 1, 2 so on till n as long as the second order once exist.

One can go for, one can go for defining covariance variance matrix. Then the covariance variance matrix that is denoted by in the big summation notation of the random variable capital X. That is defined as it is in the big summation notation it is a matrix whose elements are; the first element is variance of X 1 and the first row second element it is a covariance of X 1 and X 2 like that so, on the first row the last element that is covariance of X 1 with X n.

Now, coming to the second row; second row is covariance of X 2 with X 1, second row second element that is diagonal element that is variance of X 2; like that you can keep writing. The last element in the in the second row, that is covariance of X 2 with X like that you can fill up the last row with the first column that is X n with X 1. The last row second column that is X n with X 2 so on the last row last element that is a diagonal element that is variance of X n.

So, this matrix is n cross n order; we are creating a covariance variance matrix for n dimensional random variable. Therefore, this matrix is always n cross n whose diagonal elements are covariance of individual random variables and the other elements are covariance of X i into Xj for the i th row and j th column.

By using the property of covariance of X comma Y is same as covariance of Y comma X; you can conclude that this matrix is a symmetric matrix. All the diagonal elements are the variance that is nothing, but covariance of X i's with the X i's; therefore, it is becomes a variance of X i's.

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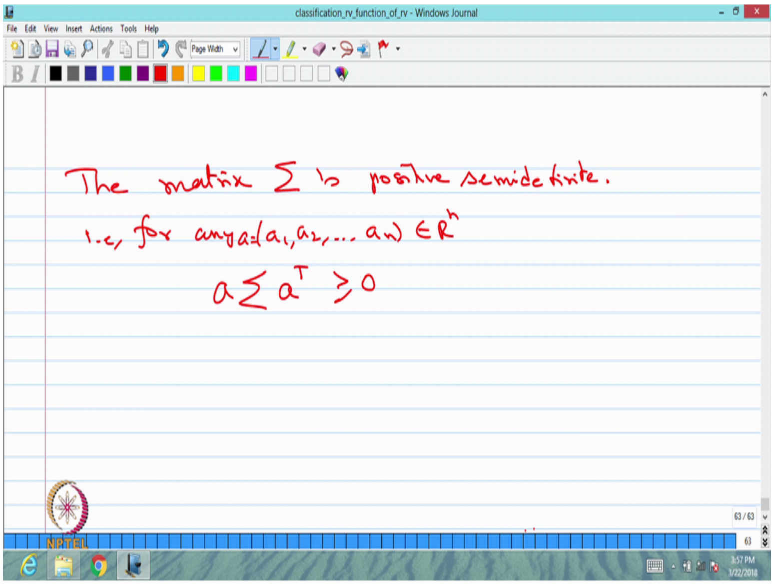

So, this matrix has the very important property that is the matrix is positive semi definite; that means, for an a 1 comma a 2 and so, on a n belonging to R n the vector. Suppose I denote this as the vector suppose I denote this as the vector a suppose I denote this as a vector a whose elements are a 1, a 2, a n; belonging to R n, a vector multiplied by the matrix; then a vector transpose this value is always going to be greater or equal to 0. That is called the matrix is positive semi definite, the covariance variance matrix of random vector is always positive semi definite.

We will give a one simple example; how to compute the expected value of a random vector and covariance variance matrix.

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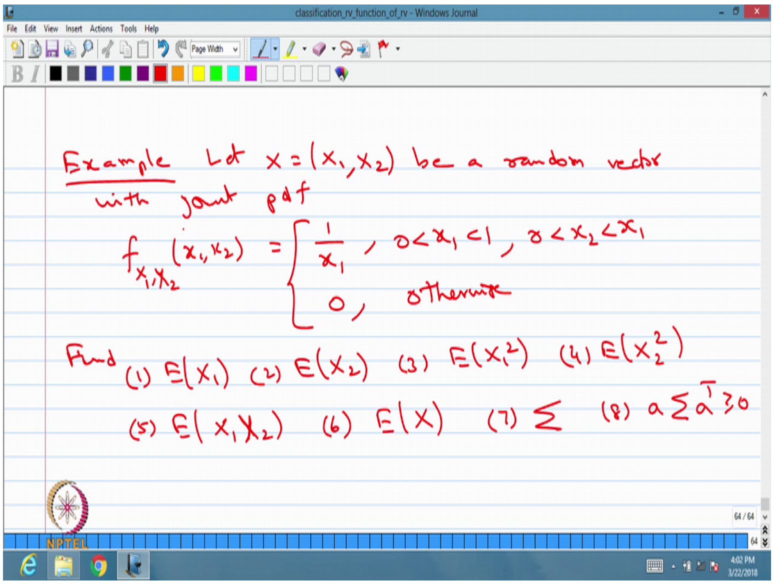

Example let capital X is the vector whose elements are X 1 comma X 2; be a random vector with joint probability density function is given by ; that means, both the random variables X 1 and X 2 are continuous type random variable. Therefore, we are defining the joint probability density function of x 1 comma x 2; that is 1 divided by x 1; when x 1 takes a value 0 to 1 whereas, x 2 takes the value 0 to x 1; 0 otherwise. So, this is the joint probability density function of the random vector X 1 comma X 2. You can verify double integration of a joint probability density function has to be 1.

Let us find expectation of X 1, then we will go for finding expectation of X 2, then we can go for finding expectation of X 1 square, then you can go for expectation of X 2 square, then we can go for expectation of X 1 X 2, then we can go for expected value of the vector X. We can find out what is the covariance, variance matrix then we can go for verifying whether it satisfies a summation; a transpose is greater or equal to 0.

The last one is the, verify a times matrix a transpose is greater or equal to 0. So, all those things we can do one by one.

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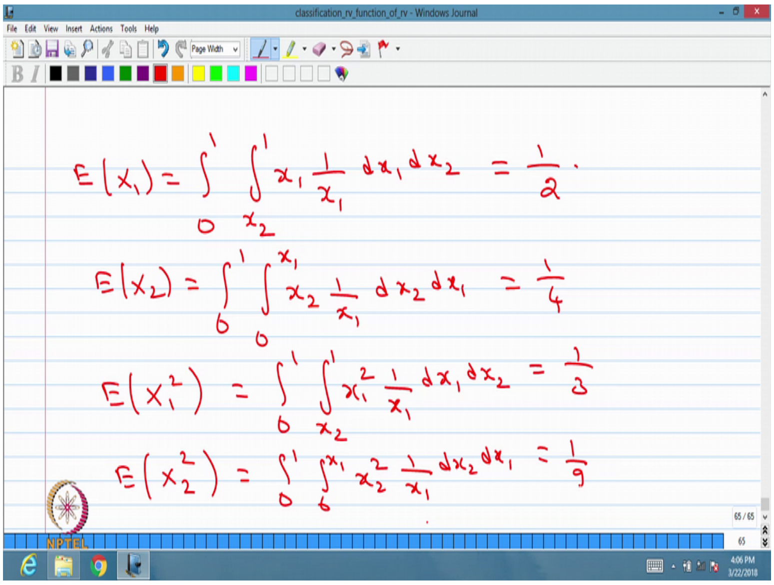

First start with the expectation of X 1; expectation of X 1, you can find the probability density function of X 1 from the joint probability density function, then you can go for the expectation or you can use the first definition that is expected value of function of a random vector. So, we are going to use that that is same as integration from 0 to 1, integration from x 2 to 1, x 1; 1 divided by x 1, dx 1, dx 2. You see that the joint probability density function is 1 divided by x 1 where x 1 is range from 0 to 1 whereas, x 2 range is 0 to x 1.

Therefore expectation of X 1 is double integration 0 to 1 x 2 to 1 x 1 times 1 divided by x 1 divided by x 1 is a joint probability density function of x 1, x 2. If you do the simplification we will get the answer 1 by 2; similarly you can go for finding expectation of X 2; that is integration 0 to 1 0, to x 1; x 2 times the joint probability density function, integration with respect to x 2, integration with respect to x 1.

Again I am using the expected value of function of a random vector; this is same as 1 divided by 4. Similarly we can go for expectation for X 1 square; that is same as same method what I have done it for expectation of X 1 that is integration 0 to 1 integration x 2 to 1 x 1 square, the joint probability density function is 1 divided by x 1 dx 1, dx 2 or you can use the change of integration and you can change the order of integration still you can go for it and you can get the answer that is 1 by 3.

Similarly, you can go for expectation of X 2 square that is again 0 to 1, 0 to x 1 x 2 square 1 divided by x 1 dx 2 dx 1; I am using the same definition again that is same as 1 divided by 9. So, till now we have got expectation of X 1, expectation of X 2, expectation of X 1 square, expectation of X 2 square.

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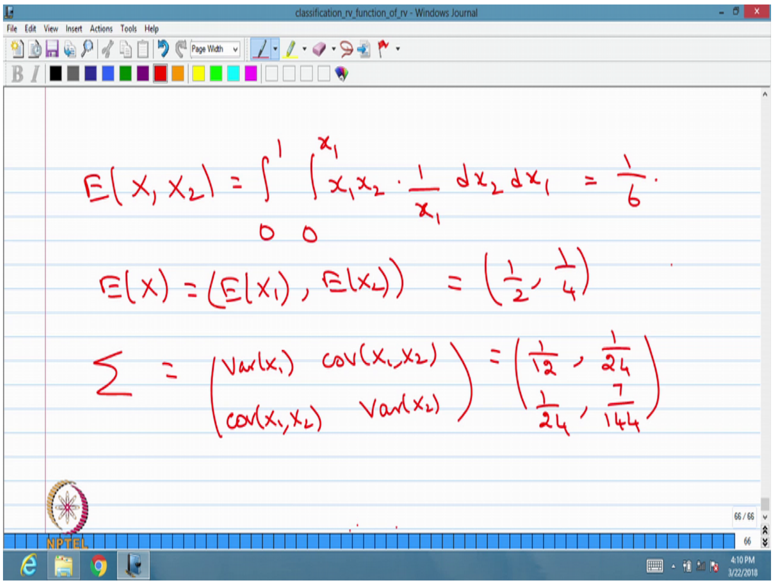

Now, we will go for finding expectation of X 1, X 2; the same technique that is expectation of X 1, X 2 that is same as integration 0 to 1, integration 0 to x 1, x 1, x 2 multiplied by the joint probability density function dx 2, dx 1. If you do the simplification you will get the answer that is 1 by 6.

So, now we are going for the next result that is expectation of a random vector; that is a vector whose elements are expectation of X 1 comma expectation of X 2; this vector. That is same as already we got the result expectation of X 1 is 1 by 2 expectation of X 2 is 1 by 4; therefore, this vector is 1 by 2 comma 1 by 4.

The next is finding the covariance variance matrix, since we have only 2 random variables which is going to be 2 cross 2 whose elements are variance of X 1, covariance of X 1 with X 2, covariance of X 2 with X 1 or X 1 with X 2 both are one and the same and variance of X 2. That is same as we got the expectation of X 1 and expectation of X 1 square. So, the variance is expectation of X 1 square minus expectation of X 1 whole square. So, if you do the simplification we will get the answer 1 by 12.

To find the covariance of X 1, X 2 you need expectation of X 1 X 2 and expectation of X 1 and expectation of X 2; so, all 3 we got it. So, substitute the values, that is 1 by 6 minus 1 by 2 into 1 by 4. So, you simplify you will get the answer that is 1 by 24. Since covariance of X 1 X 2 is 1 by 24 that is again a covariance of X 2 comma X 1 that is same as covariance of X 1 comma X 2 that is 1 divided by 24. Variance of X 2 is expectation of X 2 square minus expectation of X 2, the whole square. So, you do the simplification you will get 7 divided by 144. So, this is the covariance variance matrix for the random vector X 1 X 2.

Now, we are going to verify whether this covariance variance matrix satisfies the condition a covariance variance matrix a transpose going to be greater than or equal to 0.

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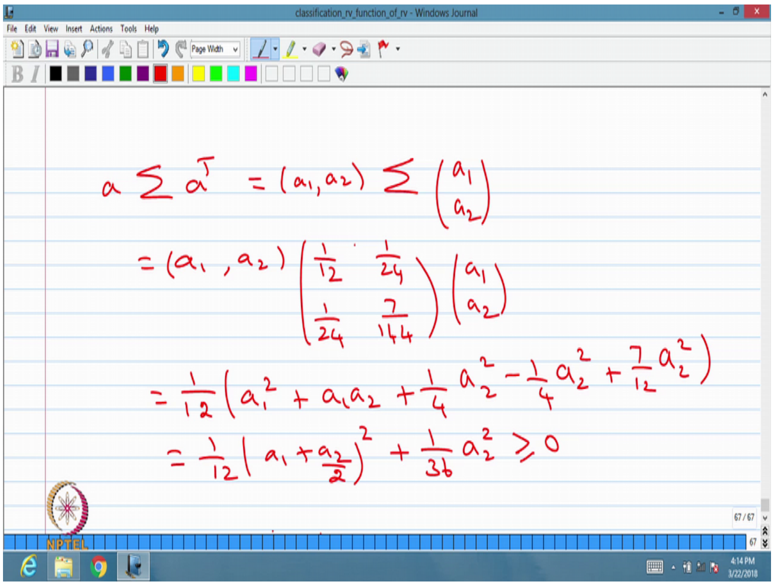

Let us compute; so, here we will go for a, with the 2 elements, a 1 comma a 2 and the matrix transpose that is a 1, a 2. So, substitute the, that is a 1 comma a 2 and the matrix value is 1 by 12, 1 by 24, 1 by 24, 7 divided by 144 multiplied by a 1, a 2 transpose vector.

Do the simplification, do the simplification first you will get 1 by 12, a 1 square plus a 1, a 2 plus 1 by 4 a 2 square minus 1 by 4, a 2 square plus 7 by 12, a 2 square and this is same as 1 by 12 a 1 plus a 2 divided by 2 whole square plus 1 by 36, a 2 square which is greater than or equal to 0. Therefore, we are concluding this particular covariance variance matrix also positive semi definite.