Introduction to Probability Theory and Stochastic Processes
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Module – 07
Cross Moments
Lecture – 35

In this module we are going to discuss a Cross Moments. In the earlier modules we have started with the basics of probability then we have discussed the single dimension random variable. Then third module we discussed the moments and equalities for a single dimension random variable, then in the fourth module we discussed the standard distributions, both the discrete and the continuous type.

And in the fifth module we discussed the two end high dimensional random variables or random vectors. In that we discussed the joint probability mass function, joint probability density function. In the same module we discussed the independent random variable source.

In the sixth module we discussed the functions of several random variables. In that we discussed the distribution of a functions of several random variables and we discussed odd statistics, then we discussed conditional distributions and random sum. In this module we are going to discuss cross moments.

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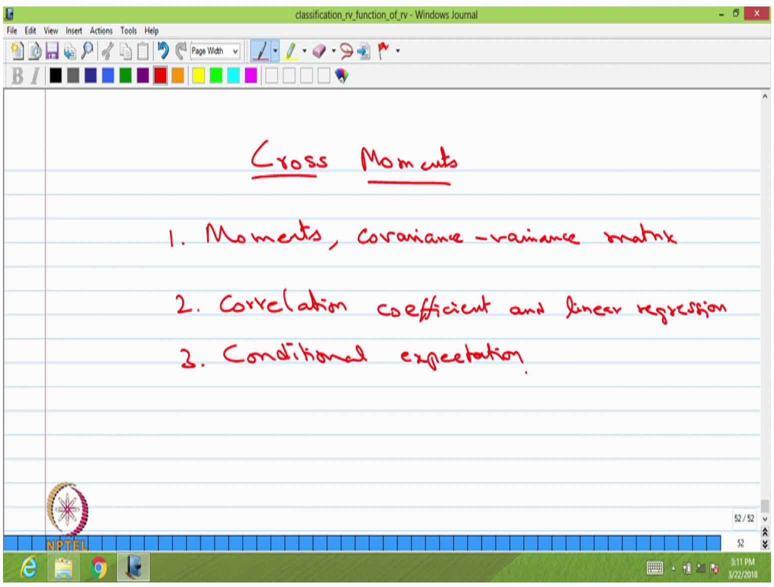

In this section we are going to discuss the cross moments with the following lectures; the first lecture we are going to discuss the various moments. The first order moment second order moment and so on, then we are going to discuss the covariance variance matrix that is in the first lecture. In the second lecture we are going to discuss the correlation coefficient and linear regression. In the third lecture we are going to discuss the conditional expectation.

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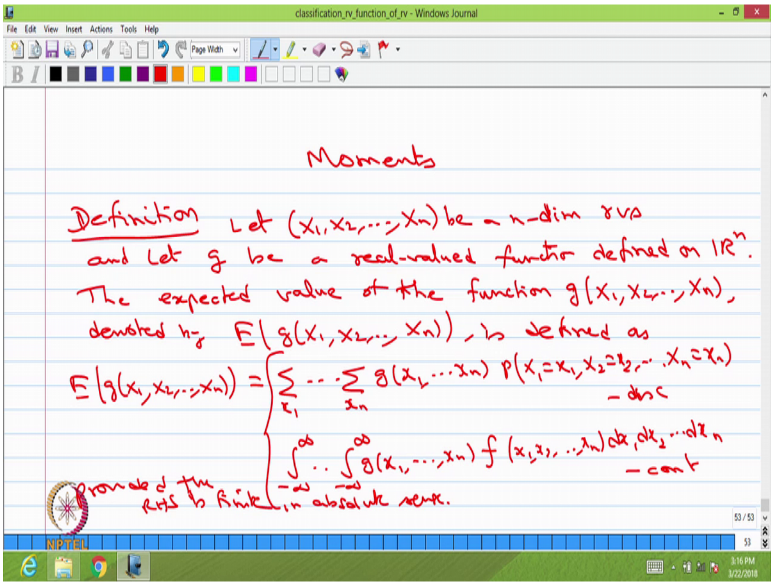

Let us start with moments, before going to the moments of the first order, second order and the cross moments for multi dimensions random variables, first let me give the definition of expected value of a function of a random vector that as a definition. So, this definition is going to be useful to compute the various moments of the several random variables.

Let X 1 X 2 and so on X n be a n dimensional random variable and let g be a real valued function defined on Rn; that means, it has a n variables. The expected value of the function g of X 1 X 2 and so on X n, that is denoted by expectation of g of X 1 X 2 and so on X of n. That is defined as expectation of g of X 1 X 2 and so on X n that is equal to suppose X 1 X 2 X n are discrete type random variables, then this is going to be summation n summations with respect to x 1 and the second summation with respect to x 2 and so on. The last summation with respect to x n of g of x 1 and so, on x n, the probability of X 1 takes a value 1 X 2 takes a value X 2 and so on, X n takes a value X n when all these random variables are of the discrete type.

If all such n random variables are of the continuous, integration the integrant g of x 1 and so on xn multiplied by the joint probability density function of x 1 x 2 and so on with the variable x 1 x 2 x n integration is with respect to x 1 x 2 and so on x when all this random variables are of the continues type. Provided when the random variables are of the discrete type random variable the summation in absolute sense it is finite or when the random variables are of the continuous type in absolute sense, the integration has to be finite quantity; that means, the provided the right hand side is finite n absolute sense.

This is the same thing we have done it for only one dimensional random variable. When x is a random variable g is a oral measurable function g of x is a random variable one can compute the r 1 can find the value that is expectation of g of x with the summation. If they are if it is a discrete type random variable integration, if it is a continues type random variable provided the summation or integration is going to be a finite quantity in absolute sense. The same provided condition is extended with the n dimensional random variable and we are finding the expected value of the function of random variables that is g.

When we say it is a g is a real valued function, we have seen that this is going to be a Borel measurable function in that we are finding the expectation of g of x 1 x 2 x again. We are not finding the distribution of g of x 1 x 2 xn, we are directly computing the expected value of the random variable g of x 1 x 2 xn. To find the various moments you can use this definition nicely so, that you can get the various moments.

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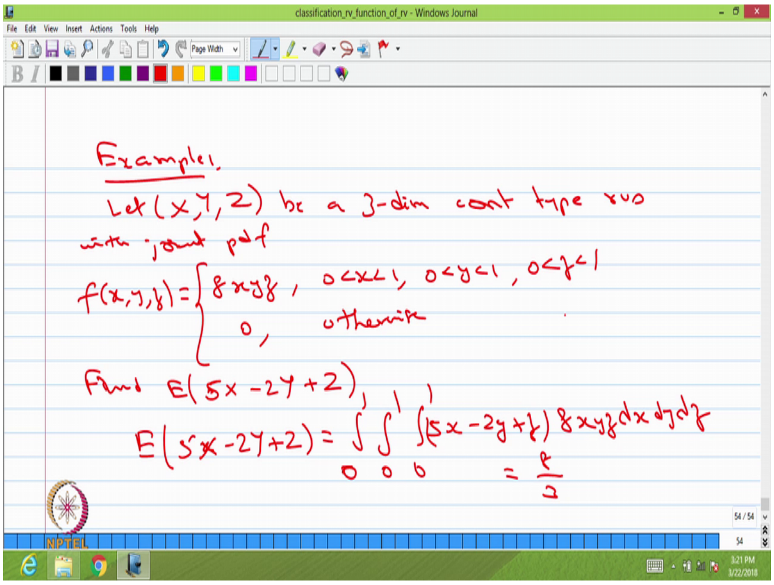

Before we go to the various moments let us give a one simple example of how to compute expected value of the function g of x 1 x 2 x that is a simple example 1. Let x comma y comma z be a 3 dimensional continuous type random variables with joint probability density function is given by joint probability density function of xyz that takes a value 8 times xyz, when x takes a value 0 to 1 y takes a value 0 to 1 and z takes a value 0 to 1. It is not 0 8 times xyz in these intervals 0 otherwise. You can verify whether this is going to be joint probability density function by integration from 0 to 1, 0 to 1, 0 to 1 8 times 8 xyz dx dy dz that is going to be 1. Therefore, this is a joint probability density function of the random variable xyz.

The question is find expectation of 5 X minus 2 Y plus Z, find expectation of 5 X minus 2 Y plus Z you can apply the previous definition. So, here the g of X 1 comma X 2 X 3 that is same as 5 X minus 2 Y plus Z. Since all are of the continuous type random variable make the 3 integration minus infinity to infinity minus infinity to infinity, that is 5 X minus 2 Y plus Z times 8 xyz dx dy dz that is going to be the expected value.

That is expectation of 5 X minus 2 Y plus Z that is same as since xyz lies between 0 to 1, it is 0 to 1, 0 to 1, 0 to 1 5 x minus 2 y plus z times 8 xyz dx dy dz you can do the integration and you can get the value that value is 8 divided by 3 fine. So, this is a very simple example in which we are finding the expectation of function of random vector.

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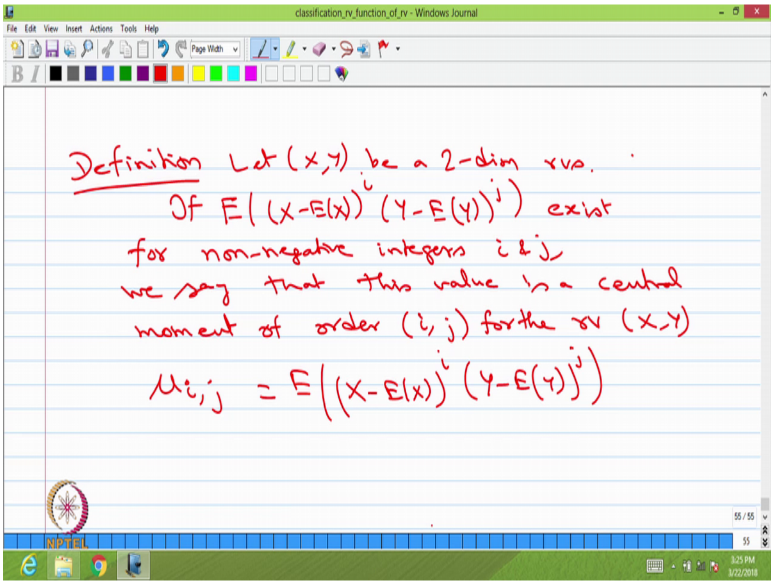

Now, we will move into the next definition we start with the 2 dimensional random variable, that X comma Y be a 2 dimensional random variables, if the expectation of the expectation of X minus expectation of X the whole power i multiplied by Y minus expectation of Y, the whole power j. If this expectation exist; that means, a in absolute sense these expectation value is going to be a finite for non negative integers i and j. We say that this value is a central moment of order i comma j for the random variable X comma Y, and it is written as it is denoted by new suffix i comma j that is expectation of X minus expectation of X, the whole power i Y minus expectation of Y whole power j

After the expectation of this function of X and Y exist, we can conclude this value is going to be the central moment of order i comma j for the random variable X comma Y, that is denoted by mu i comma j here i comma j both are non-negative integers. This is a special case of the definition which we have given the expected value of function of random vector, because this is a very special function that is 1 is x minus expectation of x power i multiplied by y minus expectation of y power j

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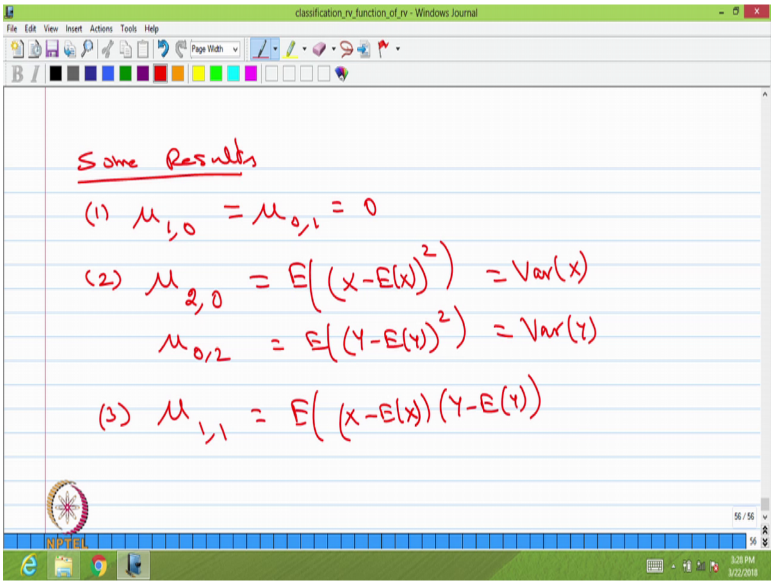

We can have a few results over this definition or remarks or some results. The first result is if you substitute i is equal to 1 and j is equal to 0 or you substitute I is equal to 0 and j is equal to 1 in the definition either you substitute i is equal 0 j is equal to 1 or i is equal to 1 j is equal to 0. In both the situation you will get the value is going to be 0.

Second result suppose you put the value i is equal to 2 and j is equal to 0 that is nothing, but the expectation of x minus expectation of X the whole power 2 and this is nothing, but variance of the random variable X.

Similarly, if you do mu of 0 comma 2 that is same as expectation of Y minus expectation of Y power 2 that is same as variance of Y. When you substitute i is equal to some integer positive integer and j is equal to some other positive integer, then you will get the central moment of order i comma j or the random variables X and Y or the random vector X comma Y. As a special case there is a next result when i is equal to 1 and j is equal to 1 that is nothing, but expectation of X minus expectation of X multiplied by Y minus expectation of Y which has the special name.

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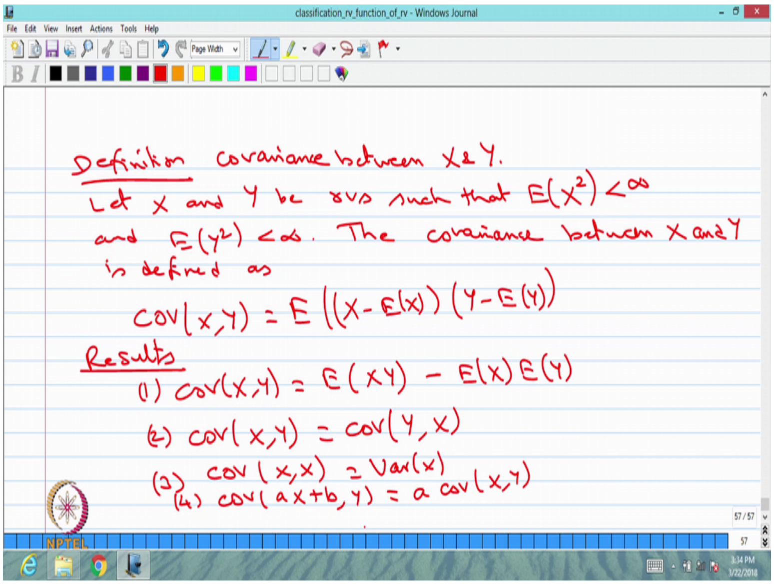

So, that I am going to make it as a next definition that is definition that is called a covariance between X and Y. Let X and Y be a random variables defined on the same probability space such that, such that the second order moment for the random variable X exist and the second order moment for the random variable Y also exist.

Then we can define the covariance between X and the random variable Y that is defined as the notation is cov X comma Y that is same as that is mu of 1 comma 1 that is expectation X minus expectation of X multiplied by Y minus expectation of Y, where expectation of X is the mean of random variable X, expectation of Y is the mean of random variable Y. First you multiply X minus expectation of X with the Y minus expectation of Y then you find the expectation. So, this is a special case of a the first definition, which we have given expected value of function of random vector.

The previous definition it is i is equal to 1 and j is equal to 1, which has a special name that is called the covariance are between these two random variables. This has the few important results that I make it as the points 1 by 1. Either you can compute expectation of X into expectation of X multiplied by Y minus expectation of Y or if you do the expansion if you expand this that is X into Y X into expectation of Y. Similarly, expectation of X into Y, then expectation of X into expectation of Y that is same as covariance of any two random variable is same as expectation of X into Y minus expectation of X into expectation of Y.

The second result if you find out the covariance of X comma Y, which is same as covariance of Y comma X; that means, by interchanging the role of X and Y, you will get the same value that is provided it exist the covariance of X comma Y which is same as covariance of Y comma X.

Third result suppose I compute covariance of X with X itself you substitute you replace Y by X in the above definition; that means, it becomes expectation of X minus expectation of X again X minus expectation of X, that is same as expectation of X minus expectation of X the whole square that is same as variance of X.

The fourth property that is the covariance of a X plus b with the other random variable Y that is same as if you expand by using the definition of covariance you will get a times covariance of X comma Y and covariance of b with y that is going to be 0 covariance of any constant with one random variable that is going to be 0. Therefore, covariance of ax plus b when a and b are constant covariance of a X plus b with y when a and b are constant that is same as a times covariance of X comma Y.

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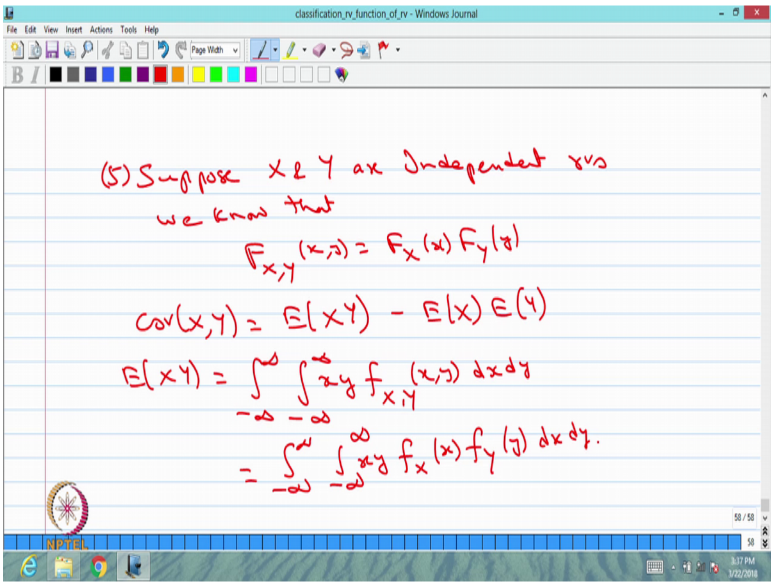

The next result number 5. Suppose the random variable X and Y are independent, suppose the random variable X and Y are independent we know that the joint CDF is same as the product of two CDF's. If they are discrete type random variable then joint probability mass function is same as the product of probability of mass functions, if they are continuous type random variable then the joint probability density function is same as product of probability density functions of X and Y.

So, here we are computing the covariance of X comma Y that is same as expectation of X into Y minus expectation of X into expectation of Y. When two random variables are independent, if you compute the expectation of x into y that is same as suppose I assumed at both the random variables are of the continuous type that is nothing, but minus infinity to infinity minus infinity to infinity x times y the joint probability density function of x comma y dx dy. I am making the assumption both the random variables are of the continuous type.

Similar derivation I can do it for discrete type random variable also. Since these two random variables are independent I can use this condition that is this is in the CDF and I can use the condition in the joint probability density function. Therefore, this is going to be minus infinity to infinity minus infinity to infinity x times y the marginal distribution or probability density function of x and y.

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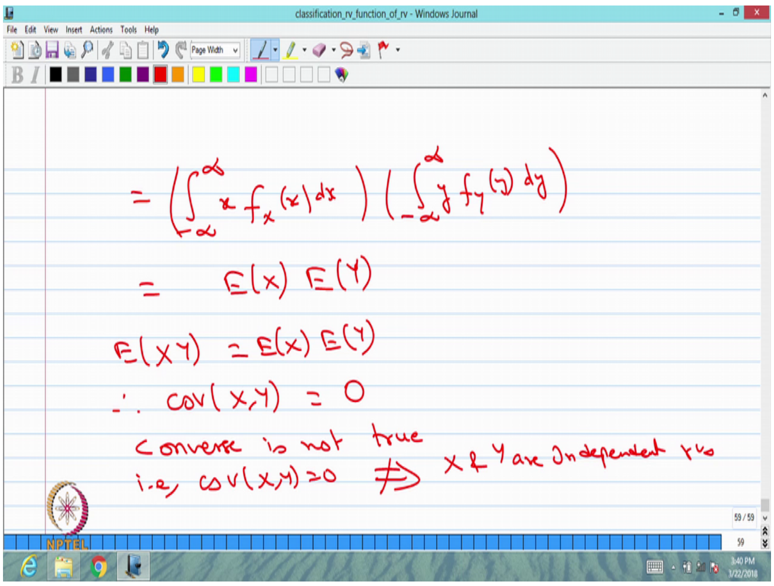

Now, the same double integration can be written as minus infinity to infinity ex times f of x dx multiplied by minus infinity to infinity y times probability density function of y with respect to y. This is because of the joint probability density function can be written as the product of probability density functions of x and y. Therefore, I can keep x f of x together similarly y and f of y together therefore, the integration becomes product of these two.

We know that that the first integration is expectation of X and the second integration of expectation of Y. That means, we have given the derivation for considering both the random variables are of the continuous type. Even if you do both the random variables are of the discrete type you will get the same conclusion that is expectation of X into Y that is same as expectation of X into expectation of Y. Therefore, if two random variables are independent, then the covariance between the random variables X and Y that is same as expectation of X Y minus expectation of X into expectation of Y. But just now we got the result expectation of XY is same as expectation of X into expectation of Y therefore, the covariance becomes 0.

This is the fifth result; that means, if two random variables are independent, the covariance is going to be 0. Converse that is the if the covariance is the covariance of two random variables X comma Y 0, that does not imply two random variables are independent; that means, converse if not true. The covariance that is covariance between any two random variable 0 that does not imply the random variable X and Y are independent random variables does not imply. Whereas, X and Y are independent random variables, then covariance of random variable X comma Y that is going to be 0.