Introduction to Probability Theory and Stochastic Processes
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Lecture – 34

Now, we will move into the next topic that is Random sum.

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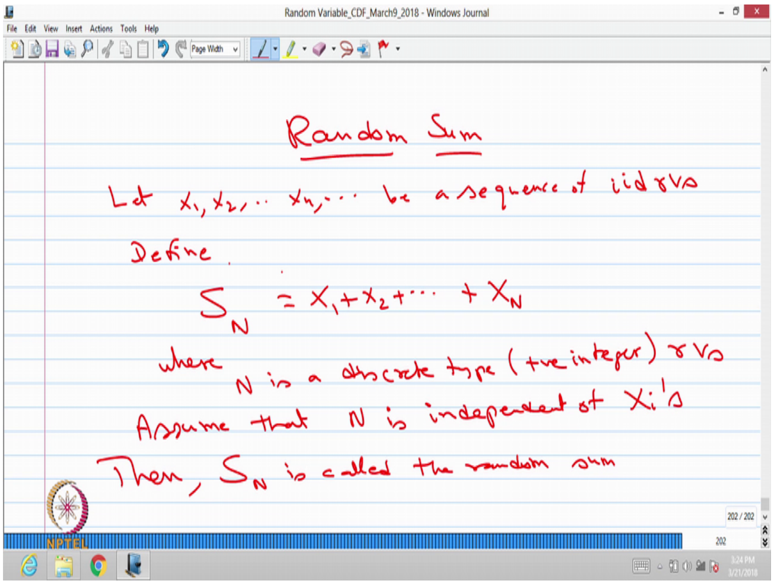

Let X 1 X 2 X n and so on be a sequence of let me start with a iid random variables be a sequence of a iid random variables. Define the new random variable as a S suffix N, that is nothing, but X 1 plus X 2 and so on plus X suffix capital N. I am going to define the random sum with the sequence of iid random variables defining the random variables X suffix N as a sum of N random variables

This is different from the earlier sum of the random variable, in the sense I am going to make a capital N which I made it suffix, that is a discrete type. In particular it is type positive integer valued random variables, N is a discrete type in particular it is a positive integer valued random variables.

I am going to make the assumptions assume that N is independent of X i's. The way we create the sum of random variables, all those random variables are mutual independent and identically distributed. And how many random variables I am going to add that is also random, that is a positive integer valued discrete type random variables which is independent of X i's.

This capital S suffix N that is called a random sum, then the S suffix capital N is called the random sum. You will come across many problems of this form, we will be adding many iid random variables and how many random variables we are going to add, that is also going to be a random variable. In that case the total number of random variables added that is a random sum. We will go for a one example for this. So, our interest is to find out the distribution of the random sum, once I know the distribution of the X is and N.

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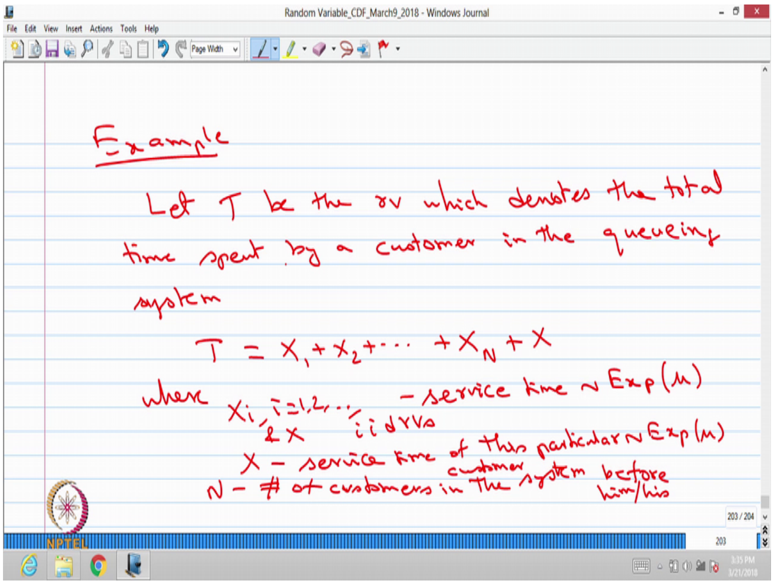

So let me start with the example, think of a some queuing system in which the people are entering in to the system, getting served and leak the place. There is the only one person do the service; that means, the number of servers in the system is only one. Because of only one server in the system whoever comes when some bodies service is going on he has to wait. He can think of queuing discipline is a first come first serve.

So, I am going to make a random variable, let capital T be the random variable which denotes the total time the total time spent by a customer; customer means any person, in the queuing system. Here I made a very simplest queuing system in which in which only one server in the system, there may be any number of people can wait. So, after the service is over then only they can leave the system.

We are interested to find out the distribution of t if I supply the information about how much time taken for the customers service, and what is the distribution of a number of customers in the system. That means, I am going to represent this capital T as a random sum in the form of X 1 plus X 2 plus and so on plus Xn plus capital X. Where each X i's i from 1 to n or the service time the random variable denotes the service time for ith customer.

And I make the assumption the service time follows exponential distribution with the parameter mu. I will repeat the total time spent by a customer in the system that can be represented as the sum of N random variables plus X are each X i's are iid random variables, where each X i's are iid random variables with the common distribution which is exponential distribution with the parameter mu where xi denotes the service time service time of any customer.

Whereas the capital X that is the service time of this particular customer particular customer, which is independent of all the previous service time. Therefore, all the X i's and X are independent random variable, this is also follows exponential distribution with the parameter mu. I am just giving another label for X to say that that is the service time for his own therefore, the capital T is going to be a random sum, because you never know how many customers are going to be before him when he enter in to the system.

So, N is the random variable. So, I can write the small n as the capital N. We can write small n as the capital N and Xi's are going to be iid random variables including the capital X. So, now, I can go for capital N is the number of customers in the system, which is independent of a service times of all the customers. My interest is to find out the distribution of capital T. So, I have to supply what is the distribution of the number of customers in the system also. So, here the capital N means the number of customers in the system before him or his term, that is the number of customers in the system when he enter into the system that is capital N.

So, in this case when capital N is equal to 0; that means, nobody is in the system before he enters, when he enters nobody in the system; that means, capital T is equal to X.

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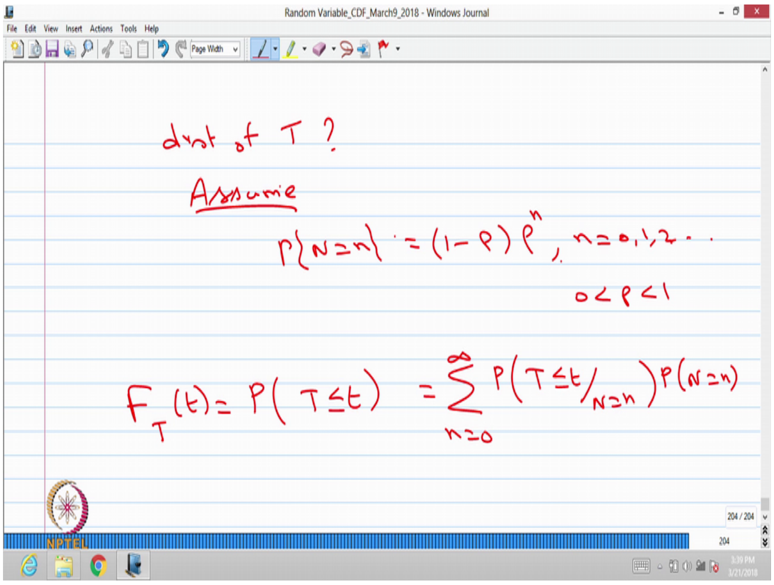

I am going to find out the distribution of a T; for that you should know the distribution of x is as well as the distribution of a n. So, we have already said the X i's the distribution of X i's are exponential distribution with the parameter mu.

Now, I am going to make the assumption of a assume that the distribution of a n that is capital N is the number of customers before he enters. So, that can be possible values are 0 1 2 and so on therefore, this is going to be; I just make the assumption that follows 1 minus rho into rho power N, where N can takes the value 0 1 and so on where the rho is open interval 0 to 1. So, this is the probability mass function of the random variable n. When n takes the value 0; obviously, the total time spent by a customer that is going to be only his own service time that is the exponential distribution.

So, let us find out the probability density function of capital T as the function of small t. It is easy to find out the conditional distribution of a capital T given takes the value then go for finding the probability density function of t. That is suppose I want to find out the CDF of the T that is the probability of P takes the value less than or equal to t, that is same as the condition probability of a T less than or equal to t given that N takes the value small n multiplied by the probability of N takes the value n for all possible values of n.

Left side we started with the CDF of the random variable t, that is same as probability T less than or equal to t, that is same as first we compute the conditional distribution of capital T given N then by multiplying into probability mass function of n for all the possible values of n we are getting the distribution of capital T by using a total probability which we discussed in the beginning for the events, but now we are applying the same concept for the random variables.

That means that finding distribution of a t by using the conditional distribution of t, by using the sort of a random sum concept then by using the total probability theorem we are getting the distribution.

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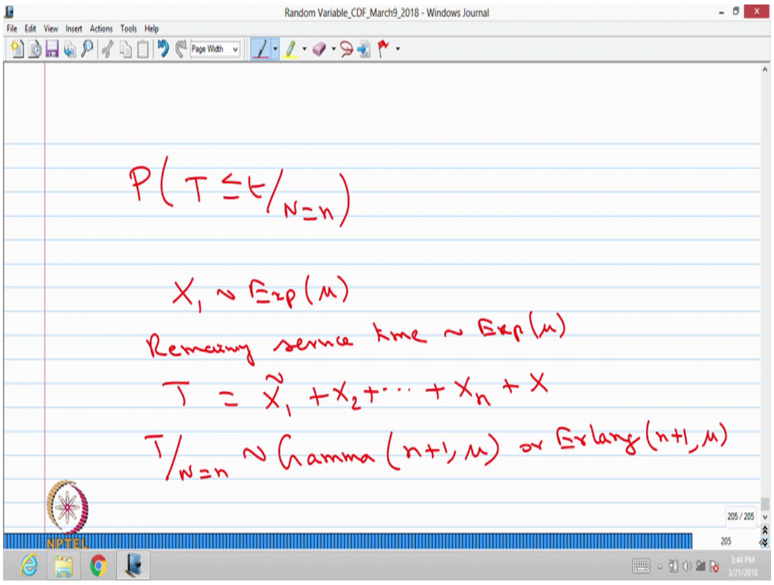

But for that first you should find out the what is the conditional distribution of t given n takes a value small n. Once you fix the total number of customers before him that is going to be n.

The total time spent by a customer is going to be the service time of all the n people plus his own service, in that there is a small issue when someone enters there is a possibility, the service of the customer who is under service it may be keep going. And it may be the remaining service time for the customer who is under service plus the first service of the second customer, third customer and so on till the nth customer then plus is on service.

So, again I am going to use the logic of a exponential distribution, since the first customer under service whose service is exponential distribution with the parameter mu. The remaining service time the remaining service time that is also going to follow exponential distribution, with the parameters mu by using the (Refer Time: 15:31) property. That is conditional probability of x greater than t plus s, given that x is greater than x that is same as probability of x is greater than t; that means, if the pass x t is arised and the conditional probability is again going to be the probability of x is greater than t, which has same distribution of exponential distribution.

Therefore the total time spent by a customer when already npq are in the system, that is going to be remaining service time of the customer under service plus plus service of n minus 1 customer plus its own service. That is capital T is X 1 sort of tilde tilde means remaining service time plus X 2 and so on plus X suffix small n plus capital X.

Each service time is exponential distribution, identical, mutually independent with the X service is one followed by the other therefore, you can conclude for a fixed N. So, this is the distribution of T for fixed N that is gamma distributed with the parameters n plus 1 with the second parameter mu. This is a one of the important properties of a gamma distribution, the gamma distribution which has the parameters whenever it is a positive integer and the other parameters mu that can be visualized as the sum of mutually independent exponentially distributed random variable with the parameter lambda.

There is a another name for this that is Erlang distribution, with the n plus 1 stages with each stage of exponential distribution with the parameter mu. So, this is a conditional distribution of T given n takes a value small n therefore, we know what is the probability density function of T given capital N takes a value n that is small t as a function of n.

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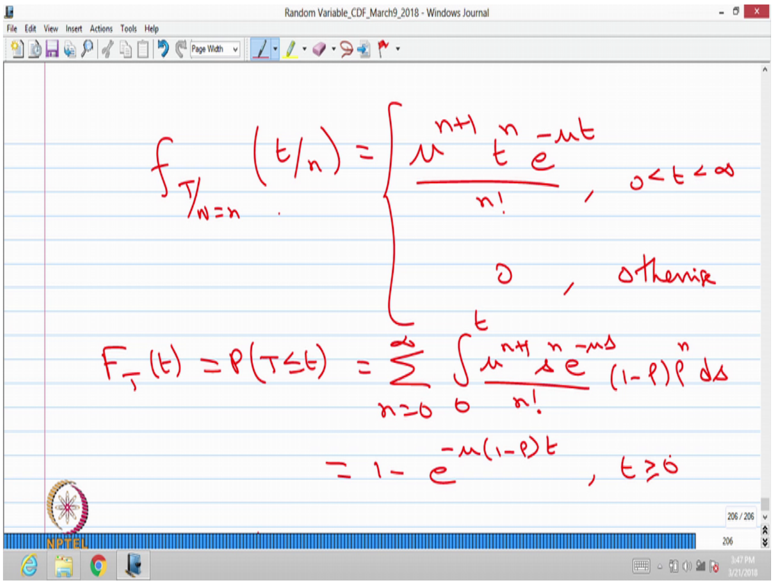

That is a mu power n plus 1 t power n e power minus mu t divided by n factorial it is a gamma of n plus 1 therefore, it is n factorial. So, this is valid when the t lies between 0 to infinity 0 otherwise. This is basically a gamma distribution with the parameters n plus 1 comma mu, now we are going back to finding out the CDF of t that is probability of T is less than or equal to t, that is same as the summation n is equal to 0 to infinity.

The above one is the conditional probability density function of t given n now, we need conditional distribution. So, therefore, it is minus instead of minus infinity you can go for 0 to small t, mu power n plus 1, s power n, e power minus mu s divided by n factorial multiplied by 1 minus rho into rho power n ds.

It is a good example of a introducing the conditional distribution random sum and a the properties of exponential distribution and the gamma distribution. If you do the simplification one can get the answer that is 1 minus e power minus mu times 1 minus rho times t.

So, this is a CDF of t for t greater than or equal to 0 otherwise. It is 0 it is a CDF of a capital T as a function of small t that is 1 minus e power minus mu times 1 minus rho times t. With this we are completing the module of a functions of several variable several random variables starting with the distributions of functions of several random variables, then a order statistics, then a as a third lecture conditional distributions and finally, random sum.