Introduction to Probability Theory and Stochastic Processes
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Lecture – 30

I will move into the one more example of a discrete type.

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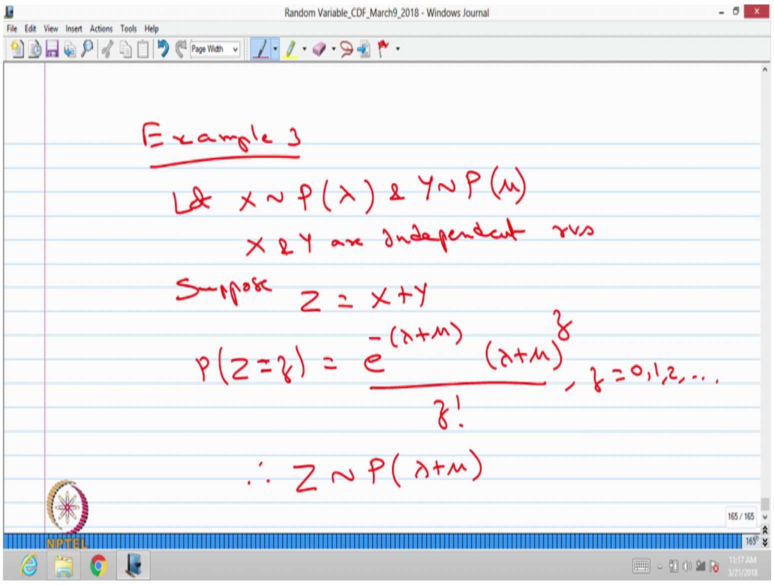

This is also going to be a very important result that is let x is Poisson distribution with a parameter lambda and y is again Poisson distribution with a parameter a mu. And I make the assumption x and y are independent random variables independent random variables.

Suppose I create a random variable Z is x plus y are similar derivation what we have done it for the binomial distribution. The similar derivation you can do and you can conclude a the probability of Z takes a value z that is going to be e power minus lambda plus mu lambda plus mu power z divided by z factorial where z can takes a value 0, 1, 2 and so on.

I am not giving the derivation we can do the similar derivation of the previous example you can get a the probability mass function of z is going to be this form other than a this z values it is going to be 0. Now we can map this with is there any standard distributions or common distribution matches we can find out.

So, this is going to be same as the probability mass function of Poisson distribution with a parameter lambda plus mu. Therefore, one can conclude a Z is also Poisson distribution with a parameter lambda plus mu. That means, if you have a two independent random variables both are Poisson distribution with some parameters then the sum is also going to be a Poisson distribution with a parameter is sum of their parameters.

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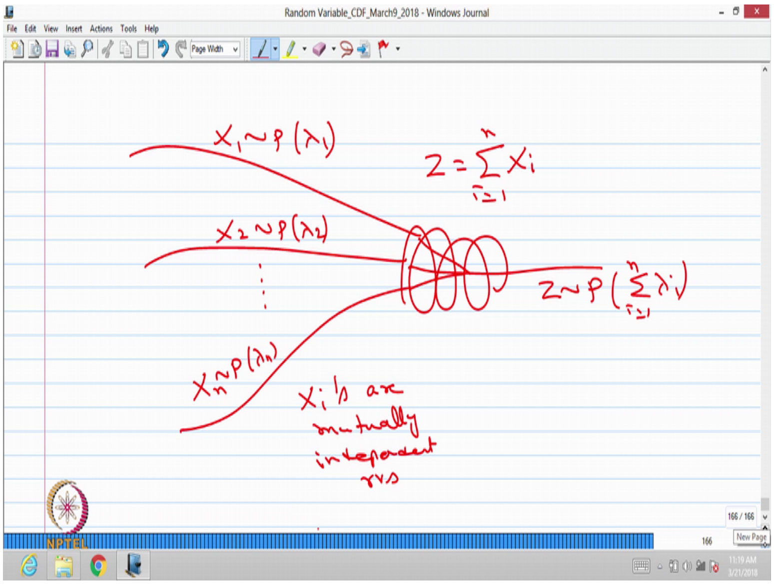

The same concept can be extended for a any n random variables; that means, if a n random variables are a mutual independent that means there is a one random variable that is the x 1 that is Poisson distributor with a parameter lambda. There is a another random variable x 2 that is also Poisson distributed with a parameter a lambda 2. Like that a I have a n-th random variable that is also Poisson distributed with a parameter lambda n.

If I make a random variable which is nothing, but sum of a X is; that means, I will land up only one with the only one random variable by summing all the random variables that is Z, and this is going to be a Poisson distribution with sum of their parameters.

As long as all the X i's are mutually independent random variables as long as all the random variables are mutually independent. Then the summation is going to be again a Poisson distribution with a parameters this is sum of lambda i's. From these we are going to give one important properties that is called Reproductive property.

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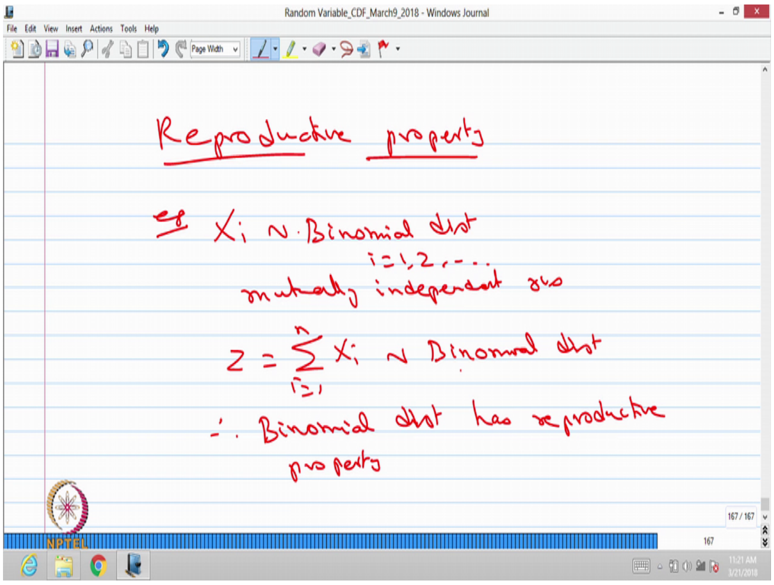

What the reproductive property says that a if you have sequence of random variables and if you make a sum of those few some of the variables out of it and all are having some distribution and after making the summation you are getting the same distribution of same as X i's, or the original sequence of random variable then we conclude this random variable has a this particular random variables has a reproductive property.

That means for example each X i's are a binomial distributed and I have a many random variables. All are mutually independent I make the assumption all the random variables are mutually independent. Then if I make a random variable as the sum of few random variables out of this collection if that is also follows a binomial distribution. So, we can conclude binomial distribution has reproductive property.

Similarly, one can say the Poisson distribution is also has a reproductive property where as the Bernoulli distribution does not have a reproductive property. Because if you have a Bernoulli distributed random variable all are mutually independent, if you make a n such random variable as a summation then that is going to be a binomial distribution no more Bernoulli distribution. Therefore, Bernoulli distribution does not have a re productive property.

Similarly, one can go for some common continuous type random variables. If you have a normal distributions all are mutually independent if you make a summation then that is also going to satisfies; the reproductive property; that means, summation is also going to be a normal distribution. So, like that we can make a list of a standard, or common distributions satisfying the reproductive property and not satisfying the reproductive property.

Now, we will move into distributions of a functions of several variable when each random variable is of the continuous type. So, for that I am going to give one important result as a theorem. After I introduce a theorem then I will go for giving some examples we are not going to prove the theorem. So, I am going to give the important result or the theorem as a some sort of results, then I am going to give some examples.

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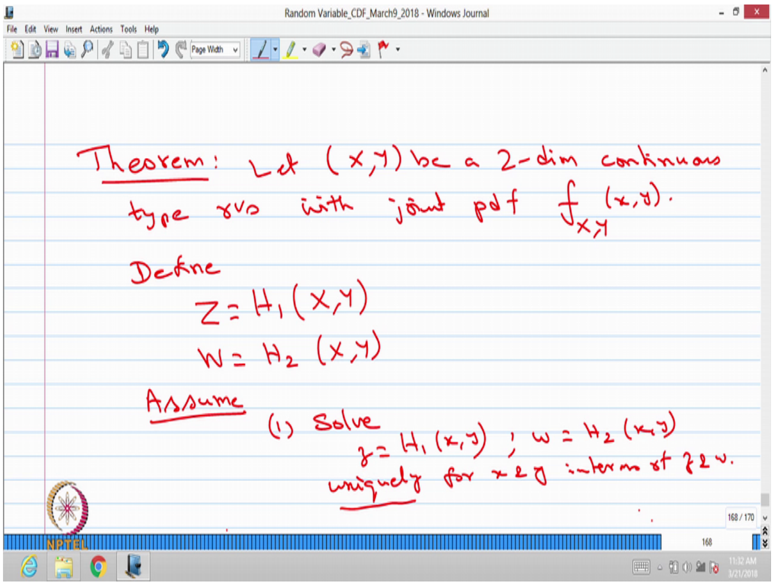

So, let me make it as the theorem we are not going to give the proof of this theorem. Let me start with this theorem for only two dimension random variable then the same concept can be extended for n dimensional. So, let me start with the two dimensional that is let x comma y be a two dimensional continuous type random variables with the joint probability density function that is small f x comma y with a variables x comma y.

I am going to define new set of random variable that is the first random variables Z is H 1 of x comma y. The another random variable W is H 2 of x comma y. We can assume that both H 1 and H 2 are a Boral measurable functions, so that Z and W are going to be a random variables.

I am going to make a few assumptions so that I can able to get the joint probability density function of Z and W directly with the help of the joint probability density function of x comma y. I am going to make a few assumptions if those assumptions are satisfied, then that makes Z is a continuous type random variable as well as W is a continuous type random variables not only that I can find the joint probability density function of Z and W with the help of the joint probability density function of x comma y, so that is what I am going to give it as theorem.

In the first assumption I can solve z as a function of x comma y and the w as a function of x comma y. This equation can be solved uniquely for x and y in terms of z and w. I can solve the same thing I am going to I am replacing capital Z by small z capital X and Y by small x and y. Therefore, whatever I made the transformation of the random variable Z is equal to H 1 of x comma y, W is equal to H 2 of x comma y. I am going to solve those with a smaller letters because I am consistently using the capital letter for the random variable. So, I am solving this equation uniquely for x and y in terms of Z and W.

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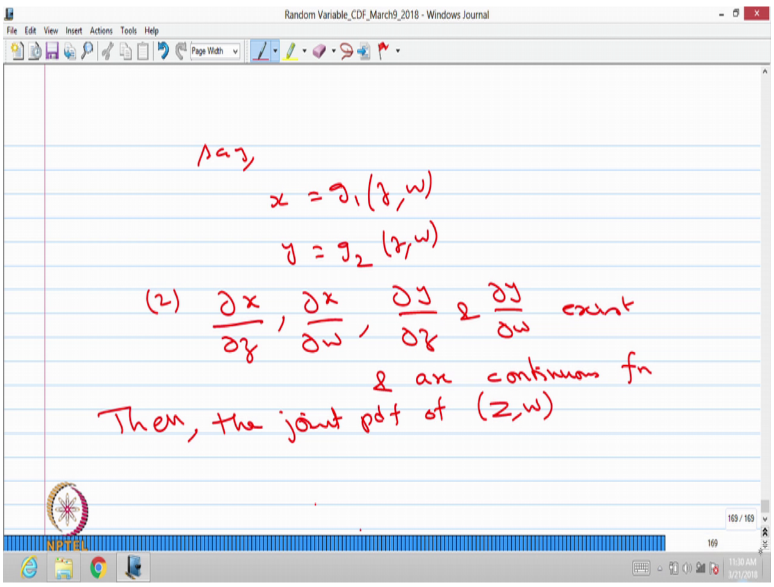

So, whatever I am getting the solution that I am going to write it as say x is going to be say the answer which I am going to get x in terms of z and w that I am going to write it as the sum function of z comma w g 1. Similarly I am going to write y as sum function of a z comma w. So, this is the after solving a those two equations ok.

The second assumption the x in terms of z and w, and y in terms of Z and w I can go for finding out the partial derivative with respect to z w for x and y. I can find the partial derivative of x with respect to z and w.

Similarly the partial derivative of y with respect to z and w here I am making the assumptions partial derivative exist not only exist all this partial derivative are a continuous functions, not only the partial derivative exist it as to be continuous functions also this is the second assumption.

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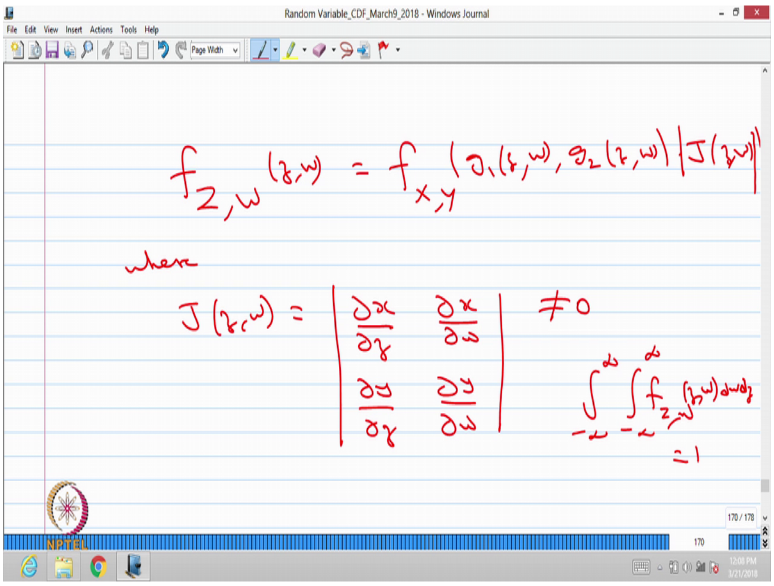

With this assumption I am going for concluding joint probability density function of Z comma W can be written as the probability density function of Z comma w as a function of a Z and w in terms of the joint probability density function of x comma y by replacing x by. If you see we made the we got by after solving x in terms of a z and w of g 1 of this y you are getting g 2. Therefore, in the joint probability density function of x comma y I am going to replace small x by g 1 of z comma w.

Similarly y I am going to replace by g 2 of Z comma w multiplied by the absolute of the determinant that is called Jacobian as a function of z comma w where I can define the Jacobian as a function of z comma w that is nothing, but the determinant of the partial derivative which we have got it partial derivative of x with respect to z, partial derivative of x with respect to w, partial derivative of y with respect to z, partial derivative of y with respect to w.

This determinant is the Jacobian where as in the probability density function of z and w you substituted the absolute of this Jacobian. This is going to be the joint probability density function of z comma w; that means, this theorem says whenever you have a continuous type random variable and you know the joint probability density function of a the continuous type random variables.

As long as these two assumptions are satisfied the word uniquely is very important if that is not satisfied then we have a another remark over it. So, here if the assumption 1 as well as the assumption 2 are satisfied, then we can directly conclude the Z comma W is going to be a continuous type variables and one can get the joint probability density function of Z comma W. By substituting an x by g 1 and y by g 2 in the joint probability density function of a x comma y with the product of a absolute of Jacobian.

The product absolute of Jacobian that is nothing, but the normalizing constant; that means, the joint probability density function of z comma w over the integration minus infinity to infinity, the joint probability density function has to be 1. So, this is going to be 1 whenever you multiply the absolute of Jacobian therefore, the absolute of Jacobian is nothing, but the normalizing constant.

There is the another remark some books use instead of a product of Jacobian they use divided by absolute of a Jacobian. In that case they make the Jacobian in the determinant form not the partial derivative of x with respect to z and w they make the partial derivative of z and w with respect to x and y. Find the determinant of Jacobian of that inverse then substitute in the formula with the divider in the denominator.

Both the results are one and the same because the result is the Jacobian matrix, Jacobian this determinant are the inverse 1 if you make a product that is going to be 1. Because you have a n dimension random variable again you are transforming another n dimension random variable by satisfying the two conditions that is you are solving a those equation uniquely and the partial directive exists and continuous that makes whether you use the Jacobian or the inverse.

Therefore, the formula changes either in the multiplication in the numerator, or multiplication in the denominator form. Because the Jacobian and the inverse Jacobian that determinant value product is always going to be 1. Now, let us go for a one a easy example to explain how this theorem works.

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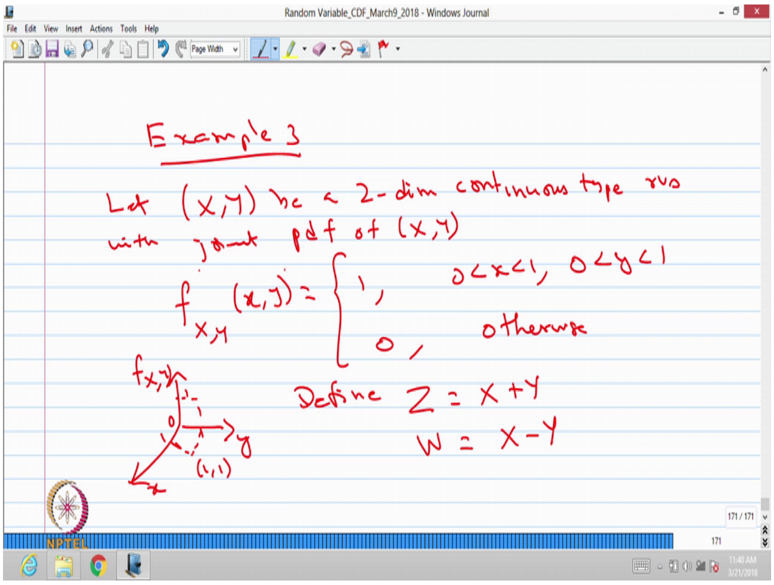

Let x comma y be a two dimensional continuous type random variables with joint probability density function of x comma y that is given as f of x comma y, that takes a value 1, when x is lies between 0 to 1 and y is lies between 0 to 1, otherwise it is 0.

So, this is the joint probability density function of x comma y you can verify you can verify by just you know this is x, this is y this is joint probability density function it takes a value 1 between the interval 0 to 1, and y is also 0 to 1. So, the in the x y plane the region is a square with the vertex 0 comma 0, 1 comma 0, 0 comma 1 and 1 comma 1.

And at the height 1 the surface is at the height 1 over the square in the x y plane. And if you find the volume below that volume below the surface that plane 1 above the square shape that is going to be 1, it is a cube volume of the cube; therefore it is easy to verify this is a joint probability density function of two dimensional continuous type random variable. The question is we are going to create a another two dimensional random variable and then we are finding the distribution of a the new set of random variables that is also two dimensional.

So, I am going to define the new set of random variable you can use the same notation Z is x plus y that is the function H 1 of x comma y. The second function that is capital H 2 of x comma y that is x minus y ok, both x and y are continuous random variable the way we have defined Z is x plus y is w is x minus y.

You can immediately say both are going to be again continuous type random variables, therefore either you can find the cdf of z comma w. If the question is find the distribution, if you know that both the random variables are of the continuous type you can find the joint probability density function. So, here we are going for finding the joint probability density function of two dimensional continuous type random variables Z and W.

We can a make sure whether the assumption of the previous theorem satisfied. If it is satisfied, then you can use the theorem and get the result if it is not satisfied. Then you cannot find the joint probability density function using that theorem to apply the theorem you have to make sure that the assumption satisfied. Now, we will go for whether the first assumption is satisfied or not.

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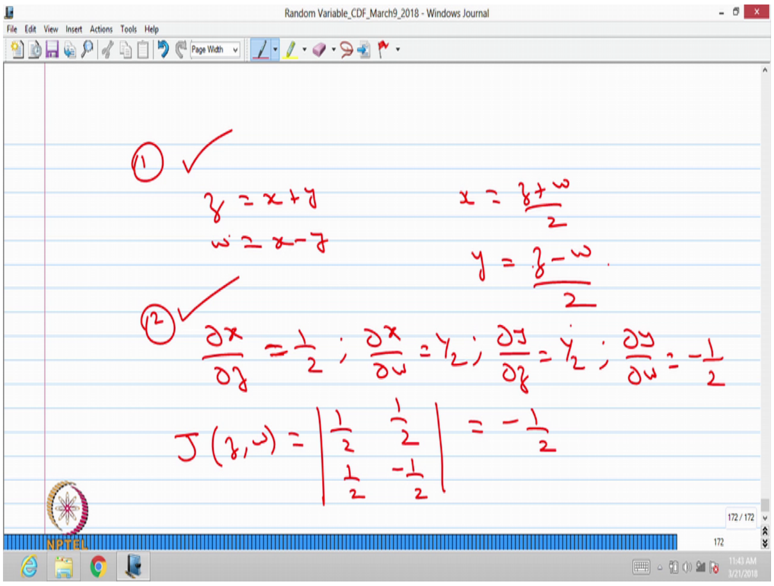

So, you try to find out that is a said z is equal to x plus y and w that is x minus y. You solve for this two equations for x and y in terms of z and w; that means, you can get x as z plus w by 2 correct. If you add this two equations y will be cancelled, so 2 x is equal to z plus w. Therefore, x is equal to z plus w by 2 y is going to be you subtract, s, x will be cancelled. So, you will get the 2 y therefore z minus w by 2, that is going to be y. So, you are able to solve this equation uniquely and you can get the answer x and y in terms of z and w. So, the first condition is satisfied.

We will go for second condition. Find out the partial derivative whether it exits or not the partial derivative with respect to z of the function x that is one by two exist which is continuous constant here that is ok. Similarly you find out the partial derivative with respect to w partial derivative of y with respect to z partial derivative of y with respect to w all are exist and are continuous functions also in particular. Here it is constant therefore; the second condition is also satisfied.

Now, we can go for writing the joint probability density function of a Z and w with the help of joint probability density function of x and y. That is oh before that we will find out the determinant of a Jacobian that is Jacobian as a function of z comma w that is determinant of I will substitute all the partial derivatives in the correct order. Whether you write like this or in the transpose ways it does not matter because at the end you are finding the determinant. That is minus 1 by 4, minus 1 by 4 therefore, it is minus 1 by 2 this is a Jacobian.

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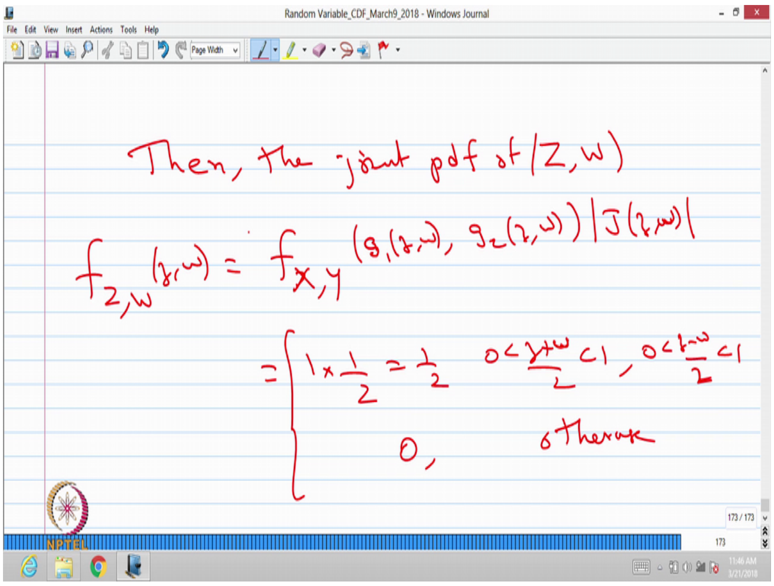

Now we can go for writing since the two assumptions are satisfied you can give the joint probability density function of z comma w as first write down the g 1 of a z comma w, g 2 of z comma w multiplied by the absolute of Jacobian. The Jacobian has to be a non zero, it is also very important condition, because if it is 0, then the probability density function will become zero no. So, as long as the Jacobian is going to be a non-zero quantity we can go for it.

Now, you substitute in this problem the joint probability density function is function of x and y is 1 between this intervals, otherwise it is 0. So, you can replace x by z plus w by 2 y by z minus w by 2 within that range of z and w lies between 0 to 1, the value is going to be 1. So, this is going to be since it is a constant you cannot substitute the x by g 1 and g 2. Therefore, this is going to be again 1, and the Jacobian quantity is a minus 1 by 2 and we have to substitute which is absolute quantity. Therefore, it is 1 by 2 multiplication provided this joint probability density function provided x lies between 0 to 1. So, here it is 0 to z plus w by 2 is less than 1.

Similarly, y is lies between 0 to 1 that is 0 less than z minus w by 2, that is has to be less than 1. So, as long as z and w satisfies these two conditions in which the joint probability density function is 1 by 2, otherwise it is 0. So, the joint probability density function is 1 by 2 when z and w satisfies 0 z plus w by 2 is less than 1, 0 less than Z minus w by 2 that is less than 1. That means, now you can think of a how the joint probability density function of z and w look like.

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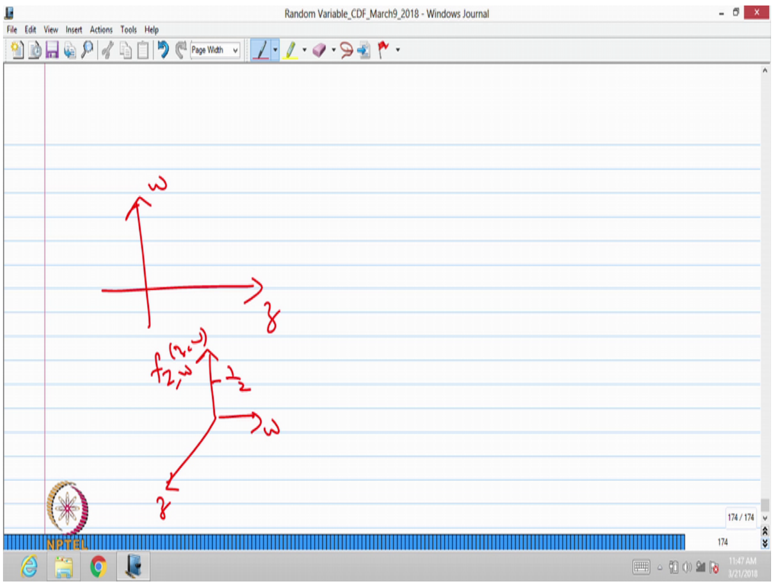

Before that we can go for what is the region of z and the w in which the joint probability density function is greater than 0 that is 1 by 2. First we will identify basically what we want is z w the joint probability density function of z and w. For that first we are making a what is the region in which the joint probability density function is going to be the value is 1 by 2. So, the region is if you simplify these two in equalities you can identify the region of z and w, z and w 0, 1, 2 and 1 minus 1 ok.

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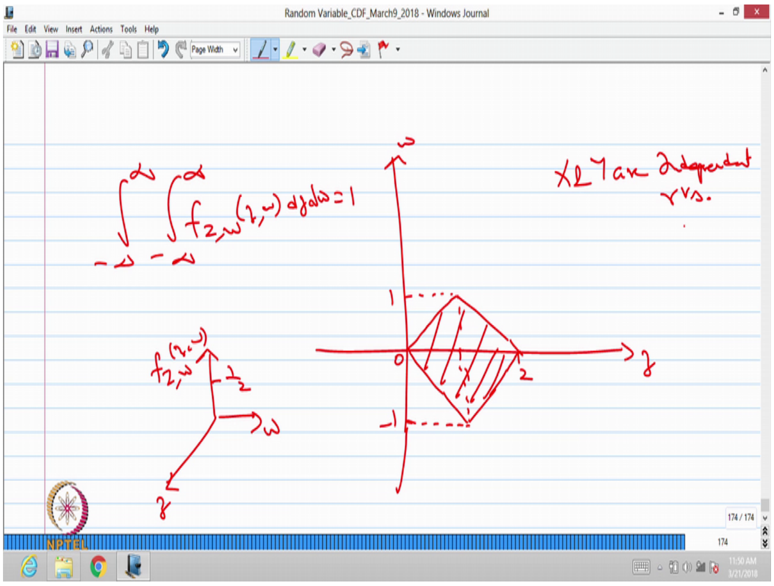

So, if you simplify those two inequalities; you can identify the region is going to be I am not drawing the diagram in correct scaling way. Just for the illustration purpose. So, this is the this shaded region is the region of a z and w; that means, z and w plane this is the region in which the joint probability density function is 1 by 2, otherwise it is 0.

Now we can verify the joint probability density function integration from minus infinity to infinity with respect to z and w is going to be 1 because the x y plane is this diagram above that it is 1 by 2. So, the volume below that surface is 1 by 2 constant over the region in which this diagram shaded region is there the volume is going to be 1.

So, this type of graphical representation is possible only for two dimensional variable not for any n dimensional random variable 3 4 and so on it is very difficult to visualize. So, this is easy to visualize one more observation in this problem given x and y you can see it. The joint probability density function is one if you find out the probability density function of x that is going to be one between the interval 0 to 1 for x.

Similarly, if you find the original distribution of y that is probability density function of y that is also one between the interval 0 to 1 of y, otherwise 0. If you multiply the probability density function of x and y that is same as joint, so we can conclude x and y or a independent random variables, where as the joint probability density function of z and w is 1 by 2 between this interval.

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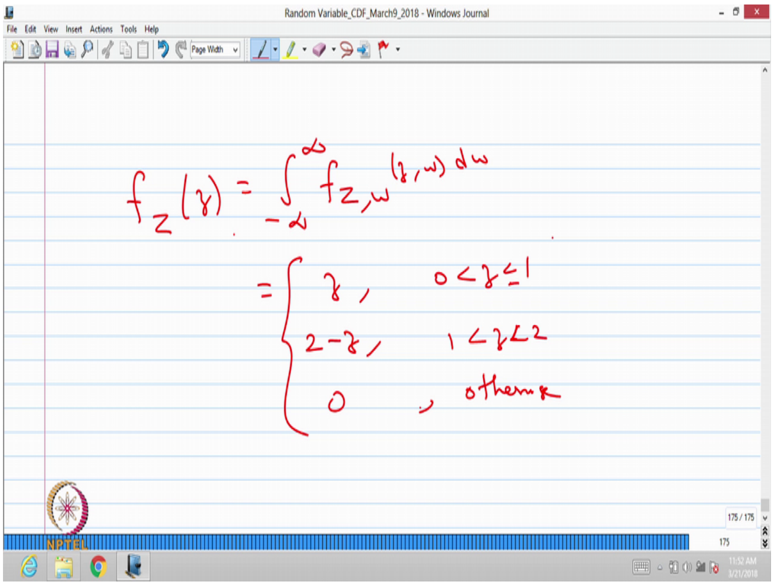

If you find out the marginal distribution of a Z if you do the simple exercise finding the probability density function of z by integrating the joint probability density function of a z and w with respect to w. One can get I am not doing the derivation by substituting the joint probability density function substitute the correct interval then integrate one can get the answer that is a z when z is lies between 0 to 1 that is 2 minus z when z is lies between 1 to 2, otherwise it is 0. So, I can make less than, or equal to here. So, this is going to be probability density function of z between the interval 0 to 1 that is z, and 1 to 2 it is 2 minus z, otherwise it is 0.

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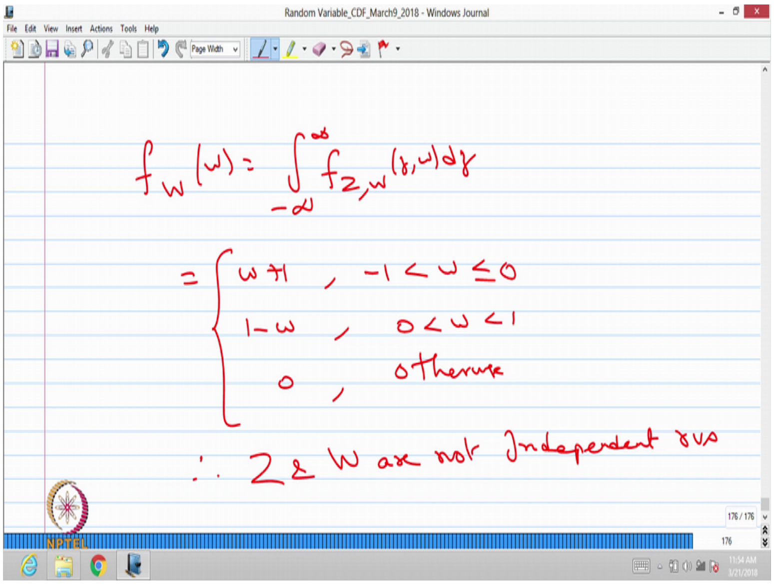

Similarly, one can compute the probability density function of w from the joint probability density function of z and w by integrating with respect to z. If you do that you will get the probability density function of w that is w plus 1, when w in the range from minus 1 to 0 and it is 1 minus w between 0 to 1 otherwise it is going to be 0.

The interval of z and w that you can get it you can feel it from the diagram itself, the range of z is 0 to 2 whereas, the range of w is minus 1 to 1. So, therefore, we are getting the probability density function like this for Z and probability density function of a w in this form. The way the probability density function of z and w is like this if you make a multiplication you would not get the value 1 by 2 that is joint probability density function of z and w.

Therefore, you can immediately conclude z and w are not independent random variables x and y are independent random variable the way we defined z is x plus y w is x minus y they are not independent random variables. So, with this example we are explaining how the theorem works.

But sometimes the assumption first assumption that is solve uniquely it may not satisfy; that means, you may have a more than one set of values instead of z and w in terms of x and y uniquely in that case for every set of pairs you have to identify what is the density function with the corresponding Jacobian. And you have to keep adding how many pairs of solution you are going to get those many summations you have to make it to get the joint probability density function of z and w.

It is similar to what we have done it with the function of a random variable for a continuous type whenever it is not satisfying whenever the function is a monotonically increasing, or decreasing or decreasing or increasing form the same technique he has to be applied for the multidimensional random variable.