**Introduction to Probability Theory and Stochastic Processes**
**Prof. S. Dharmaraja**
**Department of Mathematics**
**Indian Institute of Technology, Delhi**

**Module - 06**
**Functions of Several Random Variables**
**Lecture – 29**

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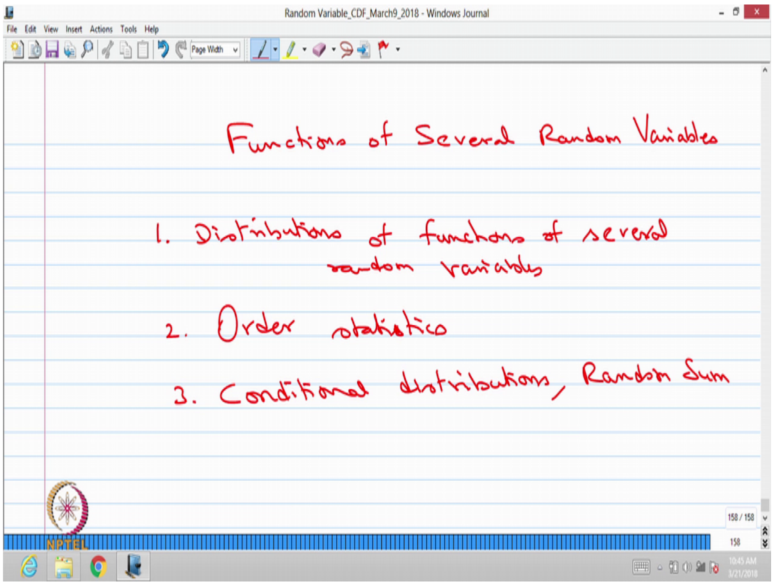


In this week we are going to discuss the module on functions of several random variables. In the last module we discussed more than one random variable together that form a random vector, and we discuss the joint distribution of n dimensional random variables. Now, in this module we are going to discuss functions of a random variables. In this topic we are going to discuss three aspects one is the distributions of a functions of several random variables. Then the second lecture we are going to discuss order statistics, this is also a one special type of functions of several random variables. Third we are going to discuss conditional distribution.

The way we discussed conditional probability of a event, we are going to discuss conditional distributions, when we discuss several random variables, and followed we are going to discuss random sum. So, these are all the three topics which we are going to cover it in this module, in functions of several random variables.

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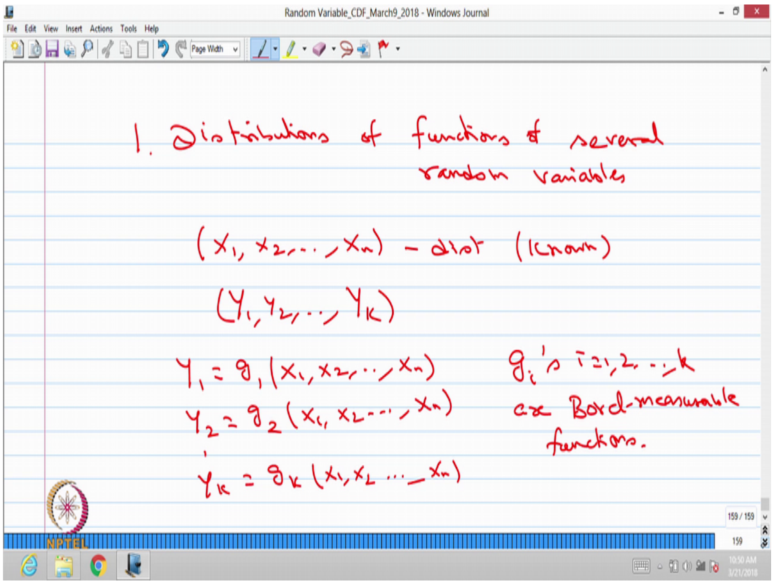


The first lecture is on distributions of functions of several random variables. In any random experiment we create n random variables, but later when we start solving the problem. So, we may feel we need some more random variables which is other then what we have started in the beginning. We can always create a new random variable from the scratch; by the definition of random variable then we can get the n random variable.

But it is easy to create new set of random variable with the help of the earlier set of random variable which we have already created; that means, we can create a some sort of composite function of random variables, such a way that we will land up the new set of random variables.

In that case our interest is to find out what is the distribution of a the new set of random variables, when we know the distribution of the earlier set of random variables; that means, through the distribution of earlier set of random variables we are going to find the distribution of a new set of random variable. When I use the word earlier set of random variables and new set of random variable, I am using the notation called the earlier set of random variables is X 1 X 2 X n; that means, each one is a random variable we have a n random variables are together jointly, therefore, this is second dimension random variable.

The new set of random variable I say Y 1 Y 2 Y suffix k; that means, the new set of random variable need not be the same size of the earlier set of random variable. So, we know the distribution of a this random variable, that is known. We are going to find out what is the distribution of Y 1 Y 2 Y k when we know the distribution of X 1 X 2 X n. For that what we are going to do? We are going to create the relation of a Y i's in terms of X i's; that means, Y i you can think of it is a composite function with the n random variables.

Similarly, you can think of Y 2 as a another random variable, which is a function of a X 1 X 2 X n random variable, like that we have k random variables; that means, we obviously, X i's are the random variable, Y i is, i is equal to 1 to k are going to be random variable, provided the function g is as to be Boral measurable function.

So, we make the assumptions gi's where i is equal to 1 to k, or a Boral measurable functions, the same thing what we have done in it in the 1 dimensional random variable. When X is the random variable g is a Boral measurable function, then Y is equal to g of X. That is also going to be a random variable the same concept we are using for the several random variables. Therefore, when xis are known with the distribution and gi's are Boral measurable functions then Yi's are going to be the random variables, therefore, Y 1 Y 2 Yk is the k dimensional random variables.

The question is how to find the distribution of Yi's, that I am going to give it as the result.

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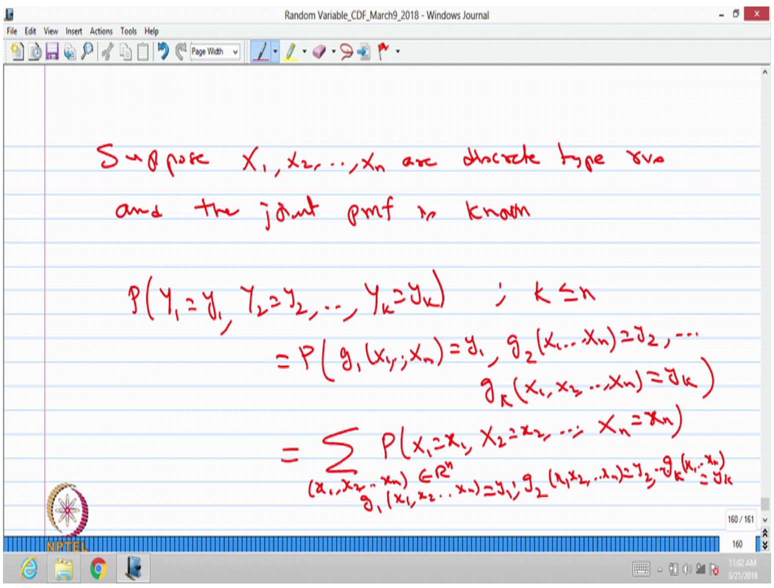

Suppose X is X 1 X 2 Xn, or discrete type random variables, and the joint probability mass function is known as, I am discussing the concept of distributions of a several random variable for discrete type random variable first, then later I am going to discuss further continuous type random variable.

Since we know the joint probability mass function of X is you can directly write down the probability mass function of Y 1 Y 2 and Y ks jointly, that is same as probability of; you can replace Y 1 by g 1 as a function of X 1 X n, that takes a value small Y 1. Y 2 takes a value small Y 2; that means, g 2 X 1 X 2 X n that takes a value Y 2 and so on.

And for the gk of X 1 X 2 and so on Xn, that takes a value Yk, that is same as all the summation of the probability of X 1 takes a value small X 1. X 2 takes a value small X 2, and so on X n takes a value small Xn; such that all the X is X 1 X 2 Xn belonging to Rn, not only that g 1 of X 1 X 2 Xn, that is same as Y 1.

Similarly, g 2 of a X 1 X 2 Xn, that is same as Y 2 and so on g k of X 1 X 2 Xn, that is same as Yk. If you make a summation over a this conditions; that means, all the X is belonging to Rn, and gone of X 1 X 2 Xn that is same as Y 1. G 2 of X 1 X 2 X n that is same as Y 2; like that gk of X 1 X 2 Xn that is same as Yk, finding of the joint probability of a X 1 takes a value small X 1 X 2 takes a value small X 2 and so on.

Xn takes a value X that is going to be the probability of the random variable Y 1 takes a value small Y 1. Similarly the random variable Y 2 takes a value small Y 2 and so on, random variable Y k takes a value Y k. So, this is a Probability mass function for k dimensional random variable. So, this is the joint probability mass function of a k dimensional random variables Y 1 Y 2 and so on Y k.

So, this is the way one can find the distribution of a functions of a several random variable when the random variables are of the discrete type. And gs are Boral measurable functions so that Y is also going to be discrete random variable one can get the joint distribution in this way. Let us go for one simple example.

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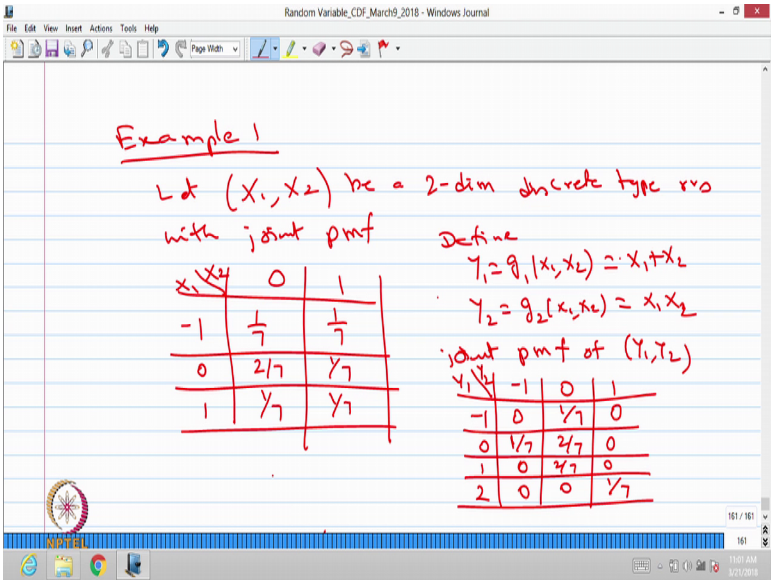


How it works the example 1? Let X 1 comma X 2 be a 2 dimensional discrete type random variables with joint probability mass function which is given by X 1 X 2, the possible values of X 1 is minus 1 0 and 1.

And the possible values of X 2, that is 0 and 1, the joint probability mass values for possible values of X 1 and X 2 this is 1 by 7, and this is also 1 by 7 this is 2 by 7, and this is 1 by 7 1 by 7 again 1 by 7. If you sum all the values it is going to be 1 2 3 4 5 5 plus 2 7 summation is one, and if you make a row sum and column sum, you will get the marginal distribution of X 1 and X 2.

Now, we are going to define the new random variables, that is Y 1 as a func g 1 of X 1 comma X 2, that is X 1 plus X 2, that is Y 1, we are defining a second random variable as a g 2 of X 1 comma X 2, that is product of X 1 X 2, X 1 X 2 are of the discrete type random variable and Y 1 is X 1 plus X 2 Y 2 is equal to X 1 into X 2. Therefore, you will get a Y 1 and Y 2 are discrete type random variables.

Now, our interest is to find out the joint probability mass function of Y 1 comma Y 2. Since it is a discrete type you can make a table so Y 1 Y 2. Since X 1 takes a value minus 1 0 1 X 2 takes a value 0 and 1 Y 1 takes a value minus 1 0 1 and 2 so, minus 1 0 1 and 2. Similarly one can find the possible values of Y 2, that is minus 1 0 and 1 1. Now you can start filling up, suppose Y 1 takes a value minus 1 Y 2 takes a value minus 1, what are all the possibilities in the X 1 X 2? So, that the Y 1 is going to be minus 1 Y 2 is going to be minus 1.

There is no way you will get this possibility, therefore, the probability of empty set is 0, whereas when Y 1 is minus 1 Y 2 is equal to 0, that is possible with the probability 1 by 7. Similarly Y 1 is minus 1 Y 2 is plus 1, that is not possible therefore, the probability is 0.

Similarly, you can fill up other elements 1 by 7, this is 2 by 7 you can verify 0, this is 0 2 by 7 again there is no possibility therefore, the probability is 0. There is no possibility this is 1 by 7 again you can cross check whether this is going to be the whole summation is double summation over Y 1 Y 2 as to be 1 1 by 7, 1 by 7, 2 by 7, 2 by 7, 1 by 7, therefore, the addition is one.

Here also you can find the marginal distribution of a Y 1 and Y 2 from the joint distribution. So, since this is the discrete type with the 2 dimensional random variable; and again we make a Y 1 Y 2 as a 2 dimensional random variable, we are getting the joint probability mass function of Y 1 Y 2. Suppose if it is a n dimensional and ys are k dimensional, still you can able to find as long as k is less than or equal to n, as long as k is less than or equal to n, you can get the joint probability mass function.

So, that can be written in the previous slide; k has to be less than or equal to n you can get the joint probability mass function of Y is where i is running from 1 to k.

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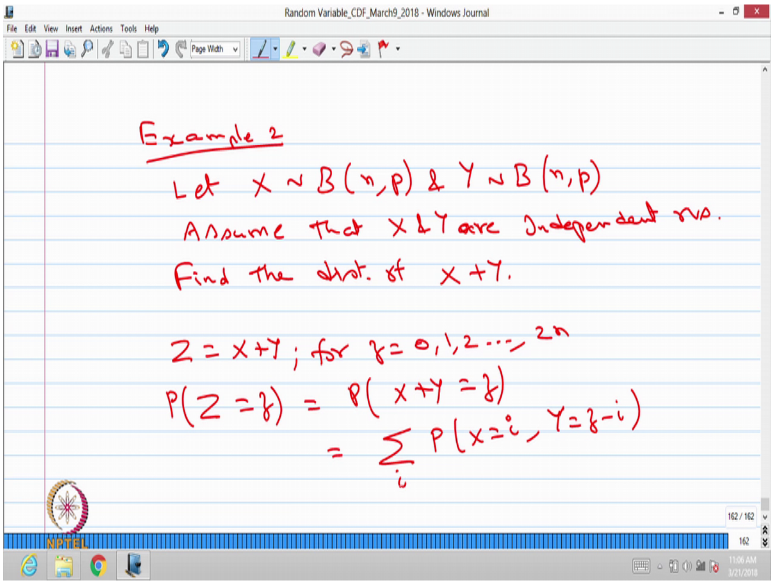


This is a very easiest example. Now we will go for the little different example 2, let X be a random variable which is binomial distributor with a parameters n comma p. And another random variable Y that is also binomial distributed with a parameter n comma same p. And I make the assumption assume that the random variables X and Y are independent.

I assume that X and Y are independent random variables. The question is find the distribution of X plus Y, find the distribution of X plus Y. It is easy to do because a X is a discrete type random variable, Y is a discrete type random variable, and we have not supplied the joint probability mass function of X and Y. Whereas, we make the assumption both are independent random variable.

So, you can use the independent concept if two random are independent then the joint probability mass function is same as product of probability mass function of X and Y. So, you can use that concept. Therefore, I can directly compute the distribution of X plus Y, I will take Z as X plus Y. Since X is a binomial distribution, the possible values are 0 1 2 and n, and Y is also binomial with the possible values are 0 1 2 and so on till n, therefore, the possible values of z is going to be 0 1 and so on till 2 n.

So, one can find the probability mass function of the random variable z for z takes a value 0 1 2 and so on till 2 n. So, this is the probability mass function which we are going to find, which is going to be a positive and all others points it is going to be it is 0.

So, we will be read about a when z takes a value 0 1 2 so, on till 2 n. So, this is same as, I can replace Z by X plus Y takes a value small z. That is same as, if I introduce a one dummy index i, that is a probability of X takes a value i and Y takes a value z minus i, and all possible values of i, I can get the probability of X plus Y is equal to Z. I am just replacing probability of X plus Y is equal to Z, that is same as for all possible values of I with a summation the joint probability of X takes a value i ze Y takes a value z minus i, that is going to be same as probability of X plus Y is equal to Z.

I am not going to write what are all the possible values of I very clearly, because for some I the Y may not be within the range of 0 to n or X cannot be in the range of 0 to n, therefore, whenever the X takes a values 0 to n as well as Y takes a value za 0 to n, then only you will have a some sort of joint probability mass function, otherwise those probabilities are going to be 0, therefore, I am not going to write what is the possibility values of y I leave it as it is.

Now I am going to use the independent concept; that means, the probability of X takes a value i, and Y takes a value z minus i, that is same as since these 2 random variables are independent. I can make it probability of X takes a value i multiplied by probability of Y takes a value z minus i, that is what I made the assumption this 2 random variables are independent otherwise I can proceed further.

So, either to solve this problem, I should have given both the random variables joint probability mass function for X comma Y or I would have made the assumption X and Y are independent, then only I can able to find the distribution of X plus Y.

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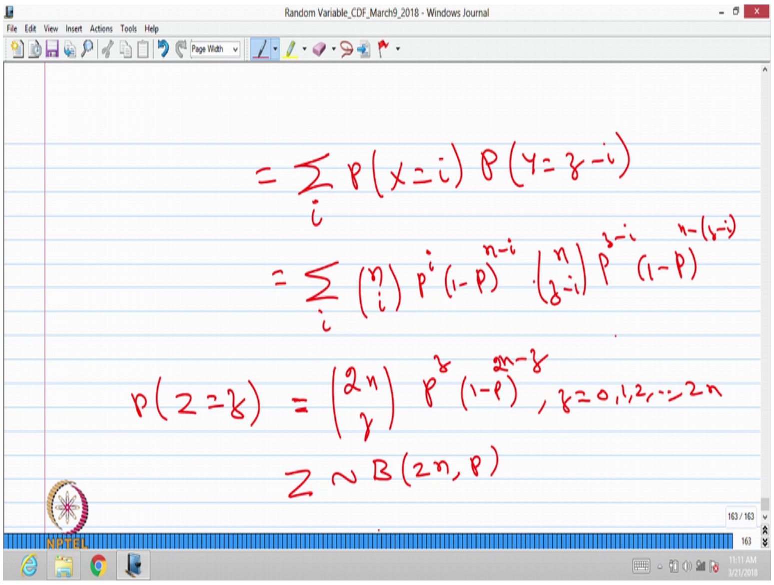

So, this is same as summation over i probability of X takes a value i probability of Y takes a value z minus i.

Now, you can use binomial distribution probability mass function for X as well as Y. That is what we have not discuss many problems in the earlier modules, when we started discussing the standard distributions, because the whole course we will be using those distribution again and again. So now, those who remember probability mass function of binomial distribution, you can directly write that is nci p power i 1 minus p power n minus i. This is a probability mass function for a binomial distribution with the parameters n comma p.

Similarly, you can write probability mass function of Y, that is nc z minus I p power z minus i, 1 minus p power n minus z minus i. You can go for simplification; this is p power i this is p power z minus i eh you can take the p power z outside. Similarly n minus z plus i and this is 1 minus p power n minus i. So, you can do some simplification, and after simplification you can get a the result that is 2 n z p power z 1 minus p power n minus z (Refer Time: 25:31), yeah sorry, it is 2 n minus z, the way I said p power i p power z minus i that is killed you will get p power z.

Similarly, 1 minus p power n minus z minus i, if you simplify you will get 1 minus p power 2 n minus z. The only thing is the summation over i nci with the multiplication nc z minus i, that is same as 2 ncz, that is the result which we are using summation of nci multiplied with the ncz minus i, that is same as a 2 n c z.

So, this is the probability mass function of the random variable z. And possible values of z is 0 1 2 and so on, till 2 n by seeing the probability mass function, which is greater than 0 in this value 0. Otherwise you can conclude the random variable z, which is also binomial distributor with the parameters, you have to map the probability mass function of this with the probability mass function of binomial distribution, then you will conclude that is parametric 2 n comma p.

So, this is the use of a named or standard or common distribution, you do not know the distribution of z you are finding the distribution of z. After you get the probability mass function, this is same as probability mass function of a binomial distribution, therefore, we conclude z is also binomial distributor with a parameters 2 n comma p. Now, one can discuss some vertex scenario.

Suppose X takes the binomial distribution with the parameter n comma p, whereas, Y takes binomial distribution with some other parameter instead of n, suppose you treat some other positive integer m, but the next parameter is same p let me write.

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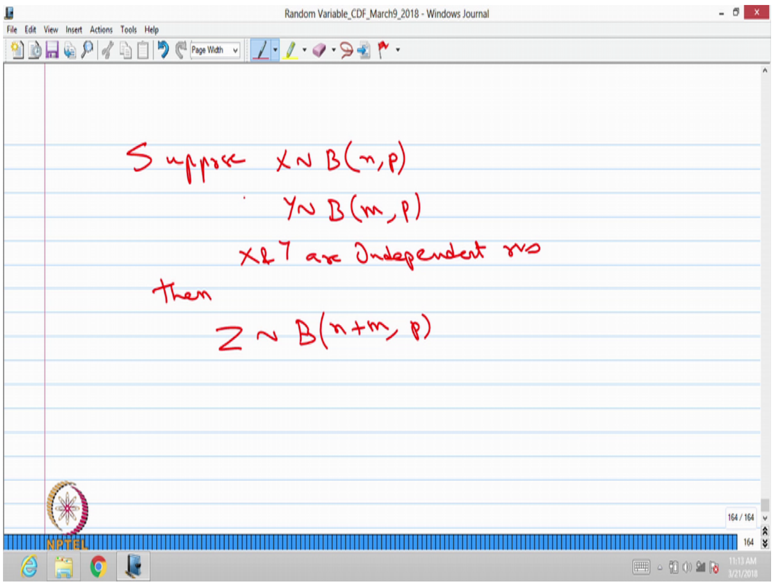


Suppose X is binomial distribution with n comma p, and Y is binomial distribution with some other parameter m comma p, and again I making the summation both the random variables are independent. Then also you will get binomial distribution with a parameters n plus m comma p; that means, if you have a two independent binomial distribution with a different n and m. Whereas, the probability of success in each Bernoullic trial p is same for both, the random variables then the sum is also going to be a binomial distribution with the parameters sum of those first parameter comma the second parameter is p.

That means, this can be extended for any n random variables, suppose each random variables is binomial distributor with a some number some positive integer with p. And all are going to be a different number for first parameter, then the sum is going to be again binomial distribution, with sum of first parameters with second parameters p.

So, this can be generalized into any n mutually independent random variables. Since we have two random variable we are using the word independent random variable. Once you have more than two random variables, for any n mutually independent random variables; each one is binomial distributor with the same probability of success p, with the different parameters in the first one then the sum is also going to be binomial distribution.