Introduction to Probability Theory and Stochastic Processes

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Lecture – 26

In the module of random vectors, we started with the 2 and high dimensional random variables, and we discussed the joint distributions, first we discussed the CDF of 2 dimensional random variable then we have generalized this in to the n dimensional random variable, how the CDF look like with the several variables.

Then we started discussing joint probability mass function; whenever the underlined random variables are of the discrete type; that means, we have n dimensional random vector with each random variable is going to be of the discrete type.

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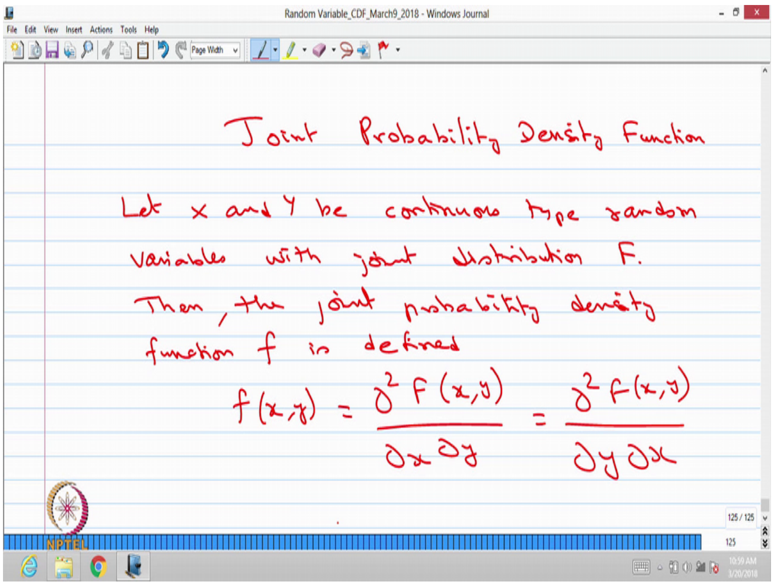

Now, we are coming in to joint probability density function. Let me start with the 2 dimensional random variable.

Let X and Y be continuous type random variables with joint distribution, that is capital F. That is basically a function of X comma Y. Then the joint probability density function, small f is defined as a function of x comma y by taking partial derivative of CDF with respect to x and y the second and the partial derivative which is same as the partial derivative of CDF with respect to y x.

Since we have a continuous type random variable, the CDF is continuous function in both X and Y. Since it is a continuous function, both and x and y by taking a partial derivative with respect to x and y, that is going to be the probability density function of x comma y.

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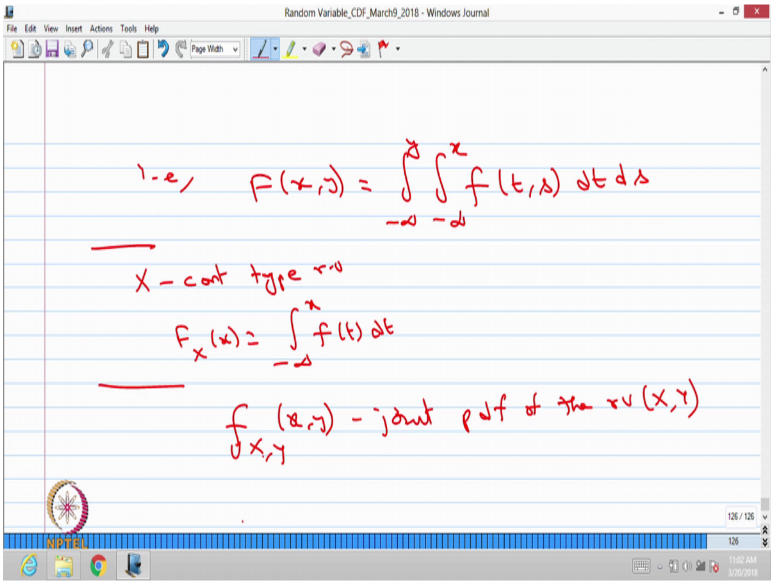

That is same as the CDF of 2 dimensional continuous type random variable can be written in the form of a doubled integration with respect to x and y; that means, since both the random variables are of the continuous type, the CDF of 2 dimensional continuous type random variable is a continuous function in both x and y, one can write CDF in the form of double integration from minus infinity to y minus infinity to x, the sum integrant small f t comma s dt ds.

Here the integrant is the joint probability density function. It is similar to one dimensional random variable, in which the X is of the continuous type random variable then the CDF of x can be written in the form of integration of sum integrant, and that is a probability density function of the random variable x. The same way we are writing a for a 2 dimensional continuous type random variable therefore, the probability density function is nothing but the second order partial derivative of CDF, with respect to x and y since it is a continuous function in both X and Y, whether you change the order of partial derivative does not matter it is going to be the c.

So, once you know the joint probability density function of X comma Y, one can find the probability density function of one random variable; that means, from the joint probability density function so, here the X comma Y small f so, this is the joint probability, density function of the random variable x comma y. From the joint probability density function one can always get the marginal distribution of x comma y that is marginal distributions.

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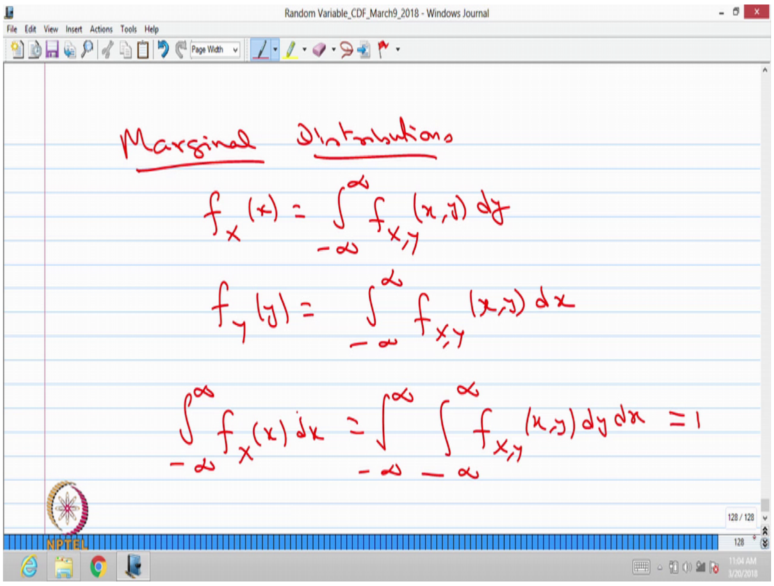

From the joint probability density function, one can get the marginal distribution of one random variable by integrating the joint probability density function with respect to y.

Similarly, we can get the probability density function of the random variable y by integrating the joint probability density function of x comma y with respect to x. The same thing we have done in for discrete type random variable. So, this is the continuous type random variable, therefore, the marginal distribution of x comma x and y can be obtained from the joint distribution of the random variable x comma y.

One can verify how this is going to be the probability density function of x, and how this is going to be the probability density function of y, because the this function you have F of x is going to be always greater or equal to 0. And if you integrate minus infinity to infinity the probability density function of x, that is going to be double integration minus infinity to infinity the joint probability density function of x comma y.

Therefore, we know that the double integration from minus infinity to infinity joint probability density function, this is going to be 1.

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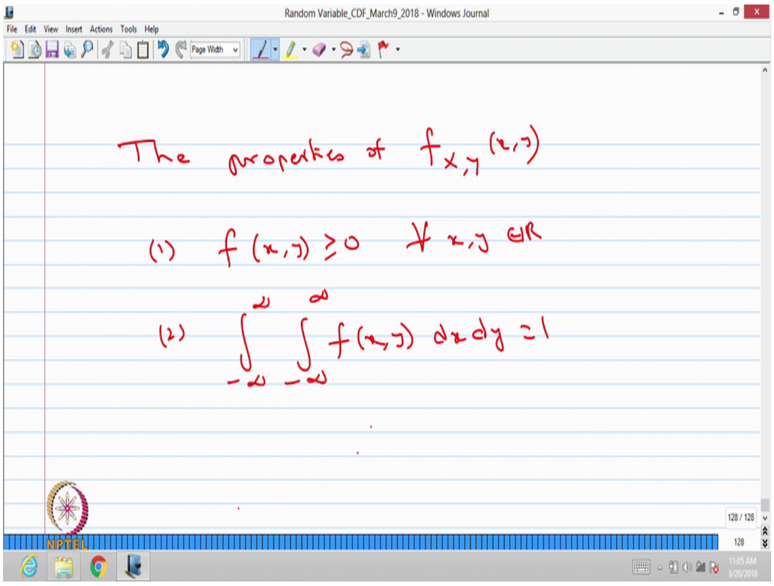

Therefore it is the properties of the joint probability density function the properties of joint probability density function is, this is always going to be greater or equal to 0, for all x comma y belonging to real and the second condition, from minus infinity to infinity the double integration, the joint probability density function is always going to be 1. That means, if you have any real valued function with the 2 variables satisfying these 2 conditions may be a joint probability density function of some 2 dimensional random variables.

If you have a 2 dimensional random variables of continuous type, then which has the joint probability density function that satisfies this 2 properties. The first property gives the value is going to be always greater than equal to 1, and has a double integration is going to be 1 that is nothing but the volume below the surface of F of x comma y that is going to be 1.

The way we say for the single random variable of continuous type the probability density function is greater or equal to 0, then the integration 1, means the area below the curve that is going to be one the same also here for a 2 dimensional random variable. The volume below the surface that surface is F of x comma y, that is going to be 1. So, only one can visualize or make a graphical representation for a single dimensional random variable and 2 dimensional random variable not more dimensions. Therefore, we started the explaining 2 dimensional random variable with the graphical representation not for any more and any more dimension.

The same concept can be extended for n dimensional random variable.

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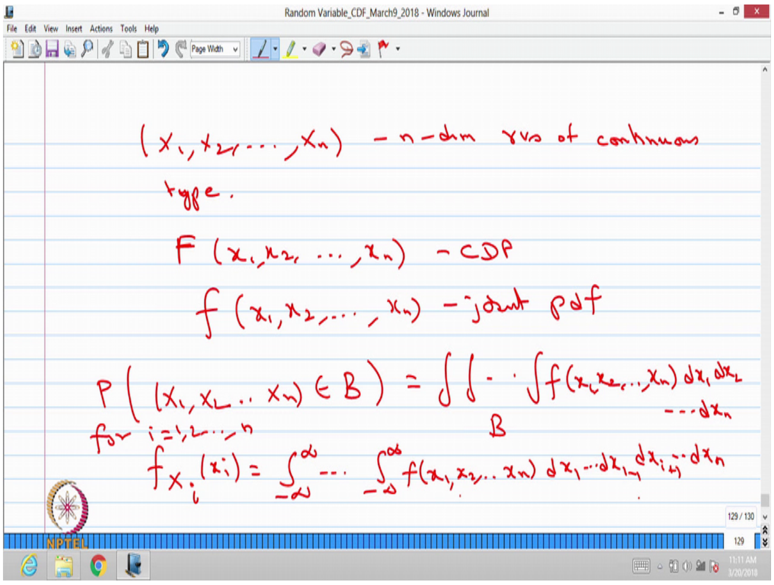

That means this is n dimensional random variables of continuous type; means each random variable is a continuous type random variable, therefore, we get n dimensional random variables of the continuous type; that means, it has the CDF with the n variables. This is the CDF of the n dimensional random variable. We have a the joint probability density function with the n variables x 1 x 2 and so on.

This is the joint probability density function. Whenever you need to find out probability of x belonging to X 1 X 2 X n belonging to some Boral set some capital B, which is in the rn, the B is belonging to, rn then that is nothing but finding out the probability is nothing but the n dimensional integration over the Boral set, capital B of the joint probability density function for n dimensional random variable, if you know the each random variable of the continuous type then you have a CDF.

Similarly you have a joint probability density function, and you can always find probability of all the n dimensional random variable takes a values in the Boral set, where Boral set is the from the rn that is nothing but n dimensional integration over a capital B of joint probability density function with respect to all the variables.

The way we got the probability belonging to the B, one can find the probability density function of any one random variable for I is equal 1 to n, you can always find the marginal distribution of any one random variable from the joint distribution of the n dimensional by integrating the joint probability density function of n dimensional random variable, with respect to x 1 xi minus 1 xi plus 1 x n, you can always get the marginal distribution of any one random variable from the joint distribution of n dimensional random variable which is of the continuous type. Now we will move into one simple example how one can describe the 2 dimensional random variable of the continuous type.

Then how one can discuss the marginal distribution and finding out the probability sense upon.

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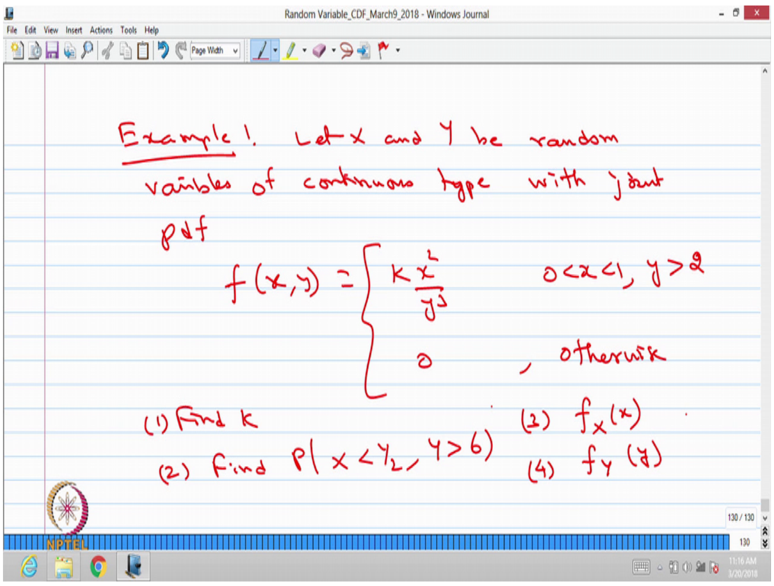

Let us start with the easy example, the example is let x and y be random variables of continuous type with the joint probability, density function is given by F of x comma y that takes a value, some constant times x square divided by y cube, whenever x takes a value from 0 to 1, and y takes a value greater than 2, otherwise it is going to be 0. So, you can think of as some surface with the 3 dimension plane, x is one coordinate, y is another coordinate, and z that is the joint probability density function.

So that means, you have a surface which is greater than 0, the value is going to be greater than or equal to 0 in between x lies between 0 to 1, and y is from 2 to infinity in that the function is going to be greater than 0 that is k times x square by y cube. Otherwise it is 0 therefore, the volume below that it is going to be 1.

Now, the question is since I have made consent times x square by y cube. First you have to find out what is a k in which the given function if the 2 variables is going to be joined from probability density function, after that you have to find out some more results so, the first question is find k. So, that this is going to be joint probability density function of 2 dimensional random variables of continuous type.

Second question find probability of x is less than 1 by 2, and y is greater than 6, that is the second question is similar to identifying the probability of x lies between x belonging to some Boral set. The third question find out the probability density function of the random variable x. The 4th question is find out the probability density function of the random variable y.

So, these are all the 4 questions for this problem it is a easy problem. So, let us go for finding out what is value of k in which it is going to be joint probability density function. We know that the property of joint probability density function is always going to be greater or equal to 0, and double integration has to be 1.

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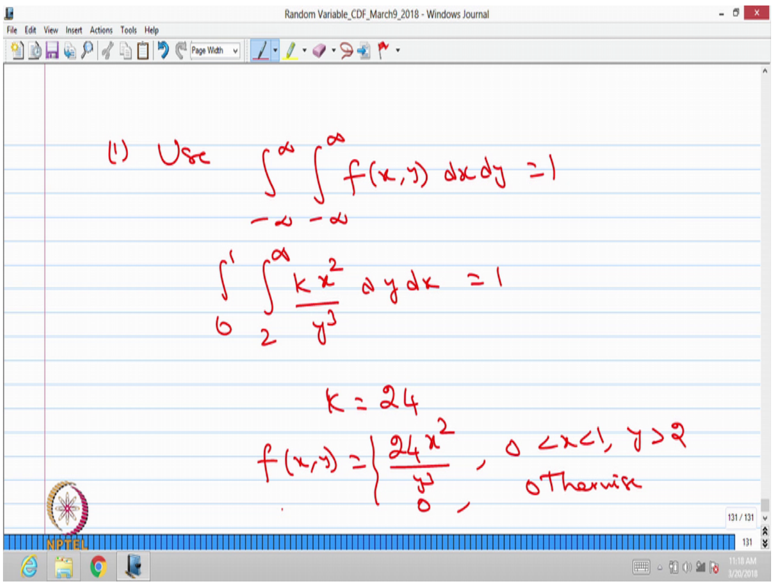

So, use double integration, the joint probability density function is going to be 1, but this joint probability density function is greater than 0 between some interval. So, we can make out the interval 0 to 1 and 2 to infinity, the function is k times x square by y cube dy dx that is equal to 1 0 to 1, 2 to infinity k times x square by y cube dy dx has to be 1, you do the simple calculation or this integration. You can come to the conclusion the k value has to be 24. Therefore, the joint probability density function is 24 times x square divided by y cube, when x lies between 0 to 2, 1 y is greater than 2, otherwise is 0.

So, we have answered the first question, second one find out the probability of x lies between x is less than 1 by 2 and y is greater than 6.

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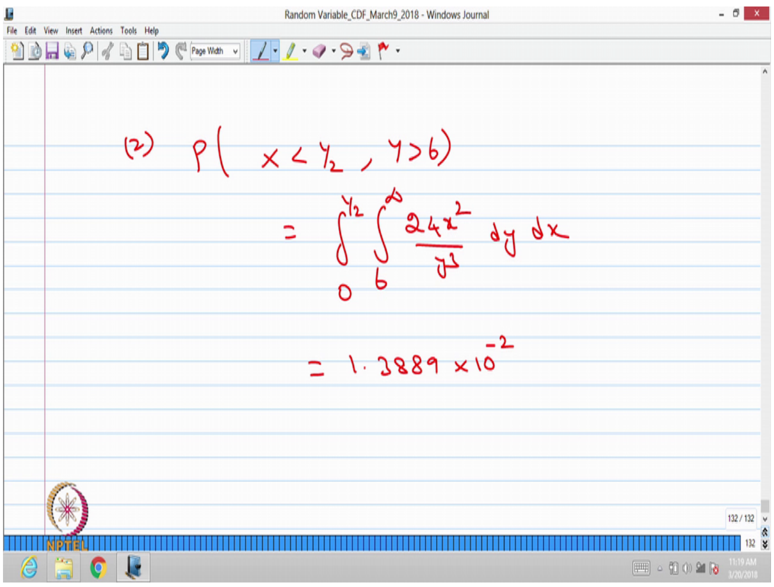

So, this is nothing but the double integration of the joint probability density function with respect to y, between 6 to infinity with respect to x it is from 0 to 1 by 2. Here also if you do integration and do the simplification, you can get the answer it is 1.3889 multiplied by 10 power minus 2, am not spending time on the integration.

You can do the integration and you can get the answer, that is am using the concept of the probability of am using this concept of probability of n int n random variable, belonging to the Boral set is nothing but integration of n dimensional random variable over the Boral set the same concept is used to compute the probability of x less than 1 by 2 and y greater than 6. The same way am going to use the finding out the marginal distribution from the joint distribution or the next question. So, the next question is find out the marginal distribution of x similarly marginal distribution of y.

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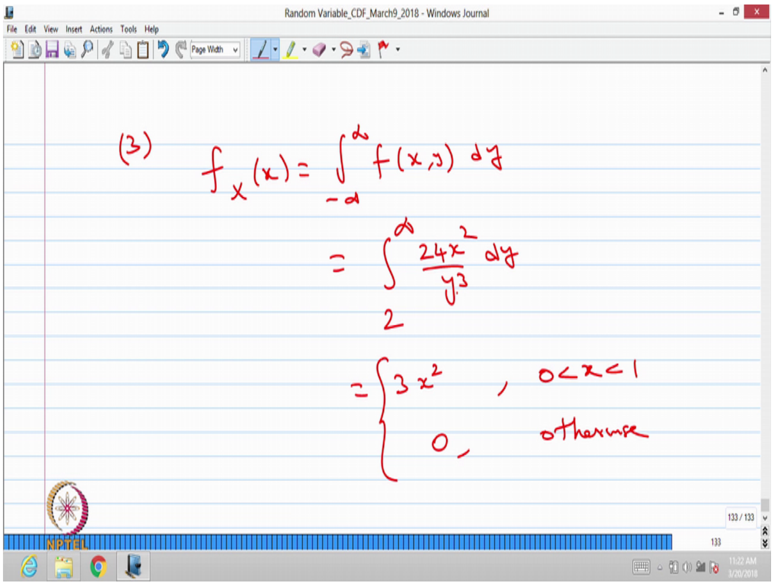

So, third question the probability density function of x is nothing but integration of the joint probability density function with respect to y.

And we know that the probability density function is greater than 0, when y is lies between 2 to infinity therefore, this is nothing but 2 to infinity the joint probability density function is 24 x square by y cube with respect to y. And if you do the little simplification, you can get the answer that is 3 times x square. And if you recall the joint probability density function is lies between 0 to 1 for x and for y it is 2 to infinity, therefore, this probability density function is going to be 3 x square when x is lies between 0 to 1, otherwise it is 0. This is a very easy problem in which the interval of x does not involve y.

Sometimes you may have a complication of the interval of x is a function of y, or interval of y may be a function of x also. So, you have to use the calculus of several variable and the integration concepts correctly to get the marginal distribution of x.

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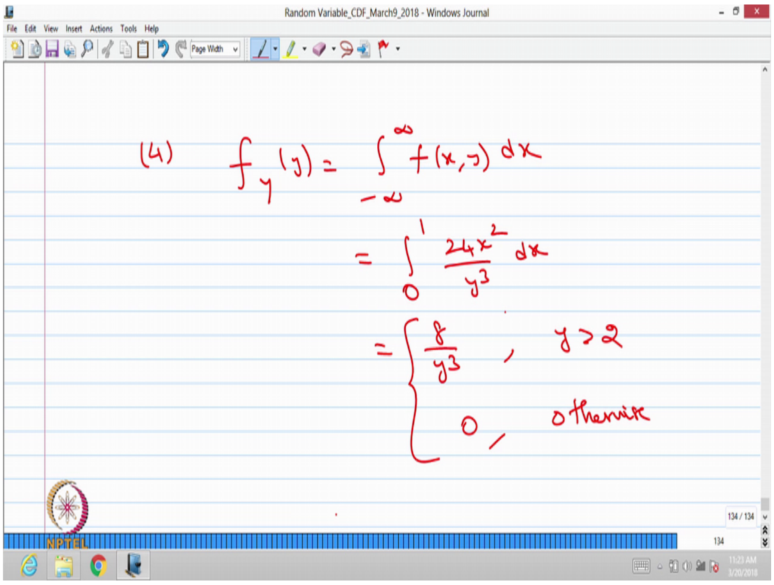

Similarly, we can go for to find the probability density function of y that is nothing but integration of joint probability density function with respect to x. So, here the joint probability density function is greater than 0, when x lies between 0 to 1. So, 0 to 1, and the joint probability density function is 24 x square divided by y cube with respect to x. That is same as if you do the little simplification, you will get 8 divided by y cube and a if you recall the range of y is 2 to infinity. Therefore, y is greater than 2; the probability density function of y is 8 divided by y cube, 0 otherwise.

Otherwise means the probability density function is 0 between minus infinity to 2 from 2 to infinity the value is 8 divided by y cube will go for second problem example 2.

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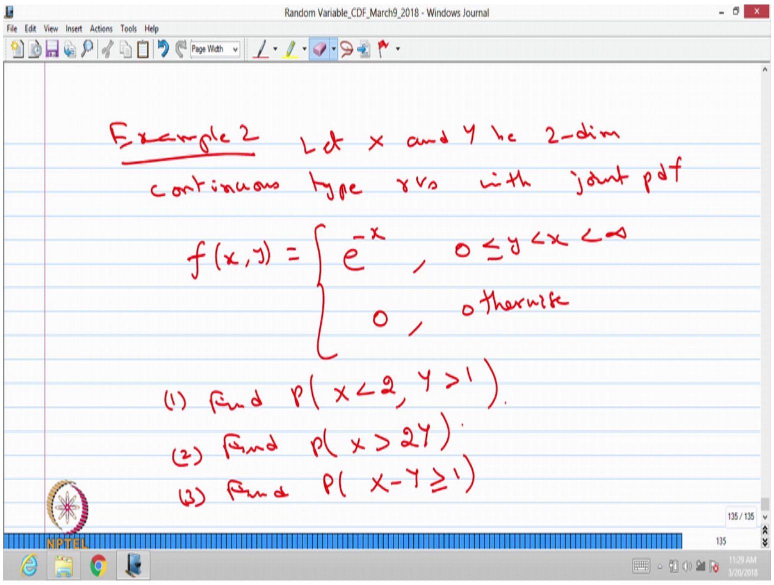

In this problem let me discuss x and y be 2 dimensional continuous type random variables with joint probability density function is given by F of small f. Whenever I use a small f that is a joint probability density function if it is a capital F ah; that means, it is a CDF cumulated distribution function. So, the here the joint probability density function it takes a value e power minus x, when y is lies between 0 to x.

And x is lies between y to infinity the range of y and range of x is a function of other variable, 0 otherwise; that means, if you visualize the graphical representation of this joint probability density function, x axis y axis, and z axis joint probability density function. So, between the y is lies between 0 to you can make a equal also when y is between the 0 to x; that means, y is equal to 0 and y is equal to x. So, in that when x lies between y to infinity you have e power minus x surface; that means, at the volume below e power minus x between this region in the xy plane that volume is going to be one you know e power minus x how it goes.

So, as x tends to infinity it asymptotically touches 0, therefore, the surface e power minus x goes down over the region of 0 less than or equal to y less than x less than infinity, and the volume below that it is going to be 1, the question is here, find probability of x is less than 2 and y is greater than 1. And the second question find the probability of x is greater than 2 times y, and the third question find the probability of x minus y is greater than or equal to 1. It is a very simple problem, the joint probability density function is given finding out the probability of a different Boral sets in r 2.

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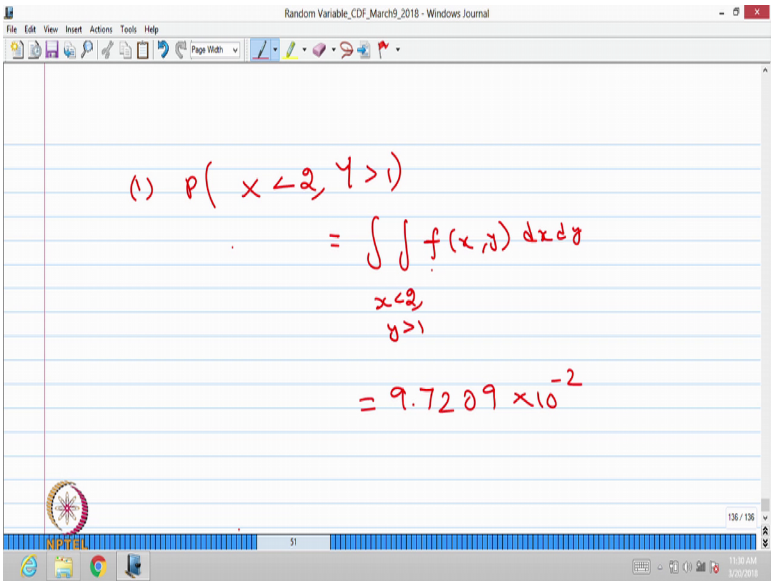

Will go for finding the first one find the probability of x is lesser than 2, y is greater than 1 that is nothing.

But the double integration of joint probability density function dx dy, where the integration is over x is lesser than 2 and y is greater than 1. You can always get this integration and we can get the final answer. So, the final answer is 7.7209 times 10 power minus 2, am not evaluating this integration, the integration is over x is less than 2 and y is greater than 1; will go for the second problem.

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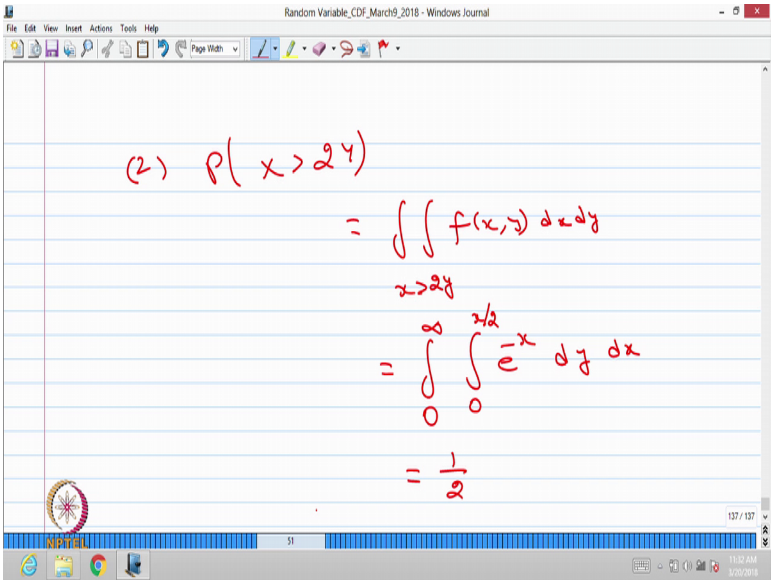

Find the probability of x is greater than 2 times y, this is also in the similar way double integration, the joint probability density function over x is greater than 2 y. That is same as the double integration the function is e power minus x with respect to y that is from 0 to x by 2, because x is greater than 2 y.

So, the range of y is 0 to x by 2 range of x, that is from 0 to infinity; that means, the joint probability density function integration over x is greater than 2 y, where our joint probability density function is greater than 0, when 0 less than or equal to by less than x less than infinity therefore, you will get the integration with respect to y is 0 to x by 2 and integration with respect to x is 0 to infinity. If you do the simplification you can get the answer 1 by 2. So, this is the probability of x is greater than 2 times.

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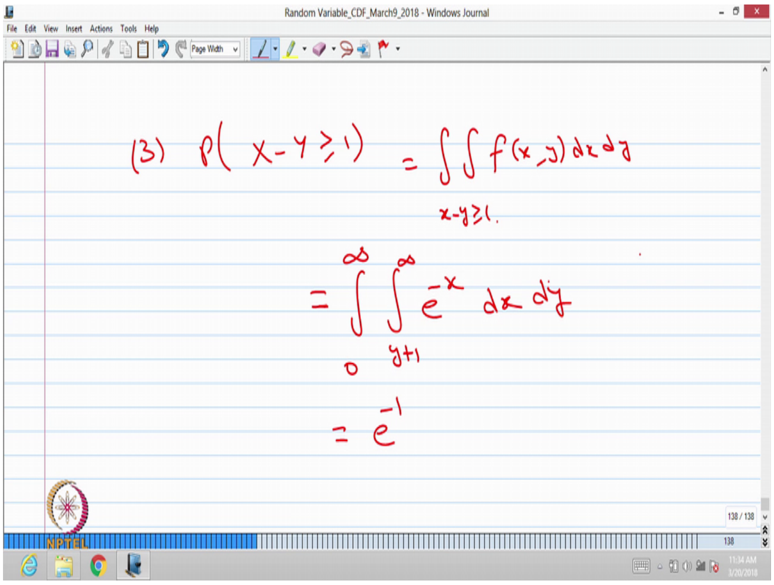

The third problem, the probability of x minus 1 which is greater than or equal to; again the same concept; that is double integration of joint probability density function with respect to x and y, over x minus y has to be greater than or equal to 1. That is same as the double integration the joint probability density function is e power minus x, with respect to x, the x range is from y plus 1 to infinity, and the y range is from 0 to infinity.

The x range is from y plus 1 to infinity, and y range is from 0 to infinity. If you do the simplification you will get e power minus 1. So, this is way one can find the probability of any Boral set by integrating a the joint probability density function over the range.