Introduction to Probability Theory and Stochastic Processes
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Module - 04
Standard Distributions
Lecture - 18

We have already finished 3 models starting with the basics of probability, then the second model is random variable, and a third model is a moments and inequalities. Now, we are moving to the 4th model that is called as standard distributions.

In this model we are going to discuss various standard distributions which includes a standard discrete random variables distributions, and distributions of standard continuous type random variables. Standard means whenever we solve the real world problems some distributions comes very often. So, those distribution we call it is a common distribution or standard distributions or frequently we come across the same distribution again and again so those distribution has some name therefore, we call those distribution as a standard distribution or some common distributions.

So, first let me discuss few standard common discrete type random variables whose distributions and also the moments in particular mean and variance. And similarly we will discuss later some common a continuous type random variables and their distributions, also the mean and variance for those distributions. Then we will discuss some of the problems which has the underlined distributions.

So, in this model we are going to make 3 lectures, that is the title of the model is standard distributions.

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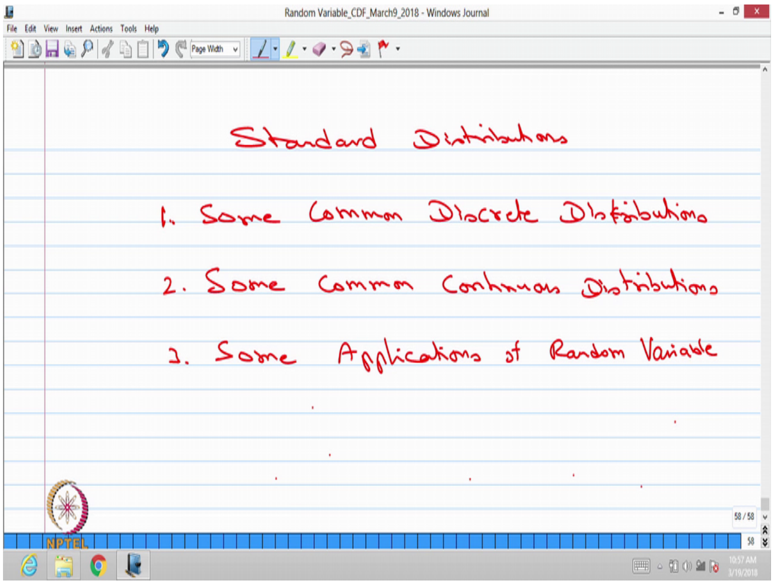

In this first we are going to discuss some common discrete distributions, second we are going to discuss some common continuous distributions, third we are going to discuss some applications of random variable. That means we are going to discuss some problems which are related to these common discrete and continuous distributions.

Let us start with some common discrete distributions.

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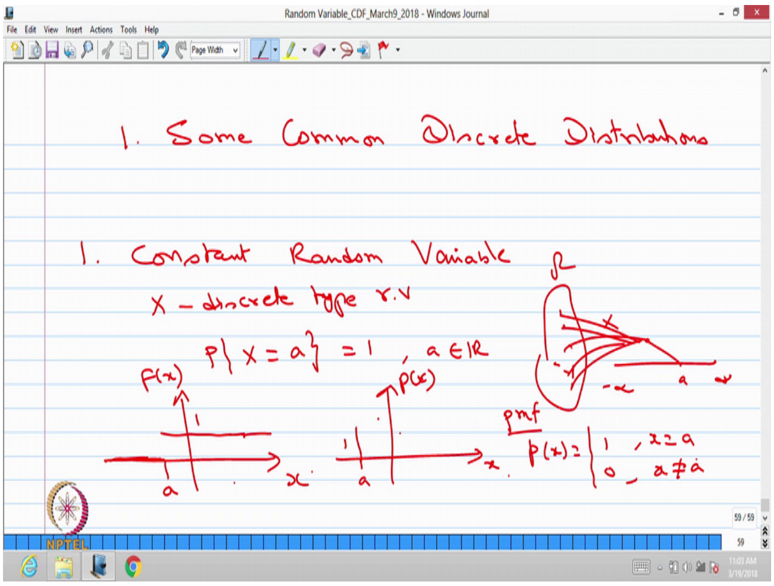

The first one which we are going to discuss that is called constant random variable. A random variable X is said to be a its a discrete type random variable, X is a discrete type random variable. It is said to be a constant random variable whenever the probability mass function of that random variable X takes one single value a with probability 1. a can be any the real, that means, the whole unit mass is accumulated at the point X is equal to e.

That means, in a random experiment you may have a sample space with the finite elements or countably infinite or uncountably many elements the way the real valued function is mapped from omega 2 or it is a many to one function. Therefore, the probability of X is equal to a is nothing but the collection of all possible outcome which gives the value a that is this is omega it has so many elements.

So, the mapping from omega 2 are at the point a for all the points for all possible outcomes the mapping is a, where a is belonging to real line; that means, the probability of X equal to a is nothing but it is a probability of p of omega. From the kolmogorov axiomatic definition we know that the p of omega is 1 therefore, it is going to be a probability of X equal to a is equal to 1; that means, all other points if you go for the inverse image whose mass is going to be 0.

You can draw the cdf of this random variable. Suppose a takes some negative value then the f of the cdf is going to be 0 till a at the point a it becomes 1, and it will be 1 till infinity that means, the cdf has a only one jump and the jump value is 1. It satisfies all the properties of the cdf, therefore you can conclude this is a cdf of the random variable and the if you draw the probability mass function for this discrete type random variable at x is equal to a it has the value one and all other point and the mass is 0.

That means in the whole unit mass is a, accumulated at only one point therefore, it is called a constant random variable that means, a any constant can be represented as a random variable with the probability 1 at that point. It is a very important result the cdf has only one jump and the probability mass function is at only one point with the value one otherwise it is 0. Therefore, the probability p of x can be written in a easy way it is equal to 1 when x is equal to a, it is equal to 0 when x is not equal to a.

So, this is a probability mass function and this is a cdf and this diagram gives the probability mass function in a graphical form.

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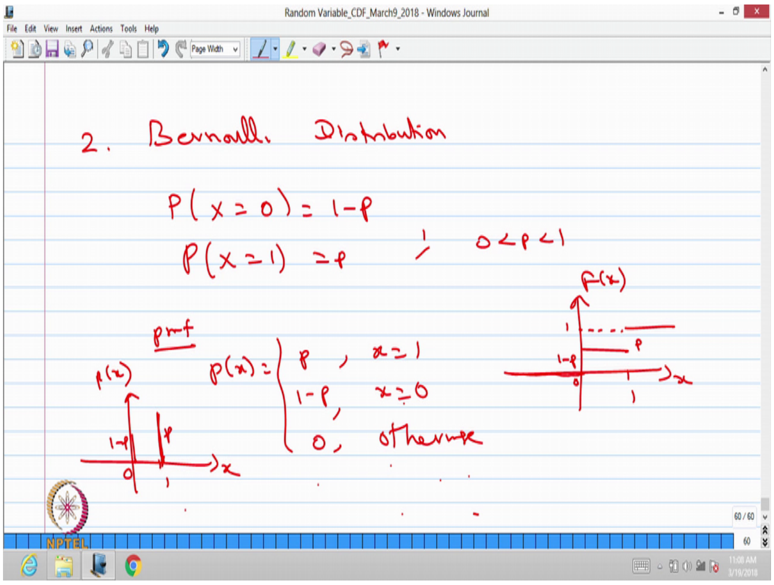

Now, we will go to the second one that is called Bernoulli distribution. A discrete type random variable is said to be Bernoulli distributed random variable when the probability mass function takes a value 0 is 1 minus p and the probability of X takes the value one that is p, where p is lies between 0 to 1 otherwise it is 0. That means, the probability mass function p of X it takes a value p of X is p when X takes a value 1 and 1 minus p when X takes a value 0 otherwise it is 0.

You can draw the cdf this discrete type random variable. X takes a value 0, so till 0, it is 0 at 0 there is a jump of a height 1 minus p. At the point 1 it has a another jump that jump value is p therefore, it I just want. That means, a for this discrete type random variable the cdf has a 2 jumps, one jump is at 0 with the jump value 1 minus p and the next jump at the point 1 with the jump value P. Therefore, this is a discrete type random variable and you can draw the probability mass function also the similar way.

So, at X is equal to 0 the height is 1 minus p height 1 minus p at X is equal to 1 it has a jump p. It depends on p is going to be less than 1 by 2 or greater than 1 by 2 accordingly you will have a 1 minus p with the shorter height and the p is taller comparing to the mass at the point is 0. Whenever, the random experiment has a 2 possibilities we call it as a Bernoulli trial suppose you treat one possibility as a success and other possibility as a failure then we usually denote the success probability with the probability p and the failure probability with 1 minus p then that type of trial is called a Bernoulli trials. A discrete type random variable with the probability mass function of this form we call it as a that random variable is bernouli distributed always the p has to be open interval 0 to 1 if p is equal to 0 or 1, then it becomes a constant.

Therefore it is has to be always the open interval 0 to 1 will which gives a Bernoulli distributed random variable X.

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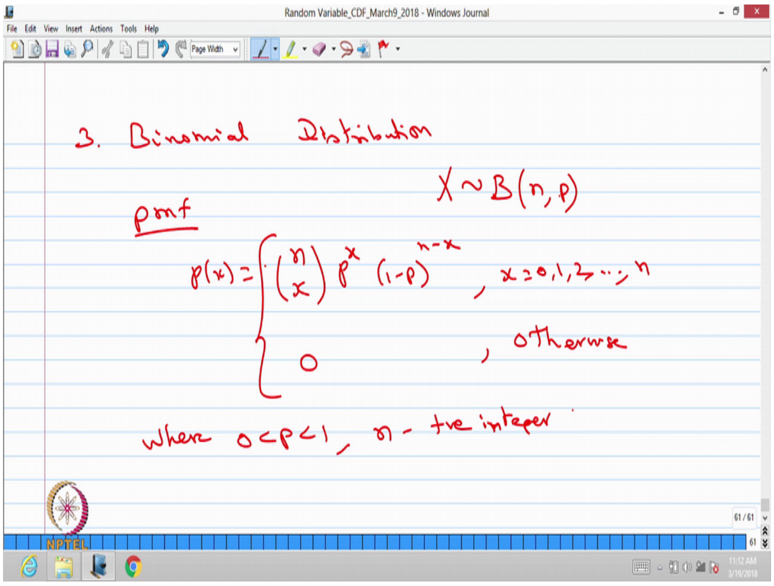

Now, we will go to the third one that is binomial distribution. When we say the given random variable is a binomial distributed, this also discrete type whenever a discrete type random variable whose probability mass function is of the form p of x that is equal to n c x p power x 1 minus p power n minus x, when x takes a value 0 1 2 and so on till n otherwise it is 0. Then we call or we say the random variable which is a discrete type random variable has binomial distributed random variable.

Since this is a probability mass function this is always greater or equal to 0 and if you make a summation over x from 0 to n that is going to be 1. And here the p value is always lies between 0 to 1 and n is the positive integer, the probability mass function of the form n c x, p power x 1 minus p power n minus x.

Once you know the value of p n d n you know the probability distribution of this random variable. Therefore, we call n and B are the parameters of the binomial distribution. So, we usually write the notation x follows this inter means of follows capital B ; that means, for the binomial distribution and here the parameters or n and p therefore, n come of p. So, whenever we write x till the capital B within bracket n come of p that means, the random variable x follows binomial distribution with the parameters n and p. If you supply the value of n which is a positive integer and small p which lies between open interval 0 to 1 then you are known with the probability distribution of this random variable.

The probability mass function has this form. So, you can make out the cdf of binomial distribution has a n plus 1 jump points and the corresponding jump values are n c x p power x 1 minus p power n minus x. So, this is a discrete type random variable. So, you can see the cdf, at x is equal to 0 is a first jump at x is equal to 1 it has the second jump and x is equal to 2 it has another jump and so on, at X is equal n it has the last jump and which gives the value one1.

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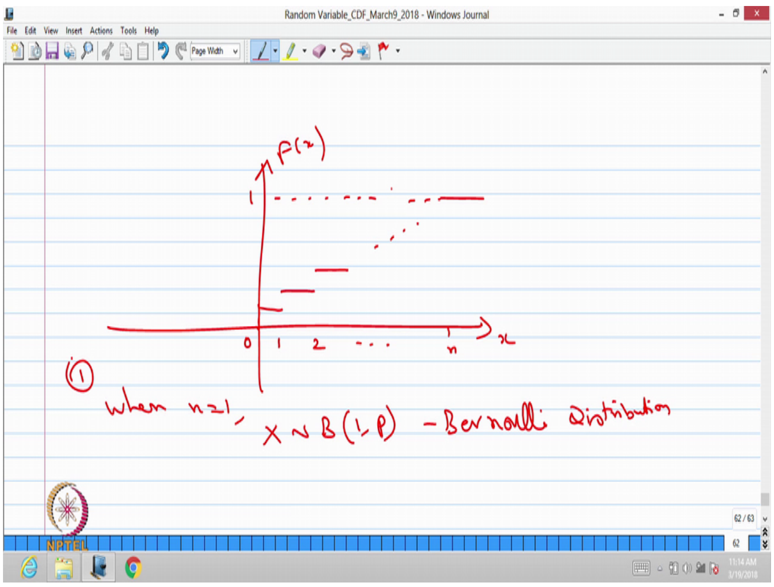

So, this cdf is a right continuous function with the n plus 1 jump points and the jump values are the n c x p power x 1 minus B power n minus x. As a special case when n is equal to 1 then the same random variable is going to be say it as a binomial distribution with the parameter 1 comma p that is nothing but Bernoulli distribution. When n is equal to 1 you will land up a binomial distribution. So, the probability mass function is going to be at x is equal to 0 it is 1 minus B at x is equal to one the probability mass is a p. Therefore, it is a Bernoulli distributed random variable.

One can create a binomial distribution with the help of Bernoulli distribution that is whenever you have a random variable second remark we can treat this as the first remark.

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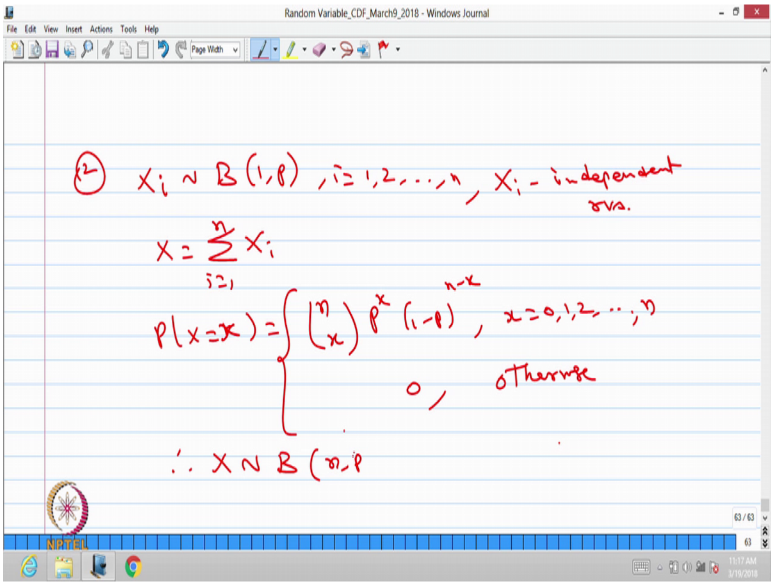

The second remark whenever you have a Bernoulli distributed random variable with the parameter p for n such random variables. If you create a one random variable as a sum of X i's, i is equal to 1 to n; that means, each random variable is Bernoulli; that means, either it takes a value 0 or 1 and if you create a random variable of a sum of a n such Bernoulli distributed random variable. And you can make the one assumption all are independent random variables.

In detailed what is the meaning of those n random variables are independent and what are all the properties going to be satisfied that we will be discuss later, but now you can keep the assumption X i's are independent random variable; that means, it is a they are mutually independent random variable. Then one can conclude the capital X is a summation of X i's, so now, the possible values of X is going to B 0 to n, because each X i at takes a value 0 or 1 therefore, sum of n such Bernoulli distributed random variable the possible values are going to be X. And you can get the probability mass function it is x takes a value small x that is going to be n c x p power x 1 minus p power n minus x when x takes a value 0 or 1 or 2 so on till n.

That means, suppose you say that x takes a value small x; that means, out of n such Bernoulli trials you are getting x with a probability p and the remaining n minus x you got the failure that is 1 minus p power n minus x with the possibilities of n c X i's therefore, the probability of X equal to small x is going to be n c x, p power x 1 minus p power n minus x. So, this is going to be a probability mass function of binomial distribution therefore, one can conclude if you have a n independent Bernoulli distributed random variables with the same parameter p for the Bernoulli distribution then the sum of a n independent Bernoulli distributed random variable becomes binomial distribution with a parameters n and p.

So, you can say X follows a binomial distribution with the parameters n comma p. So, this is a way one can create the binomial distribution with the help of a Bernoulli distribution. As a third remark we can go for finding what is the mgf of a binomial distribution.

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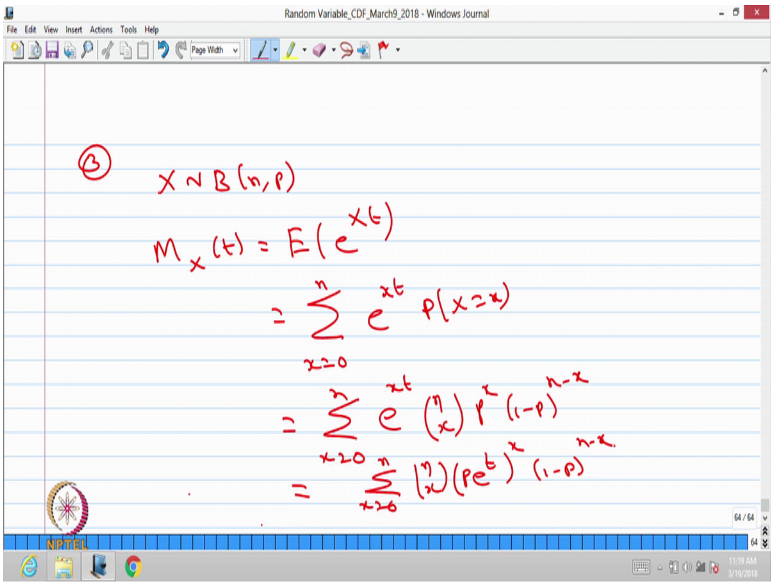

Suppose X follows a binomial distribution the parameter n and p we can find mgf of binomial distribution that is nothing but expectation of e power x times t that is same as since it is a discrete type random variable. So, it is going to be summation e power x small x t and the probability of X takes a value small x and the possible values of x are from 0 to n. So, you substitute all the values that is x equal to 0 to n e power x times t and their probability of x equal to small X is a n c x, p power x 1 minus p power n minus x.

Now, also you do not need to expand and simplify and so on you can keep p power x and e power x t together. Therefore, this is nothing but summation x is equal to 0 to n, n c x, e power t times p whole power x and 1 minus p power n minus x.

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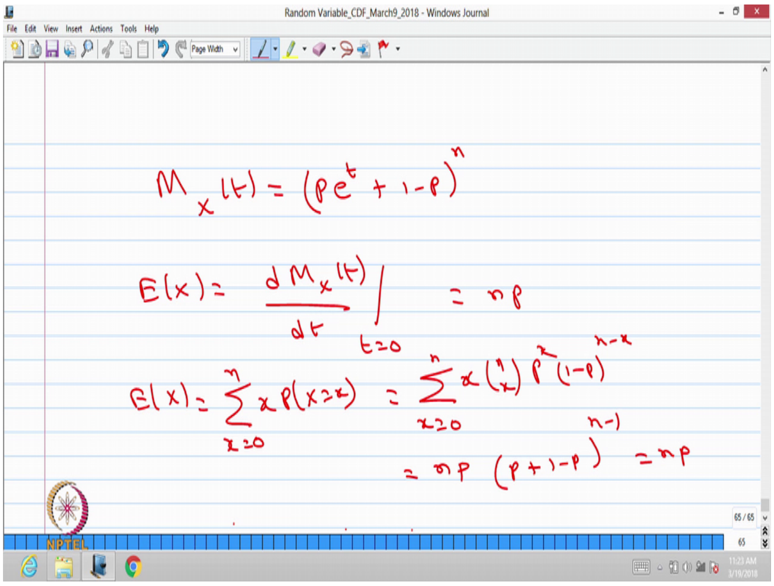

Therefore the mgf of a binomial distribution is p e power t plus 1 minus p the whole power n using a binomial summation you can get a mgf of binomial distribution is p time e power t plus 1 minus p power n.

What is the use of mgf? Once you know the mgf you can get the all the movements by successive derivative. If you want to find out only mean you can use the mean definition and you can get the mean suppose you need many moments then better you can find the mgf first then successive derivative and I have already explained from the moment generating function how to get the moments odia. So, we can use that and find out the first order moment, second order moment, any nth order moment.

So, in particular if you want to find out the mean of this random variable if you want to find out the mean of this random variable, that is nothing but if you use the derivative one derivative of mgf with respect to t then substitute a t equal to 0 will give the mean of the binomial distributed random variable. So, by doing the derivative and so on you can get the value n into p. This is a one way of finding mean of the binomial distribution or you can find out from the scratch that is that is a summation of x times probability of x equal to x where x takes the value 0 to n. So, that is same as a summation x equal to 0 to n x times n c x, p power x 1 minus p power n minus x.

You can cancel x with n c x one term therefore, you will get factorial n divided by n minus x factorial and x minus 1 factorial. So, now, you can keep one n and one p outside. Therefore, the simplified quantity becomes n minus 1 c x minus 1 p power x minus 1 and one minus p power n minus x. So, that is nothing but, that is nothing but p plus one minus p power n minus 1 and you know the that value is 1. Therefore, this is going to be n p. So, there are 2 ways we can find out the expectation either from the scratch by the definition method or by the mgf method.

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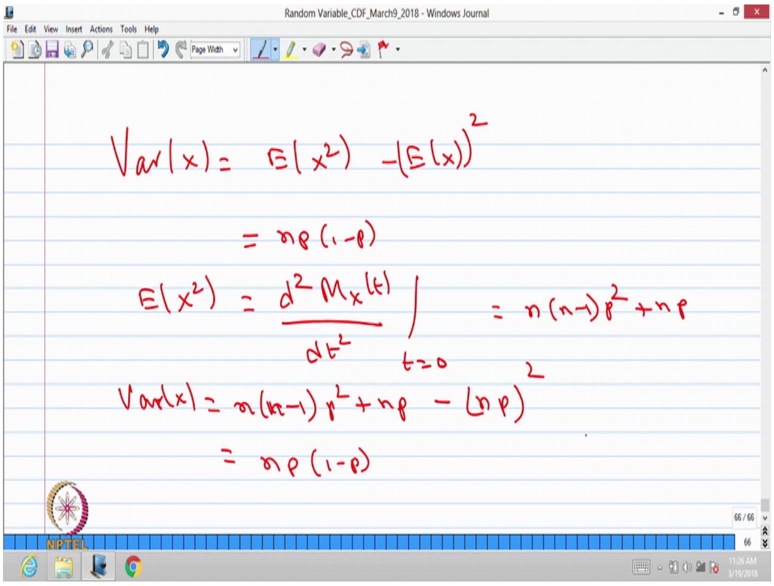

The same way one can compute the variance. So, the variance of X that is going to be either by using the definition for that you have to find out the expectation of X square and already you know the value of expectation of X you can substitute and get the value or you can do the second derivative of the mgf then through that you can get the variance. So, if you do the little simplification you can get the variance of X is going to be n into p into 1 minus p.

For that you can do the other method that is a you can find expectation of X square that is nothing but second derivative of mgf of X with respect to t twice, then substitute a t equal to 0 that some simplification will give the value that is n into n minus 1 p square plus n p.

So, once you know the expectation of X square and the substitute in the variance formula that is a n into n minus 1 p square plus n into p minus n p the whole square. So, you simplify that you will get the value that is n into p into 1 minus p. So, this is a way one can get the variance of X for the binomial distributed random variable.

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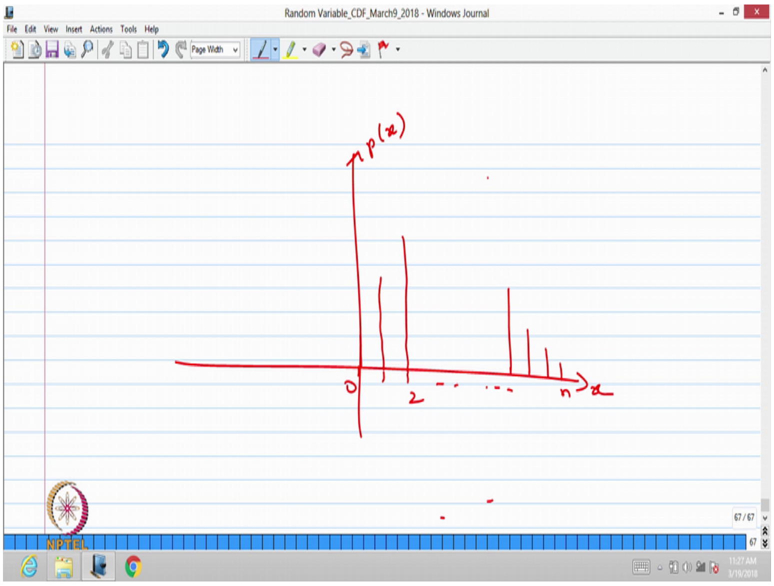

There is another observation if you draw the probability mass function of a binomial distribution. For n is equal to X is equal to 0 you will have some height, for X is equal to 1 you will have another height, for X is equal to 2 you will have a more height and so on then it will be keep decreasing.

So, this is based on the value of p you will have a increasing for the different values of x, then decrease. For p is equal to less than of you will have a more increase in the first half then it goes decrease. For p is greater than 1 by 2 you will have less heights in the beginning and more heights in the second half. If p is equal to 1 by 2 then you will have a very symmetric of increasing and decreasing over 0 to n, and for n is equal to even you will have a 2 heights for n is equal to odd you will have only 1 height. That is the way the values are going to be keep increasing then it will decrease.

I have just to drawn one diagram for probability mass function for some n and some p. The importance of the cdf graph is from the finite number of data and if you draw the cumulative distribution graph, if that graph is same as the cdf of binomial distribution then one can conclude the data follows a binomial distribution. Not only from the cdf one can do the observation from the probability mass function also. That means, if you draw the histogram of the values of the data over 0 to n, and the probability the histogram look like the probability mass function at some point if it is almost same then you can conclude the data also follows binomial distribution. Therefore, one should always know the cdf and the probability mass function for a discrete type random variable.

Similarly for a continuous type random variable one should know the cdf as well as the probability density function of a continuous type random variable. So, both the things are very useful when you have a data in the first then you can identify what could be the distribution.