# Introduction to Probability Theory and Stochastic Processes

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## Lecture - 17

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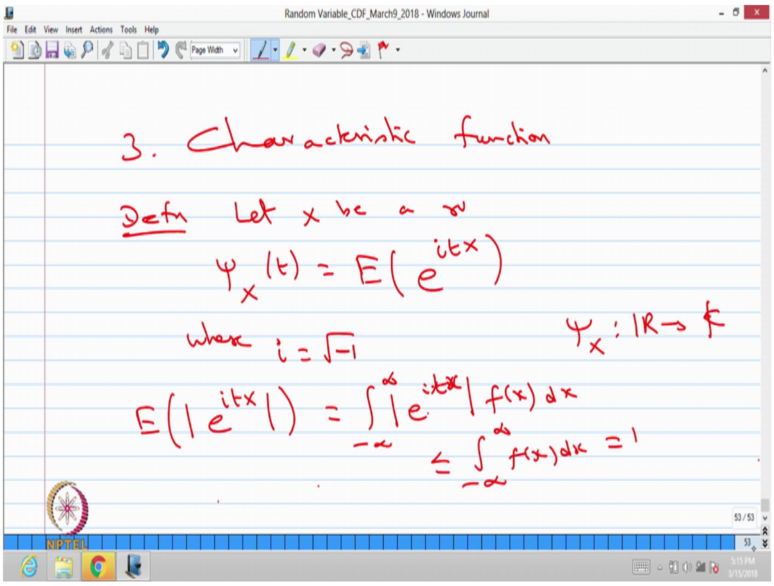


We will go for third generating function that is Characteristic Function. This is also generating function. The definition for characteristic function is as follows. Let X be a random variable. The characteristics function of the random variable X as a function of t that is psi is a notation is defined as expectation of e power i times t X, where i is square root of minus 1 is a complex one.

You compare the definition of probability generating function, moment generating function and characteristic function. Probability generating function can be defined only for the random variable which is non negative integer valued random variable. The moment generating function is valid for the random variable in which expectation of e power t X is a finite for all t in some interval including 0. Whereas, the characteristic function there is no restriction for the random variable that means, for any random variable or for all random variable one can define the characteristic function, that is expectation of e power i times t X.

Even though we use the expectation of function of random variable we are not making the provided condition expectation exist that means, the right hand side quantity is always exist. That can be proved, because once you say the expectation of a function of random variable exist provided in absolute sense which should be finite we can verify expectation of absolute of a e power i times t X.

So, here the provided condition is not necessary because the expectation of absolute of e power i times t X. This quantity is same as minus infinity to infinity absolute of e power i times t X times f of X d X. And we know that i is square root of minus 1 and absolute of e power i times t X is always less than or equal to 1 therefore, this is less than or equal to minus infinity to infinity f of x dx. Here when I go for integration from minus infinity to infinity I assume that it is a continuous type random variable therefore, I am going for integration with the probability density function as a multiplication. Suppose if it is a discrete type random variable then it is a summation absolute probability mass function.

So, here I have considered X is a continuous type random variable. And this quantity is going to be 1. So, always expectation of a absolute of e power i times t X is going to be less than or equal to 1, that is a finite quantity therefore there is no need of provided condition for characteristic function.

One more observation the earlier two functions or the real valued function whereas, this is a complex valued function. That is a psi characteristic function it is a complex valued function.

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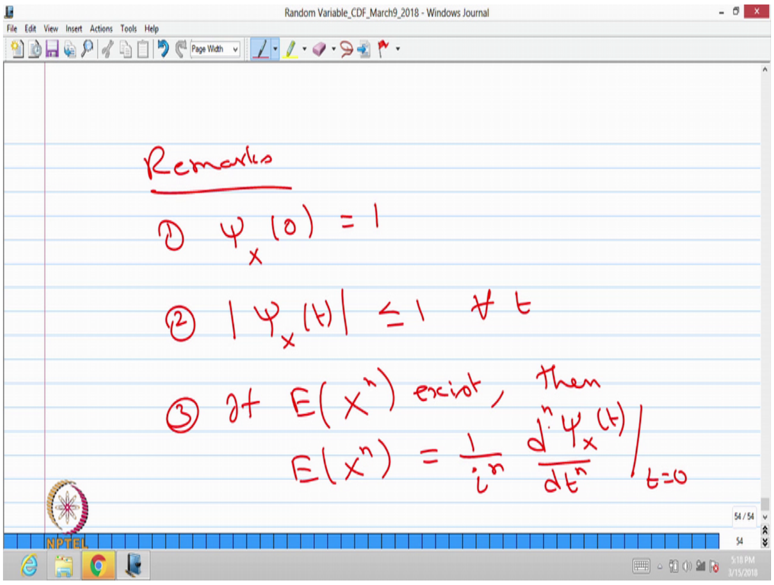


Now, we will go for few remarks over characteristic function. The first remark if you substitute t equal to 0 in the characteristic function that is same as expectation of 1 that is going to be 1.

The second remark if you find out since it is a complex valued function you can go for absolute of characteristic function, that is for all t if you take a absolute of this that is same as absolute of expectation of i times t X that is less than or equal to expectation of absolute of e power i times t X. Just now we got that result is 1 therefore, that is going to be, this is for all t the absolute of characteristic function is always less than or equal to 1 between for all t.

The third remark if expectation of X power n exist because the characteristic function that does not say the moment generating function exist or not. If the nth order moment exist then one can find the nth order moment about the origin by successive derivative of characteristic function with respect to t n times. Then substitute t equal to 0 then multiplied by i power n if the moment of nth order about the origin exist from the characteristic function by successive derivative n times substituting t equal to 0 multiplied by 1 divided by i power n one can get the nth order moment about the mean.

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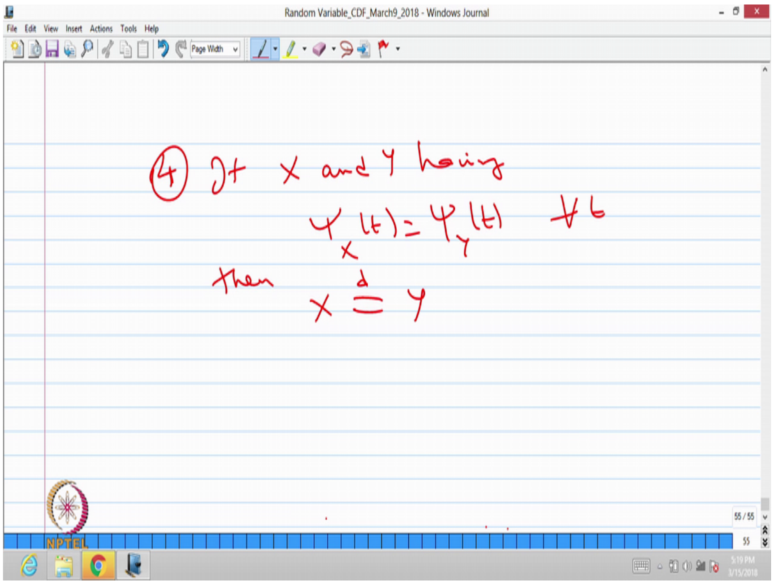


The next remark, if two random variables if two random variables having characteristic functions same for all t then you can conclude both are identically distributed is a notation, both are random variables X is equal to Y having the same distribution therefore, I write a d above the equal symbol. If two random variables having the same characteristic function for all t then we can conclude both are having the same distribution, so all this 3 generating functions this result valid if two random variables having the same thing then both are going to have the same distribution.

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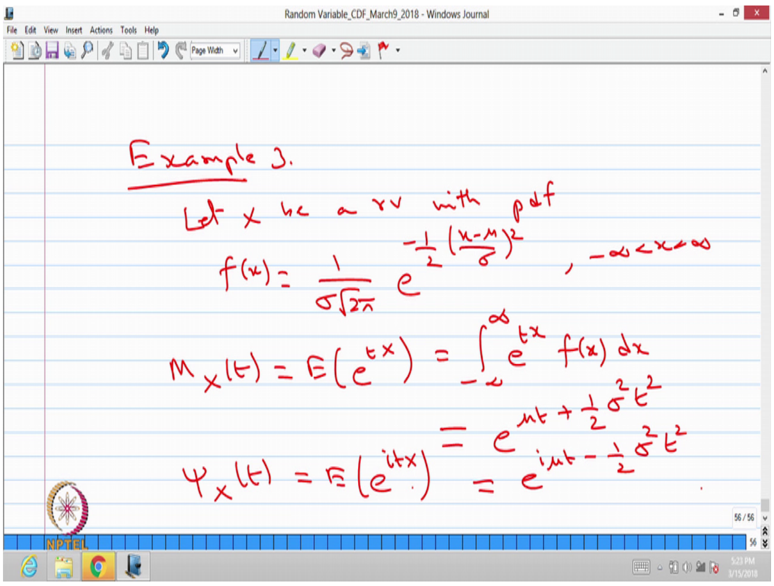

As a example, as a example already we discussed two examples, this is third example.Let X be a random variable with probability density function f of x 1 divided by sigma square root of 2 pi e power minus 1 by 2 X minus mu by sigma the whole square, where X lies between minus infinity to infinity mu is lies between minus infinity to infinity that is a parameter and the sigma is greater than 0. So, this is a probability density function of continuous type random variable, later we are going to say this is a normal distribution with the mean mu and the variance sigma square. You can find the mgf of the random variable you cannot find the probability generating function because it is a continuous type random variable. So, this is nothing but the expectation of e power t times X that is same as minus infinity to infinity e power t X you substitute the above probability density function

So, in this example we are finding both moment generating function as well as characteristic function. If you do the simplification it is going to be e power mu t plus 1 by 2 sigma square t square. I am skipping all the calculation, substitute the f of x then do the integration after simplification you can get the answer it is e power mu t plus 1 by 2 sigma square t square. This is a moment generating function of the random variable X whose probability density function is given. This, you can find the characteristic function that is expectation of e power i times t X. You see that the difference between moment generating function and the characteristic function is replacing t by i times t, where i is square root of minus one therefore, the characteristic function for the normal distribution is e power i times mu t minus 1 by 2 sigma square t square by replacing t by i times t.

So, this result of characteristic function of a normal distribution is going to be used later therefore, I am introducing finding the mgf as well as characteristic function for a normal distribution with the parameters mu and sigma square, where mu is the mean and sigma square is the variance.

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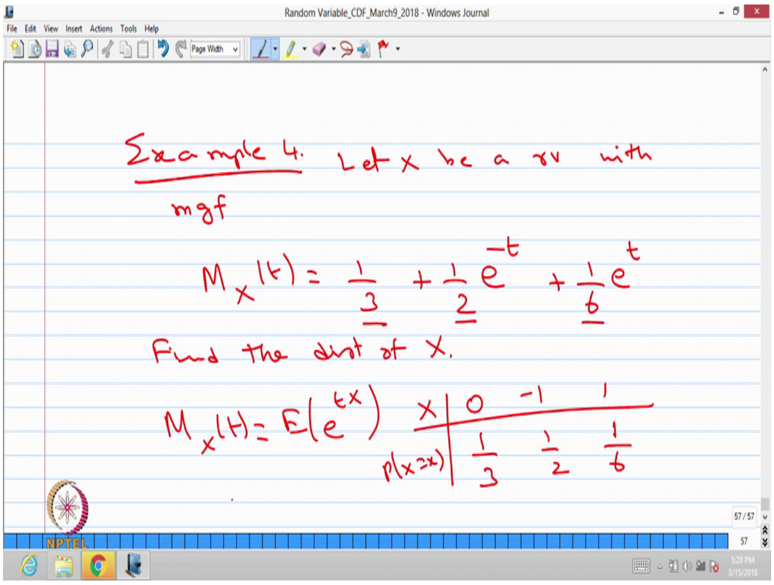


Let us go for one more example. Let X be a random variable with mgf is given by 1 by 3 plus 1 by 2 e power minus t plus 1 by 6 e power t. Find the distribution of X this a reverse problem or inverse problem by giving the mgf we have to find the distribution. Till now we know the distribution we can find the mgf provided it exist. So, here the mgf exist and the mgf is given by 1 by 3 plus 1 by 2 e power minus t plus 1 by 6 e power t. Find the distribution of X, it could be discrete type it could be continuous type or it could be mixed type.

There is a one important result if the mgf exist it is unique, and the characteristic function is always exist and it is unique if the probability generating function exist then it is unique for the random variable. With that concept if the mgf exist it is a unique and it is going to give a unique distribution of the random variable X we know that the mgf is nothing but for any random variable that is expectation of e power t times X if it is a discrete type then it is a summation of e power t X probability of X equal to small X or it is integration from minus infinity to infinity e power t X times probability density function.

By seeing the definition and by looking at the mgf you can conclude 1 by 3, 1 by 2, 1 by 6 if you had that is going to be 1 all are with the positive symbol and multiplied by 1 term nothing and another term e power minus t and the other term is e power t. And you see that it is a expectation of e power t X, from that we can conclude the random variable

X takes a value 0 minus 1 and 1 with the probability of X takes a value that is going to be 0 is 1 by 3 1 is 1 by 2 minus 1 is 1 by 2 and 1 is 1 by 6.

You can verify if this is a probability distribution then the mgf is going to be 1 by 3 times e power 0 times t plus 1 by 2 times e power t rimes minus 1 plus 1 by 6 times e power t times 1 that is same as a 1 by 3 plus 1 by 2 e power minus t plus 1 by 6 e power t therefore, the distribution of X is this table with the probability mass at the point is 0 is 1 by 3 minus 1 is 1 by 2 and 1 is 1 by 6. So, this is a discrete type random variable that gives the mgf 1 by 3 plus 1 by 2 e power minus t plus 1 by 6 e power t.

With this, 4 examples we have completed generating functions namely probability generating function moment generating function and characteristic function. So, with this we are completing the module on moments and inequality starting with mean and variance then higher order moments and moment inequalities, and finally generating functions.