Introduction to Probability Theory and Stochastic Processes
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Lecture - 16

So, we have already discussed Mean and Variance in the first lecture, Higher Order Moments and Moments Inequalities in the second lecture, now we are moving into the third lecture on Generating Functions.

In this lecture we are going to discuss 3 important generating functions namely probability generating function, moment generating function and characteristic function. We will give the definition, some properties of this, and then one or two examples. And later we will find out the generating functions for some standard distributions in detail. So, as such now we will give the definition and the properties and one or two easy examples. So, let start with third lecture that is generating functions.

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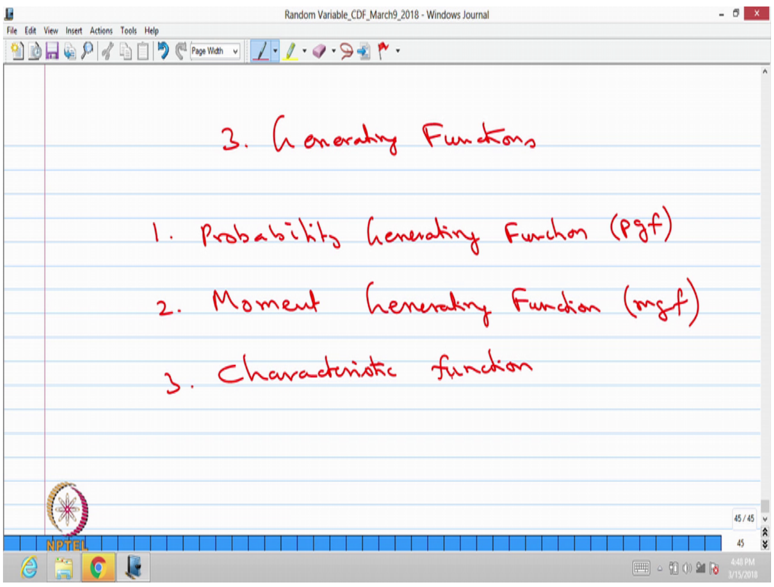

In this we are going to discuss probability generating function in short it is pgf. The second we are going to discuss moment generating function that is mgf. The third we are going to discuss characteristic function this is also generating function.

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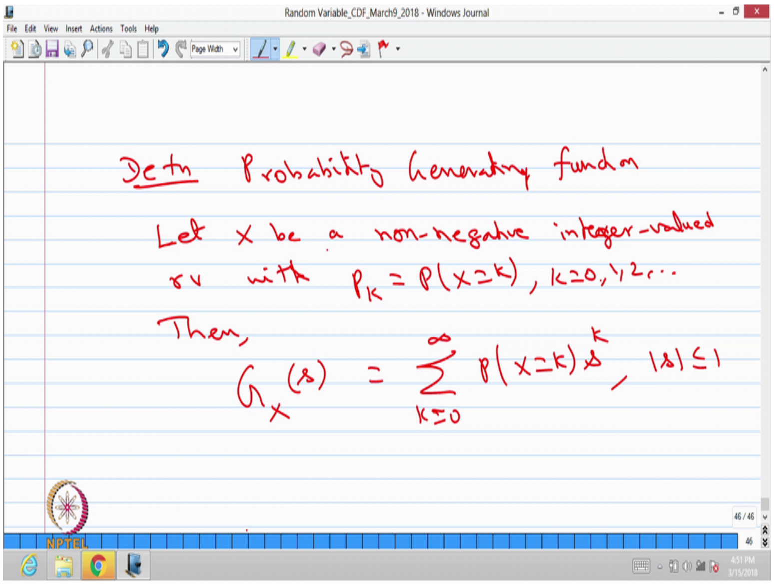

So, let us start with a first one probability generating function, the definition probability generating function. Let X be a non negative integer valued random variable that is basically a discrete type random variable in particular it is a non negative integer valued random with the probability of X takes the value k we denote it as a P suffix k, where k takes a value 0 1 2 and so. There is a possibility it may take a finite number of values also. Then one can define the probability generating function as G suffix X as a function of s that is nothing but summation probability of X takes a value k, s power k, where k takes a value from 0 to infinity, where s is the real variable which lies between minus 1 to 1.

So, this is a real valued function it is a function of s in the form of series with the probability mass at the point k multiplied by s power k. So, the right hand side is a series, the left hand side we are denoted by the G suffix X as a function of s. Since, a right hand side is a series this series converges within the interval mod s less than or equal to 1. We are not bothering about whether this series is converges or not outside this interval, what we are saying is within the interval s lies between minus 1 to 1, the right hand side series converges gives the function of s that is called the probability generating function of the random variable X, where X is a non negative integer valued random variable. That means, whenever X is a non negative integer valued random variable one can define the probability generating function as a function of s.

We will go for few remarks then we will give a some examples.

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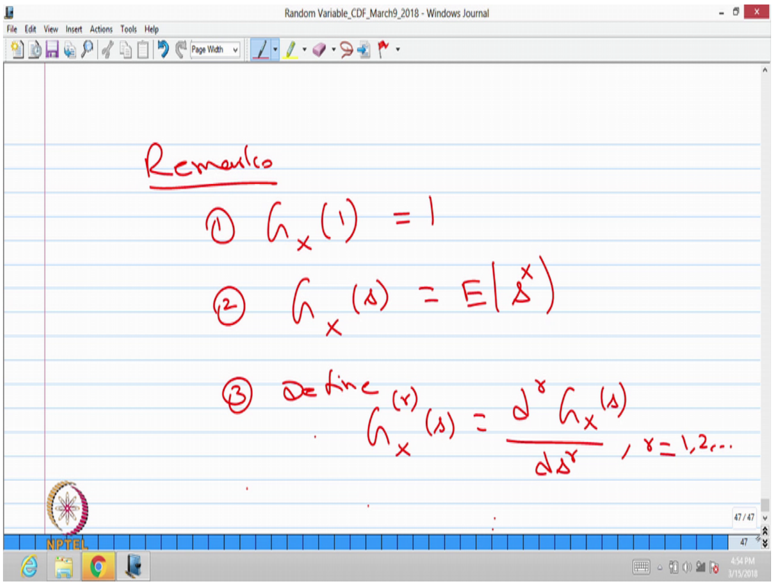

As a remarks the first remark since I said that the right hand series is converges between minus 1 to 1, if you substitute the value s is equal to one you will get the summation of probability of X equal to k that is nothing but 1 because that is a probability mass function. So, this value is going to be 1.

The second remark you can relate probability generating function with expectation of function of a random variable that is the probability generating function of X is same as the way it is summation of probabilities multiplied by s power k, one can write it is nothing, but expectation of s power random variable X.

Third remark one can relate the probabilities with the derivative of generating function. Suppose I define suppose I define the generating function with the bracket r of the function of s is nothing but rth derivative of generating function probability, generating function, with respect to s suppose I define G suffix X bracket r is the rth successive derivative of probability generating function with respect to s where r can take the value 1 2 and so on.

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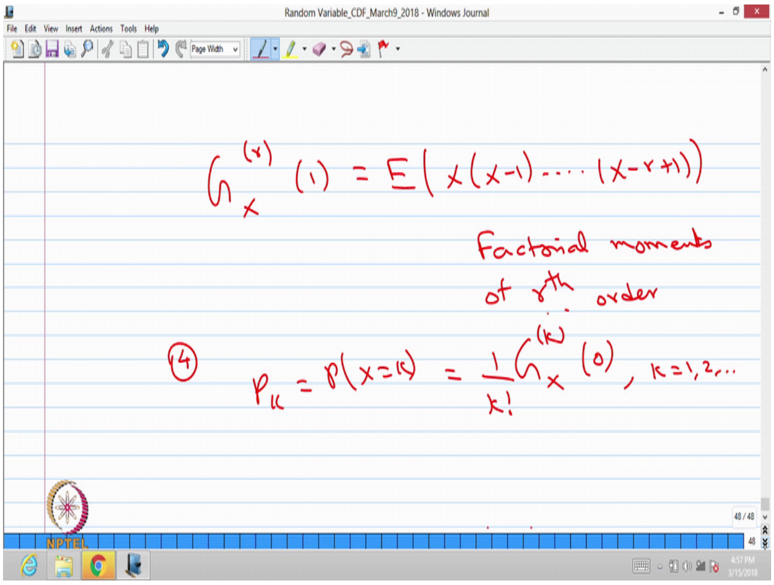

I can write down rth derivative evaluated at the point 1 that is nothing but expectation of, expectation of X into X minus 1 multiplied by X minus 2 and so on till X minus r plus 1. The way I defined G suffix X bracket r at 1 that is same as expectation of X into X minus 1 multiplied till X minus r plus 1. The right hand side is called factorial moments of rth order. Sometimes if you want to find out the variance you can find the factorial moment of a second order through that you can find the second order moment or variance.

Similarly, I can find from the probability generating function by taking the derivative I can get the probability mass at the point k, that is P suffix k that is nothing but the probability of X takes a value k that is same as the kth derivative of probability generating function evaluated at the point 0 multiplied by 1 divided by k factorial that is going to be the probability of X takes a value k, here k can take the value 1 2 and so. After you get the kth successive derivative of probability generating function substituting value s is equal to 0 multiplying by 1 divided by k factorial will give probability of X takes a value k. So, this result is by knowing the probability generating function you can get the probabilities whereas, the definition says if you know the probabilities you can get the probability generating function for a non negative integer valued random variable.

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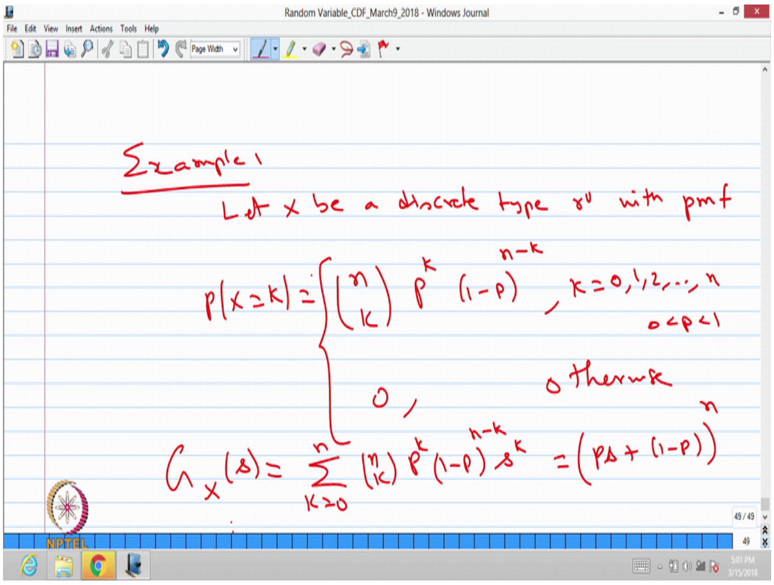

Let us go for finding the probability generating function for the random variable which is of a non negative integer valued. Example 1, let X be a discrete type random variable with probability mass function P of X is equal to k that is n c k P power k, 1 minus P power n minus k, where k takes a value 0 1 2 and so on till n, where n is a positive integer, otherwise the probability mass function is going to be 0. So, this satisfies the properties of to define the probability generating function. So, one can go for finding the probability generating function for this random variable.

So, G suffix X has a function of s is nothing but by definition probability of X equal to k s power k, summation k is equal to 0 to n, n c k P power k 1 minus P power n minus k s power k. So, here the P values is lies between 0 to 1 that I did not specified earlier. And if you see the summation this is always converges summation k is equal to 0 to n, n c k P power k 1 minus P power n minus k into s power k. So, you do not need to expand and then do the simplification. P power k and s power k you can keep it together, then it becomes a P s power k 1 minus P power n minus k and this is nothing but the binomial summation therefore, one can easily write P s plus 1 minus P the whole power n. By combining a P power k and s power k you can easily get the result P s plus 1 minus P the whole power n that is going to be the probability generating function for this random variable.

Later we are going to introduce this random variable as a binomial distribution with the parameters n and P or with the parameters P and n. P is lies between 0 to 1 open interval, n is a positive integer then the probability generating function is P s plus 1 minus P whole power n.

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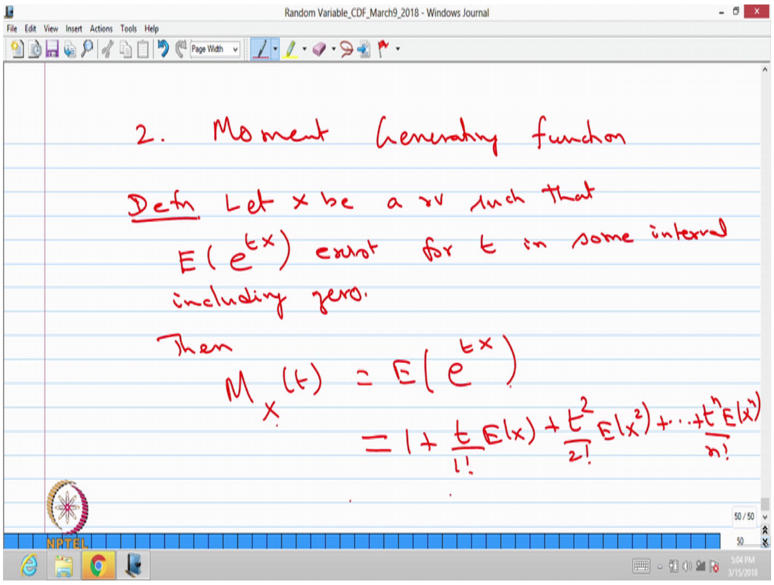

Now, we will move into the second generating function that is moment generating function. In short it is mgf. The definition is as follows. Let X be a random variable such that the expectation of e power t X exist for t in some interval including 0.

So, as long as a expectation of e power t X exist for t in some interval which include 0 in that case one can define the mgf or moment generating function of the random variable X. As, is a notation m suffix X of t that is nothing, but expectation of e power t times X, since I have written expectation of e power t X, e power t X can be expanded therefore, you will get that is same as, that is same as 1 plus t divided by factorial 1 expectation of X plus t square by 2 factorial expectation of X square and so on plus t power n by n factorial expectation of X power n. Since we said expectation of e power t X exist.

Therefore, all the moment of order n exist about the origin and this series also converges then only one can find the mgf of the random variable X. It is a function of a t if a fewer moments exist and other moments does not exist one cannot define the mgf of the random variable.

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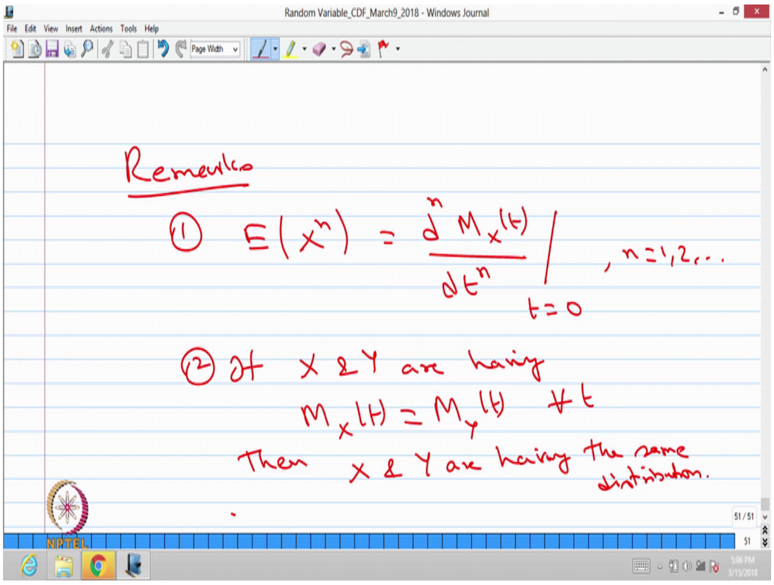

As a remark, the first remark one can get the nth order moment from the mgf by successive derivative of moment generating function n times with respect to t then substituting t equal to 0. One can get the nth order moment about the origin from the moment generating function by successive derivative n times with respect to t then substitute t equal to 0 that is same as expectation of X power n.

The second if two random variables or having mgf's are same for all t if two different random variables whose mgf's are same for all t then one can conclude both the random variables or having the same distribution. This result is valid for the probability generating function also I have not mentioned. If two random variables of probability generating functions are same for all s then both the random variables are having the same distribution.

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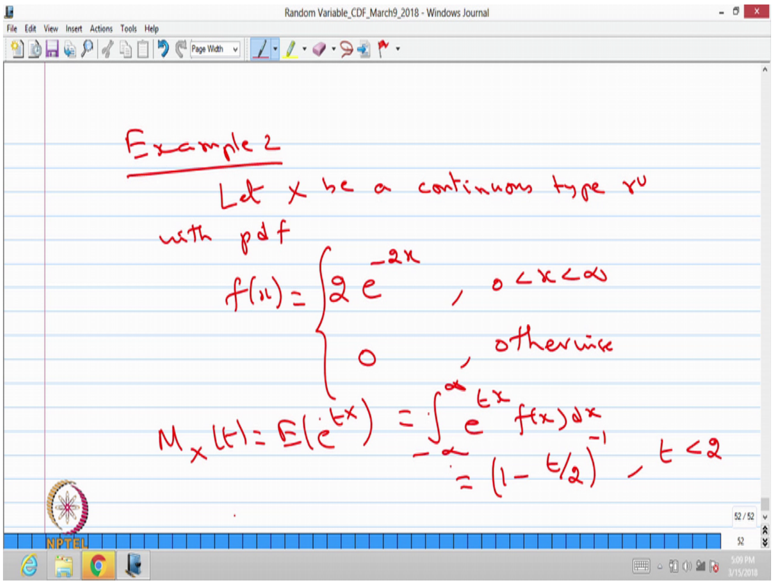

Let us go for the example finding the mgf. Let X be a continuous type random variable with probability density function f of X is 2 types, e power minus 2 X when X is lies between 0 to infinity 0 otherwise the same example which we have taken earlier for finding the mean and variance. Now, we are finding the mgf of a same random variable that is nothing but mgf as a function of t that is expectation of e power t X, that is same as minus infinity to infinity e power t X, f of x dx. Here we are doing with the assumption that the mgf expectation of e power t X exists then we are finding the mgf.

Substitute f of x is a 2 times e power minus 2 X between the interval 0 to infinity n. After simplification you can get the answer that is 1 minus t divided by 2 power minus 1, whenever this result is valid when t is less than 2; that means, the e power t X is a finite quantity for t less than 2 then only we can define the mgf and that mgf quantity is going to be 1 divided by 1 minus t divided by 2.

So, this is a very important not for all t between minus infinity to infinity the mgf exist. For this random variable the mgf exist between minus infinity to 2 and the mgf value is 1 divided by 1 minus t by 2.