Introduction to Probability Theory and Stochastic Processes

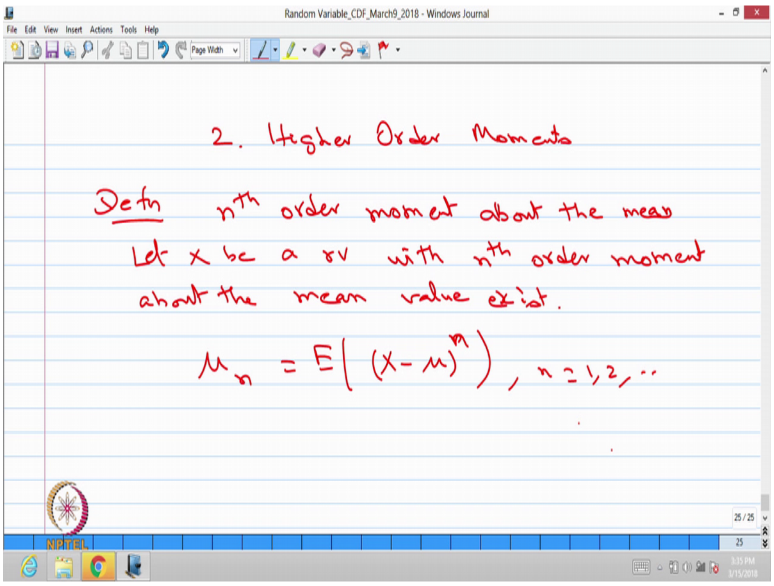
Prof. S. Dharmaraja
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture – 14

In this week we started Moments and Inequalities. Already we discussed in the last lecture mean and variance, mean is nothing but the first order moment and variance is nothing but the second order moment of the random variable. In the last class we have discussed the first order moment and the second order moment with the examples. In this lecture we are going to discuss higher order moments.

Since we have already discussed first and second order moment now we are going to discuss any nth order moment for the random variable if it exist followed by we are going to discuss the moments inequalities. So, let me start with definition of higher order moments, higher order moments.

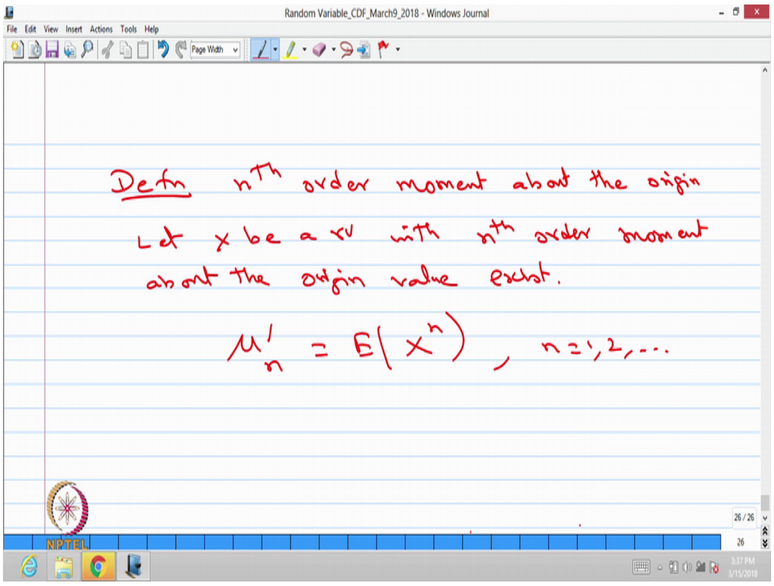
(Refer Slide Time: 00:58)



The definition that is a nth order moment about the mean. Let X be a random variable with nth order moment about the mean value exist. Then one can define with the notation mu suffix n that is nothing but expectation of X minus, the expectation of X is denoted by mu that is mean X minus mu power n that is going to be the nth order moment about the mean. Whenever it exist it can denoted by mu suffix n, whenever it exist that is the right hand side expectation exist then you can denote by the letter mu suffix n that is expectation of X minus mu power n, where n can takes the value it could be 1 2 and so on.

Obviously, if you take the value n is equal to 1 that is nothing, but the mu suffix 1 is expectation of X minus mu that is same as expectation of X minus mu that is going to be 0, when n is equal to 2, then it is nothing but the variance of the random variable X. So, provided the right hand side expectation exist then one can define the nth order moment about the mean with the dot notation mu suffix n.

(Refer Slide Time: 04:08)

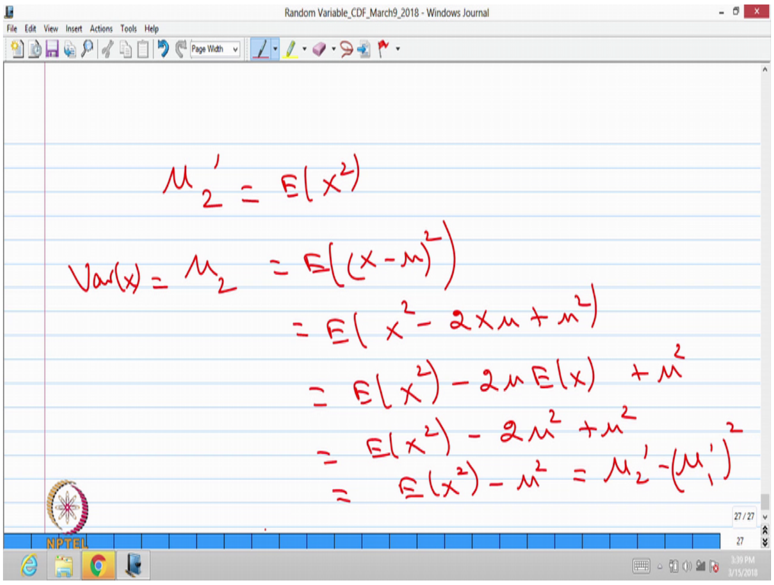


The same way I can define the nth order moment about the origin nth order moment about the origin or some books they use a word 0, both are one of the same.

Let X be a random variable with nth order moment about the origin value exist. Then one can define with the notation mu suffix n dash that is nothing, but expectation of X power n here again n can take the value 1, 2 and so on. So, when n is equal to 1 this is nothing but the mean or expectation of the random variable and two onwards it is going to be called as a nth order moment about the origin, provided the expectation exist that is very important.

One can relate the second order moment about the origin with the second order moment about the mean.

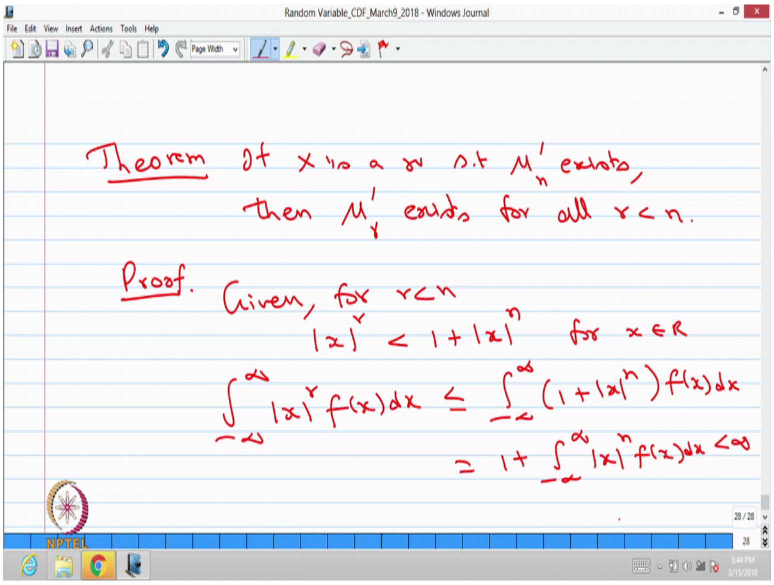
(Refer Slide Time: 05:55)



For example, mu suffix 2 dash that is expectation of X square and mu suffix 2 that is nothing but the expectation of X minus mu the wholes square. This is same as variance of X. So, the expectation of X minus mu whole square if you expand that is expectation of X square minus 2 X mu plus mu square, expectation is a linear operator so it is expectation of X square minus 2; and mu are so it is expectation of X. And mu is a constant, so mu square constant expectation of mu square that is mu square

So, when you simplify you will get expectation of X square minus this is 2 mu into mu therefore, 2 mu square plus mu square. So, that is same as expectation of X square minus mu square that is same as mu 2 dash minus mu 1 dash; that means, mu 2 is nothing but mu 2 dash minus mu 1 dash the whole square so that means, one can write central moment about the mean in terms of central moment about the origin.

(Refer Slide Time: 08:01)



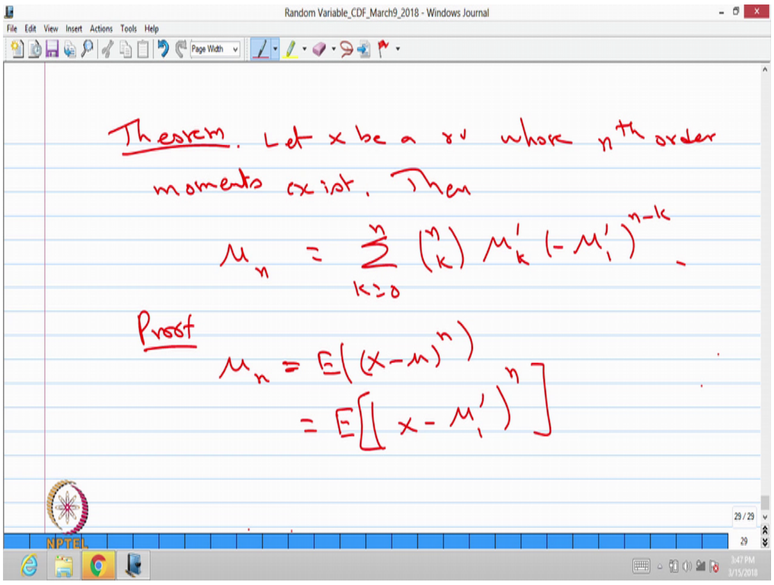
Next I am going to give the one important result as a theorem. What the theorem says if X is a random variable such that mu suffix n dash exist then mu suffix or dash exists for all r less than n that is a theorem.

Whenever for a random variable if the nth order moment about origin exist, then all the rth order moment about the origin exist for all r less than n. You can give the proof of this theorem, given mod of x power or which is lesser than 1 plus mod x power n, this is for all x belonging to real.

We can conclude suppose you consider x as a continuous type random variable, minus infinity to infinity mod x power r, f of x dx, where f of x is the probability density function of a continuous type random variable that is less than or equal to minus infinity to infinity 1 plus mod X power n times f of x dx that is same as 1 plus minus infinity to infinity mod of x power n f of x dx. And since nth order moment about the origin exist therefore, this quantity is going to be find it the integration quantity therefore, this whole quantity is going to be finite. This implies minus infinity to infinity absolute of x power x, f of x dx is a finite that is for all r which is less than n.

So, this is given you can include one more statement for r less than n, given for r less than n mod x power r which is lesser than 1 plus mod x power n this is true therefore, both side you can do the integration by multiplying f of x and given that its nth order moment about the origin exist therefore, for all r less than n the moment of r exist also.

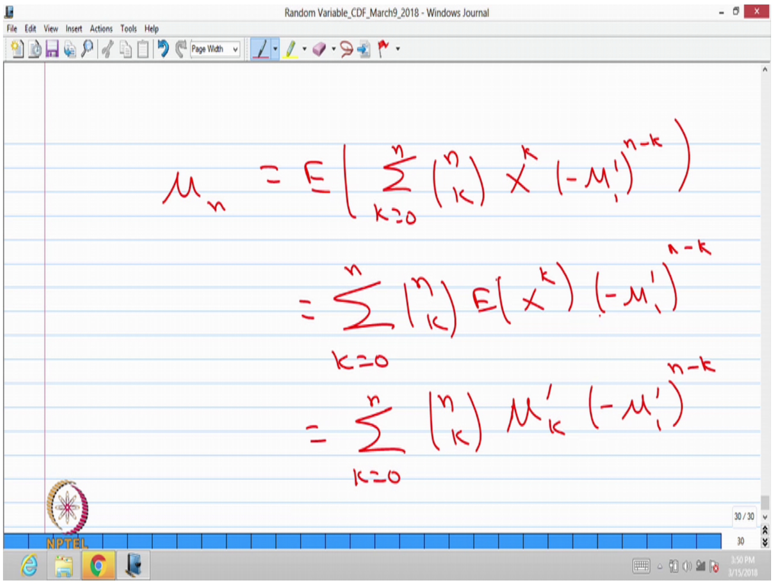
(Refer Slide Time: 11:47)



The next result as a theorem let X be a random variable whose rth order moments exist. Then one can write mu suffix n is same as summation over k is equal to 0 to n, n c k mu suffix k dash with the minus mu 1 dash power n minus k. This can be proved whenever nth order moment exist, then one can write the nth order moment about the mean is same as a function of all the previous order moments about the origin.

The proof is as follows you start with mu suffix n that is nothing but nth order moment about the mean that is X minus mu power n, that is same as the expectation of X minus 1 can write mu as a mu 1 dash that power n. Now, you can go for the binomial expansion of X minus mu 1 dash power n, that is same as expectation of summation k is equal to 0 to n, n c k X power k and minus mu 1 dash power n minus k that is same as the n c k is a constant that is not a random minus mu 1 dash that is also not random therefore, expectation can be taken inside that is summation k is equal to 0 to n, n c k expectation of X power k minus mu 1 dash power n minus k.

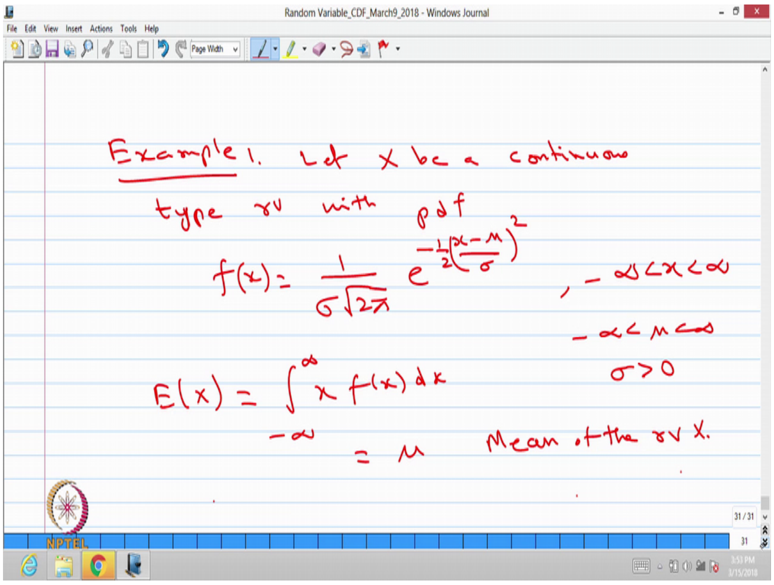
(Refer Slide Time: 14:15)



I can rewrite expectation X power k as the kth order moment about origin therefore, this is going to be summation k is equal to 0 to n, n c k mu suffix k dash multiplied by minus mu suffix 1 dash power n minus k. Because of the previous theorem when the nth order moment about mean exist that means, all the previous order also exist therefore, this is a valid statement. With the help of previous moments about the mean you can always find the moment of nth order about the origin.

In conclusion with the previous starting from first to nth order moment about the origin one can get nth order moment about the mean, one can go for one easy example of how to find the nth order moment for some random variable which is of the continuous type.

(Refer Slide Time: 16:31)

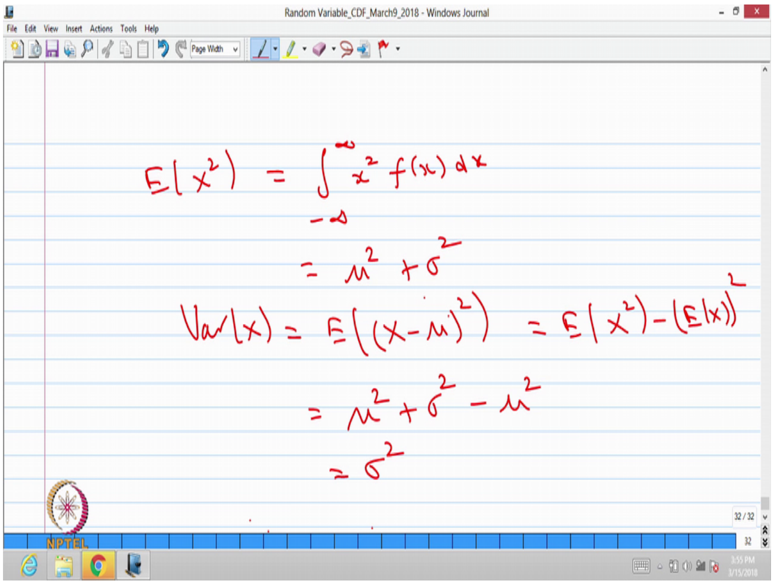


Let X be a continuous type random variable with probability density function f of x is 1 divided by sigma times root pi e power minus x minus mu divided by sigma the wholes square multiplied by 1 by 2, where x lies between minus infinity to infinity. So, this is a probability density function of a continuous type. Later we are going to call it as a normal distribution when we are discussing a standard distributions.

So, now, we will keep it as a continuous type random variable with the probability density function, 1 divided by sigma times square root of 2 pi E power minus 1 by 2 X minus mu by sigma the whole square. Always the sigma and mu values are given. One can say the mu value can lies between minus infinity to infinity whereas, the sigma quantity is always greater than 0. What is a meaning of mu and sigma? That also can be discussed.

In this example if you find out expectation of x that is minus infinity to infinity x times probability density function with the assumption that the expectation exist we will try to find the value minus infinity to infinity x times f of x dx. This is going to be after simplification you can get the answer that is mu. I am not going for the simplification of this integration as it is. If you substitute the f of x, x times f of x integration minus infinity to infinity you can get the value mu, this mu is going to be call it as a mean that is called the mean of the random variable x here.

(Refer Slide Time: 19:32)

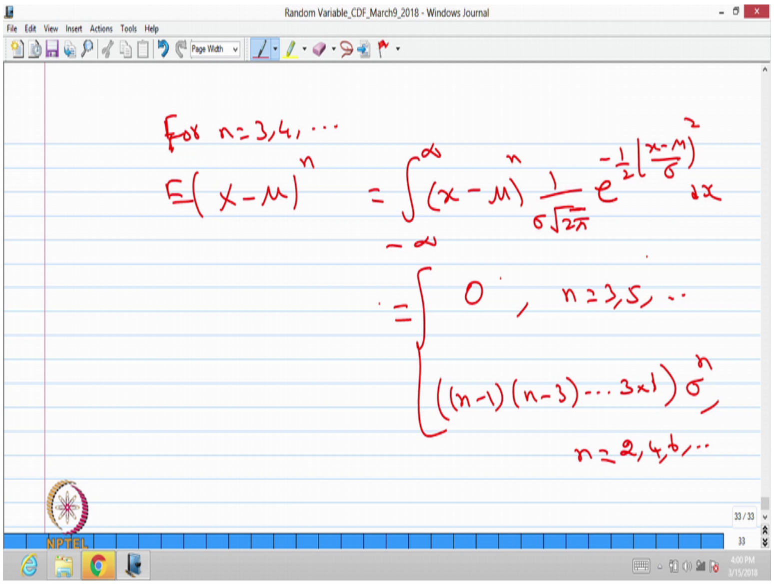


Similarly, if you compute expectation X square nothing but minus infinity to infinity x square times f of x dx. One can able to get by after some simplification you can get mu square plus sigma square substituting f of x is 1 divided by sigma times square root of 2 pi power minus 1 by 2 X minus mu by sigma whole square therefore, the variance of the random variable X that is expectation of X minus the mean is mu the whole square that is same as expectation of X square minus expectation of X the whole square.

Just now we got it E of X square is mu square plus sigma square and expectation of X that is mean that we got it as a mu that is minus mu square therefore, you get variance is sigma square. That means, for a for this continuous type random variable the mean is going to be mu and the variance is going to be sigma square.

We have another measure that is a positive square root of variance that is called as standard deviation. So, here the sigma is the standard deviation because sigma square is a variance and the positive square root of variance that is called standard deviation for this continuous type random variable, the sigma is the standard deviation and sigma square is the variance. So, we got first moment that is the mean, variance we got it a sigma square now we can go for higher order moments.

(Refer Slide Time: 21:56)



That is expectation of X minus mu power n for n is equal to 3 onwards, because for n is equal to 2 that is famous variance we got it already. So, we are computing the nth order moment about the mean from 3 onwards, that is same as minus infinity to infinity, x minus mu is a mean power n 1 divided by sigma square root of 2 pi e power minus 1 by 2, x minus mu by sigma the wholes square dx.

If you see the integration very carefully when n is a odd positive integer then the integration value is going to be 0 because e power minus 1 by 2, x minus mu divided by sigma whole square is a positive is a even function when n is odd positive integer the whole integration values is going to be 0 therefore, you can immediately conclude, this is going to be 0 for n is equal to 3 5 and so on. Now the question is what is a value when n is going to be the even positive integer. By even positive integer one can simplify this integration and you can get the answer that is n minus 1, n minus 3 and so on till on to (Refer Time: 24:03) of 3 into 1 times sigma power n when n is going to be 2 4 6 and so on.

So, for this continuous type random variable which is nothing but the normal distribution with the mean mu and the variance sigma square we are finding the nth order moment about the mean for all for all the odd powers it is going to be 0 for the even you get the expression is n minus 1 into n minus 3 and so on, 3 into 2 times sigma power n, when n takes a positive even positive integers.