Introduction to Probability Theory and Stochastic Processes
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Lecture – 13

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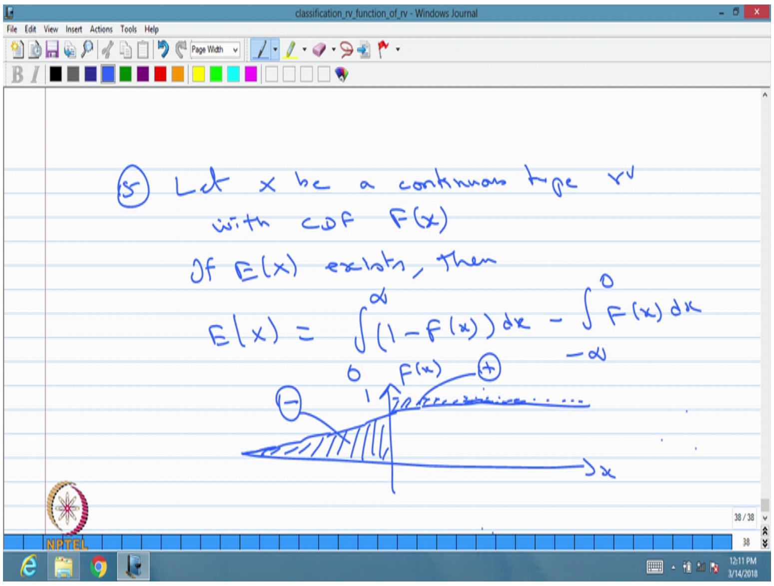

The next remark, all this remarks can be proved, but I am just giving as a remark it is for the understanding. The next remark let X be a continuous type random variable, let X be a continuous type random variable with CDF capital F of x, ok.

If the expectation X exist that means, the provided condition is satisfied then you can always find the value of expectation of X, that is E of X that is integration 0 to infinity 1 minus F of x, dx minus minus infinity to 0 F of x dx. This a very important result, whenever you have a continuous type random variable with CDF a capital F of x and the expectation exist.

Then you can always find the value with the help of CDF you do not need the probability density function of that continuous type random variable, from the definition of a expectation when x is a continuous type random variable if you know the probability density function if the expectation exist then E of X is same as minus infinity to infinity E x times of F of x, dx, where that small F of x that is the probability density function.

But now what I am saying is you do not need a probability density function with the help of CDF itself you can able to find the expectation provide that it exist.

Let me give a one nice pictorial representation of how to compute the expectation for this when x is a continuous type random variable. Recall, when x is a continuous type ransom variable the CDF is the continuous function in the whole real line from minus infinity to infinity. When x is a continuous type random variable the CDF of the continuous type random variable is a continuous function in the whole real line minus infinity to plus infinity therefore, I am just making a one CDF it is a syntactically touches one at infinity.

So, this is the CDF of a some continuous type random variable ok. 0 to infinity 1 minus F of x, dx that is same as a area below 1 till F of x between the interval 0 to infinity. So, you can shade. So, this quantity the shaded quantity is nothing but the 0 to infinity of 1 minus F of x dx, 1 it is a line F of x, 1 minus F of x 0 to infinity that is area be between the F of x with the line 1 that shaded area is a 0 to infinity 1 minus F of x dx.

Next integration minus infinity to 0 F of x, dx, that means this part. So, when you whenever the expectation exist for a continuous type random variable this sign is with the plus sign, this sign is with the negative sign. So, that area plus sign area minus that minus sign area that value is going to be the expectation value. Therefore, the expectation can be negative or positive based on the area between minus infinity to 0 that is going to be more than or less than the area between 0 to infinity of 1 minus F of x.

Suppose x takes a x is a non negative random variable that is a probability of X is greater than or equal to 0 is 1, using the first remark, yeah third remark, sorry.

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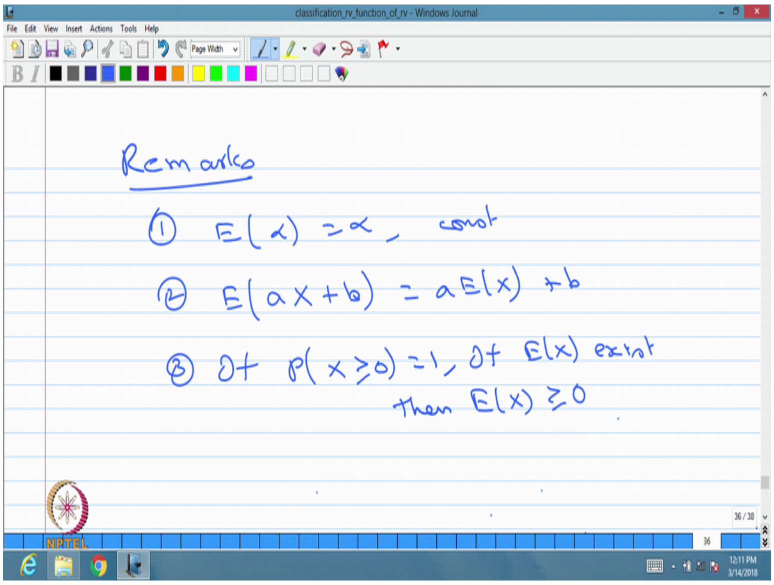

If probability of X is greater than or equal to 0 is 1 and the expectation a, if a expectation exist then the expectation of X is going to be greater than or equal to 0 that you can visualize minus infinity to 0 F of x, dx that quantity is going to be 0 because a probability of X is greater than or equal to 0 is 1. Therefore, you will get the positive quantity from the first integration and the second integration values is 0. So, this can be visualized.

There is a another remark over this remark I started with the continuous type random variable you can think of a discrete type random variable also. Suppose it takes only the positive values, then one can make a if the expectation exist then the expectation is same as in the summation form.

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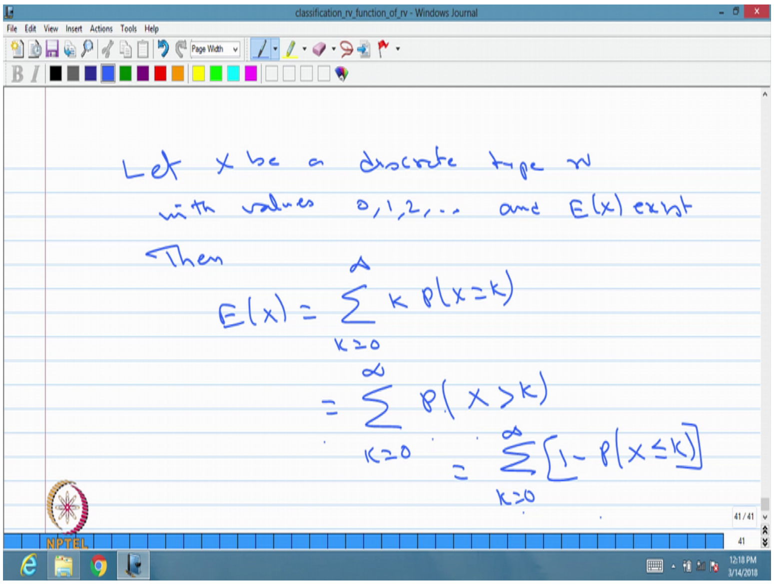

Let X be a discrete type random variable with values 0 1 2 and so on, with taking values as 0 1 2 and so on and E of X exist then we can find the expectation of X. By the definition it is a K times probability of X equal to K, where K is running from 0 to infinity by using the previous remark we can conclude that is same as summation of p of x is greater than K, where K is running from 0 to infinity.

By using the previous result summation K is equal to 0 to infinity probability of X is greater than K that is same as K equal to 0 to infinity 1 minus probability of X is less than or equal to K ok, I want to use similar logic of 1 minus F of x. So, this is probability of X is less than or equal to K is nothing but a F of random variable x at the point K.

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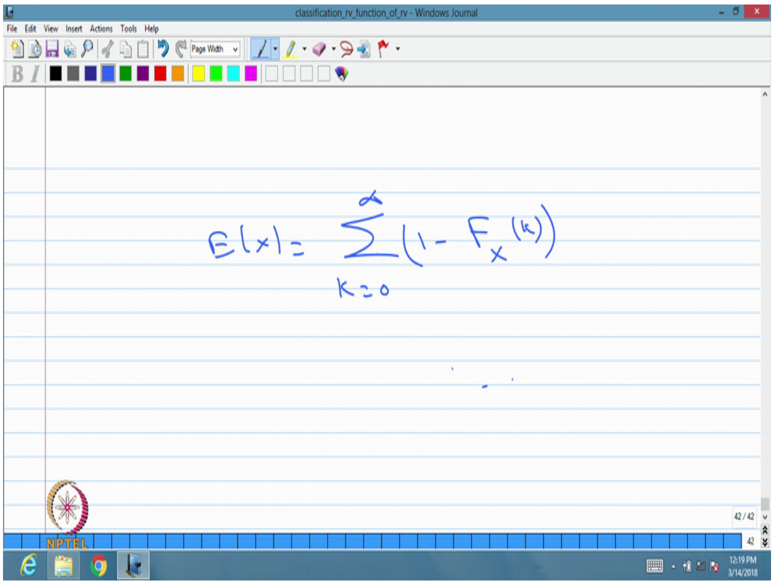

So, this is same as E of X is summation over K is equal to 0 to infinity 1 minus F of X at the point K. So, if you see the previous remarks, when X takes the only the non negative values then the second integration vanishes therefore, you will get the first one. So, first I give the remark, so for the continuous type random variable for a discrete type random variable this is going to be expectation of X provided X takes the value 0 1 2 and so on. So, this is also very important remarks for the expectation.

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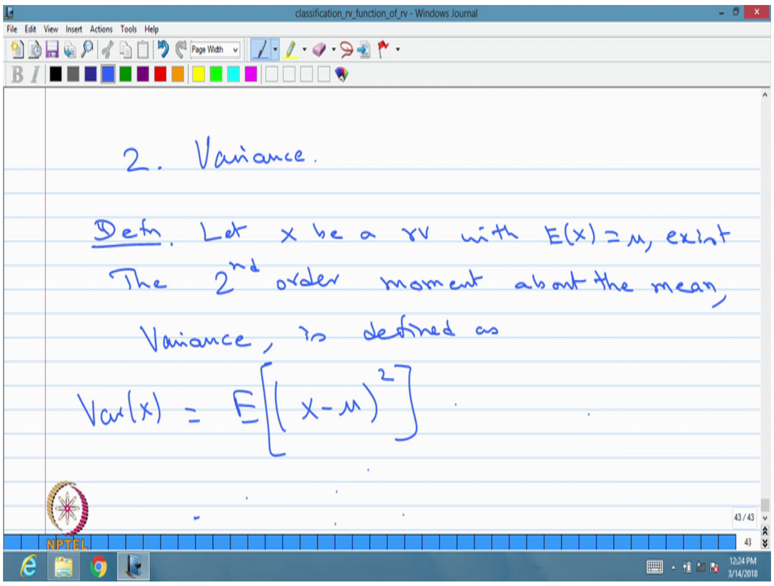

Now, we will move into the second moment that is called variance. Let me give the definition. Let X be the, X be a random variable with expectation of X that you call it as a mu which exist. I am going to give the notation for E of X that is mu which exist.

The second order moment about the mean that is nothing but variance is defined as expectation of X minus mu the wholes square that is denoted by variance of X. So, finding the value of expectation of X minus mu whole square that is going to be call it as a variance that is called a second order moment about the mean.

Since we are writing the right hand side expectation of X minus mu whole square that means, the provided the right hand side exist; that means, you can treat X minus mu whole square where mu is expectation of X. So, X minus mu whole square you can treat it as the function of x the g of x. So, right hand side is nothing, but expectation of g of x, where g of x is x minus mu whole square. You can use the previous result provided the expectation of g of x exist then the expected value that expectation of g of x, where g of x is x minus mu whole square that is going call it as a variance of X.

So, this second order moment one can define after the existence of the first order moment. If the first order moment does not exist then one cannot define the second order moment about the mean. So, here I have given the second order moment about the mean, later I am going to introduce a second order moment about the origin, that is nothing but the expectation of x square by making mu 0 or expectation of X square will be the second order moment about the origin similarly one can define nth order moment about the origin.

So, now we will discuss the variance first then later we will go for the higher order moments. So, the variance of X that is denoted by var of X that is expectation of X minus mu whole square. So, as I said earlier you do not need to find out the distribution of X minus mu whole square as long as you know the distribution of X and the expectation of X exist, and expectation of X minus mu whole square exist then the value is going to be expectation of X minus mu whole square is same as variance.

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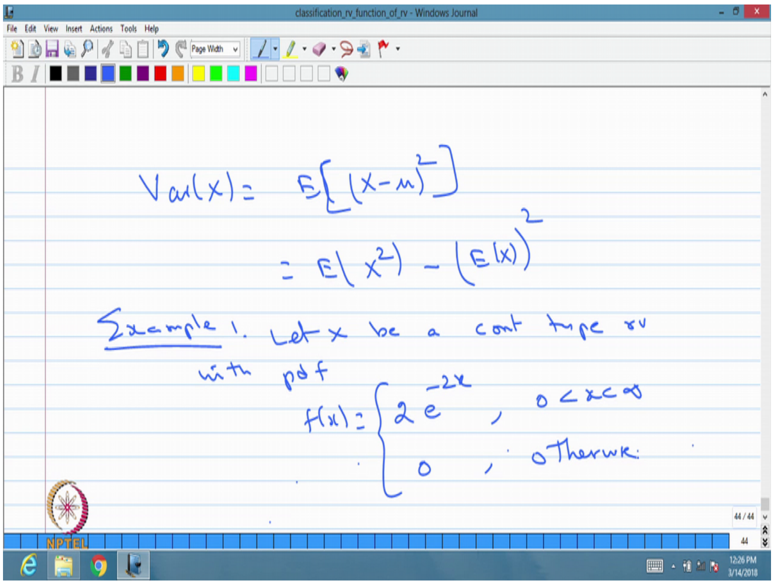

You can rewrite the variance of X as first you have X minus mu whole square, you can expand X minus mu whole square therefore, after expansion you will get E of X square minus E of X the whole square the E of X is same as mu. So, whether you compute a expectation of X minus mu whole square or you find out the expectation of X first that is mu whole square, then find out the second order moment about the origin that is E of X square then this difference is going to be the variance of X.

Similar to the expectation exist or not same way for some random variable the variance may not exist even though the first order moment exist. If the first order moment does not exist you cannot define the second order moment. So, even the first order moment exist there are some random variable in which the second order moment does not exist therefore, the variance does not exist.

So, we can go for simple examples the first example. Let X be a continuous type random variable with the probability density function f of x the same example 2 times e power minus 2 x, where x is lies between 0 to infinity, 0 otherwise.

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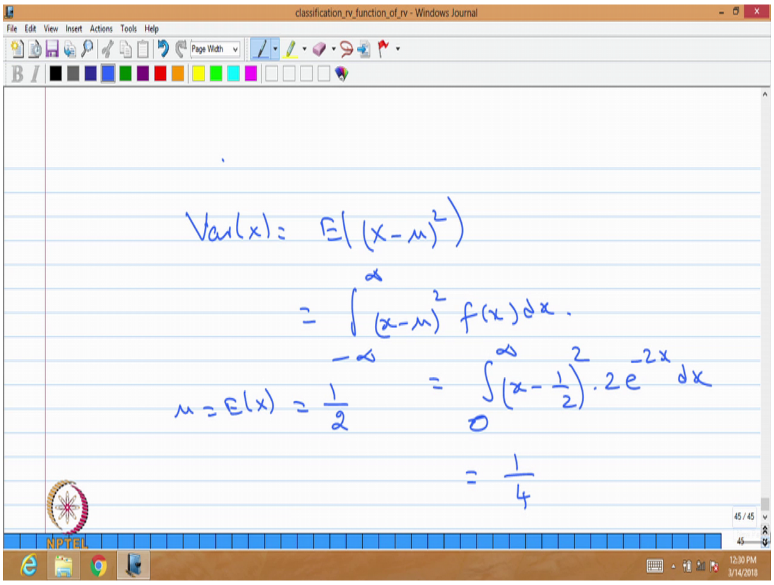

So, how to compute variance of X for this problem is variance of X? Is same as expectation of X minus mu the wholes square, that is same as integration from minus infinity to infinity x minus mu the wholes square f of x dx. With this problem the mu is same as expectation of X which we got it already that is 1 by 2 therefore, this is going to be minus infinity to 0 the f of x is 0 therefore, you can directly go for 0 to infinity x minus 1 by 2 the wholes square f of x is 2 times E power minus 2 x dx.

We are not finding the distribution of x minus mu whole square, then we are finding the expectation, no we are using a expectation of g of x, which I said it in the remark. So, x minus 1 by 2 whole square probability density function which is greater than 0 between 0 to infinity therefore, minus infinity to 0 will vanishes. So, this is a must 0 to infinity of this if this quantity is going to be find it then the existence also taken care, since x takes a non negative values you are finding this.

If you do the simplification you can get the value that is 1 by 4, if you do the simplification of this integration you will get the value 1 by 4 therefore, for the random variable which is continuous type whose probability density function is 2 times 2 power minus 2 x, when x lies between 0 to infinity the mean is going to be 1 by 2 and the variance is 1 by 4 later we are going to conclude this is going to be exponential distribution with the parameter 2. So, the mean is 1 divided by the parameter and the variance is a 1 divided by the square of the parameter. So, here the parameter is 2 therefore, it is 1 divided by 4.

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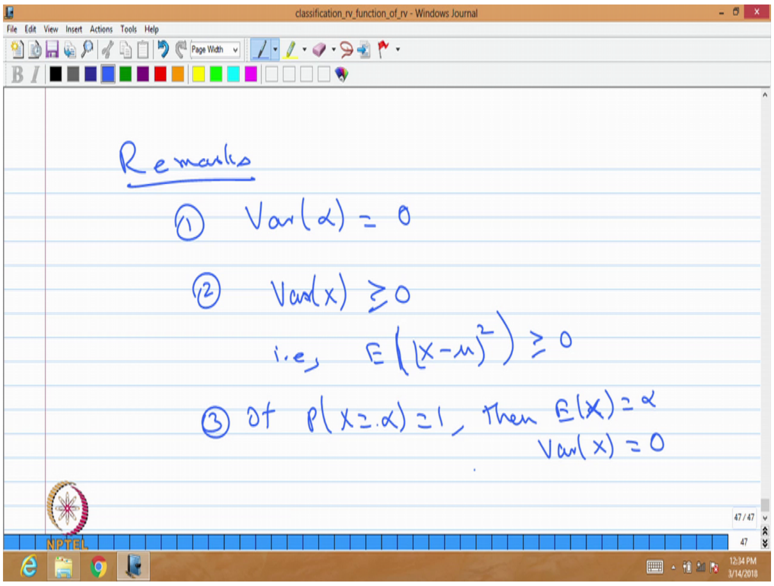

Let us go for a few remarks over the variance. The first remark variance of a constant is 0 whereas a expectation of a constant is a constant. Variance means what is a variation about the mean. So, here the mean of a constant is a constant therefore, there is no variation about the mean that is alpha therefore, the variance is 0. So, intuitively you can see the variance of constant is 0.

The second remark variance of a X if it exist then that value is always going to be greater than or equal to 0, if the variance exist for a random variable x then the variance is always greater than or equal to 0 that is expectation of X minus mu the whole square is always greater than or equal to 0.

This you can say by using the remarks on the expectation you go back when the random variable X whose probability is P of X is greater than or equal to 0 is 1, then the expectation is going to be greater than or equal to 0 with that logic E x minus mu whole square whose probability is always probability of a x minus mu whole square greater than or equal to 0 is 1 because it is a non negative random variable. Therefore, the expectation of X minus mu whole square is always going to be greater than or equal to 0.

Because, whether it is a discrete random variable or continuous type random variable this is nothing, but if it is a discrete it is a summation x i minus mu the whole square probability of X equal to x i and the probability mass function is greater than or equal to 0, x i minus mu whole square is a positive quantity therefore, if it exist then the summation is going to be positive.

If x is a continuous type random variable then x minus mu whole square probability density function integration from minus infinity to infinity again integrant is great or equal to 0 therefore, the if the expectation exist sorry if the variance exist then this quantities also in going to be greater or equal to 0.

You can combine the remark number 1 and 2 in the form of a third remark if probability of X takes a value alpha is equal to 1 that means, it is a constant probability of X takes a value alpha where alpha is a sum number that is 1. Then the mean is same as mean of the random variable same as the alpha and the variance of the random variable is going to be 0. That means, for a constant random variable the expectation is same as constant and the variance is going to be 0. And this is the if and only if condition if the variance is 0 then you can conclude that is a constant random variable when I say constant random variable the probability of X takes a value that constant is 1 therefore, it is called constant random variable. Later I am going to discuss in detail.

So, here the remark is if the variance is 0 you can conclude the probability of X takes that constant is 1, if a constant then whose expectation is the same and the variance is 0. Now, I can go to the 4th remark.

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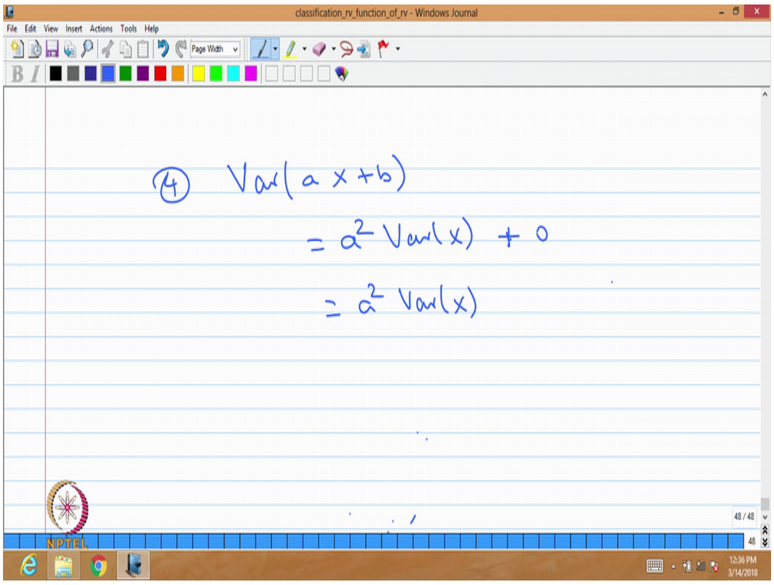

Variance of a some constant times x plus b if the variance exist, if the variance of X exist that is same as if you do the little calculation you can prove that is same as a square times variance of X plus variance of constant is 0 therefore, it is going to be a square times variance of X. Whereas, expectation of a x plus b that is a times expectation of X plus b here variance of a times x plus b when variance of X exist then it is same as a square variance of X because variance of constant is 0.