Introduction to Probability Theory and Stochastic Processes
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Module – 03
Moments and Inequalities
Lecture – 12

So, in this model, we will discuss the Moments and the Inequalities. So, we planned to give a 3 lectures in this model. So, the first lecture we will discuss the first 2 moments mean and variance and in the second lecture, we discuss the high order moments and moment inequalities and in the third lecture, we will discuss 3 important generating functions; that is a probability generating function, moment generating function and characteristic function.

Moments and inequalities; it is a very important topic in the probability or in particular random variable because most of the time finding the distribution of the random variable is a tedious task or it is a very difficult to find out the distribution of the random variable. But, it is easy to compute or easy to find the value of the mean or expectation that is basically the first order moment. Similarly finding the variance that is of the second order moment that is also easy comparing to the finding the distribution of the random variable. .

Therefore, in the probability theory course we discuss the distribution first, then we discuss the moments; then, later at the end of the course how much the moments are going to play a important role when we are going to use central limit theorem for approximating some of the random variable into normal distribution with some conditions. So, in that place, the moments are playing important role and also the moments is the easy measures; it is easy to compute in the real world problems not the distribution.

Once you know the distribution, if the moments exist we are going to tell the existence of the moments. If the moments exist we can always find it. If you know the moments of many order; then you can able to identify what could be the distribution of that random variable. That means, for a random variable if the moment exist we can able to find through the moments of some unknown random variable; it is possible to fit the correct distribution for that moments matches.

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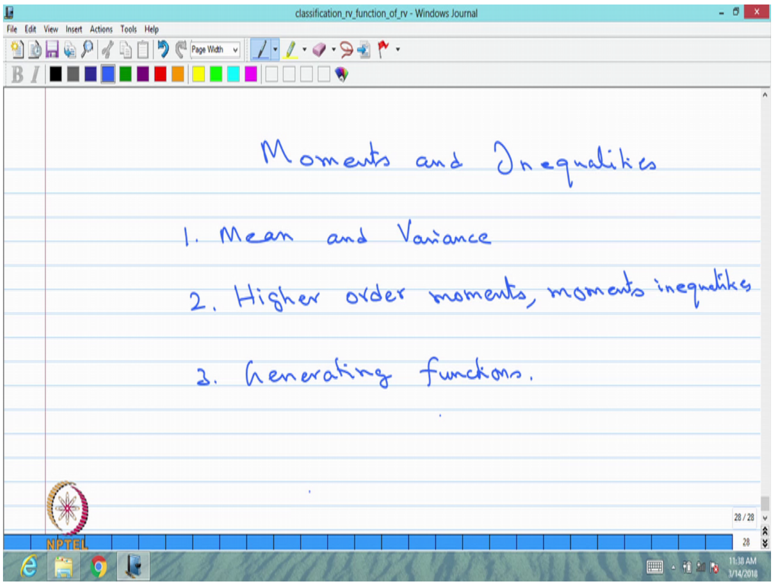

So, in that way I am going to introduce thismodule that is Moments and Inequalities. In this lectures, we start teaching first Mean and Variance. Mean is the first order moment and the Variance is the second order moment. Then, we discuss higher order moments and also we discuss moment inequalities and the third, we discuss generating functions. .

So, in this model we are going to have a 3 lectures. First one, mean and variance; second, higher order moments and moment inequalities; third, generating functions. Now, we will move into the first topic that is called Mean and Variance. This a very important measures. So, let me start with a definition of mean.

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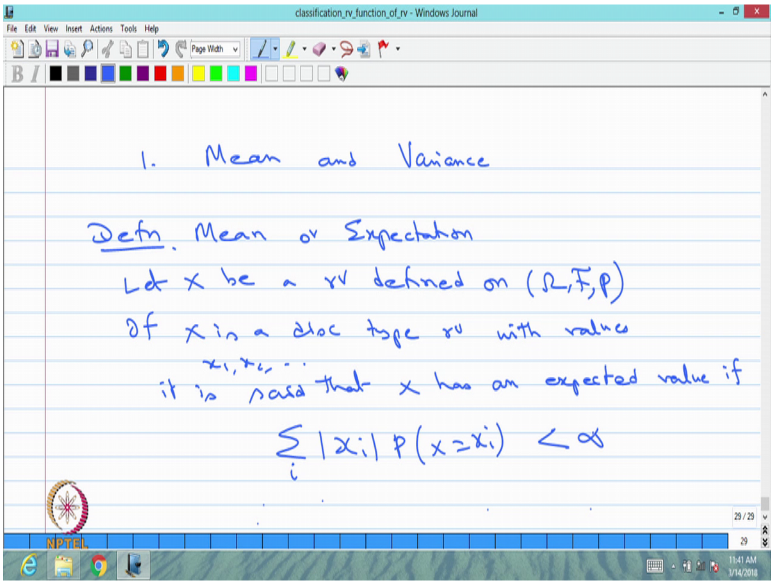

There is a another name for mean that is called Expectation. In many times we use find the value of average of some n values that is also going to be sort of mean and expectation with some conditions of that random variable. So, that we are going to discuss little later. So, as it is now I am going to write mean or expectation.

Let, X be a random variable; r v means random variable defined on the probability space omega F capital P. If X is a discrete type random variable with values x 1 x 2 and so on. It is said that X has an expected value, if the summation in absolute of x i's over the i's and the probability of X takes a value x i if the summation is finite quantity

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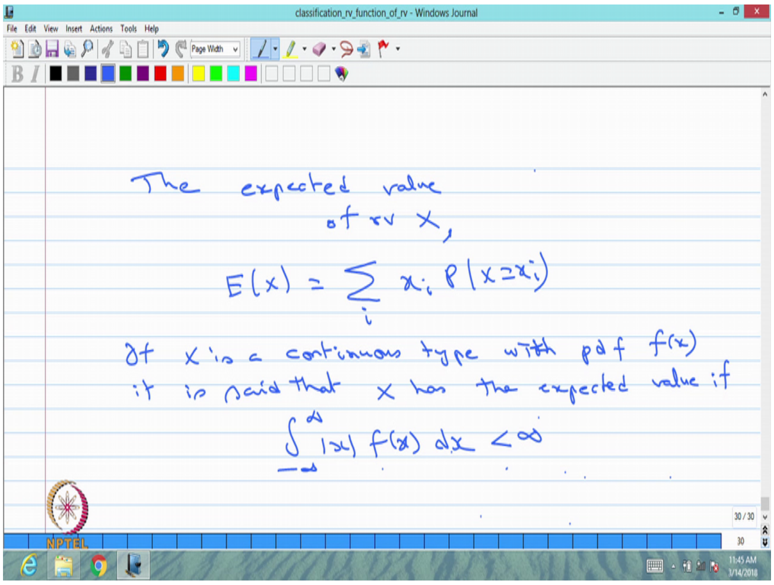

The expected value of the random variable X denoted by E of X is defined as E of X that is same as the summation over x i probability of X equal to x i summation over i. What we are saying is whenever you have a random variable defined on the probability space omega F P, if it is a discrete type random variable whose values are x i is a x 1, x 2 and so on. It is said that X has expected value if the summation over absolute of x i's multiplied by probability of X equal to x i as a finite value. Why the absolute is there? There is a possibility the x i's could be positive or negative.

So, in the absolute sense, if it is going to be finite; without the absolute that is going to be the expected value of the random variable provided in absolute sense that summation is a finite quantity. In other words, this is nothing but the series you can think of summation of x i P of X equal to x i as the a i's, whatever you have studied in the calculus course, this series converges in absolute sense.

Then, without absolute sense the similar series, you made it summation of x i with the probability of P is equal to x i that is going to be the expected value. If the series is convergence in absolute sense, then it is going to be converges here. So, we are using the calculus sequence convergence concept to conclude expected value of a discrete random variable exist.

So, now I am going for if the random variable X is continuous type. If X is a continuous type random variable with the probability density function is F of x, it is said that X has the expected value if the integration minus infinity to infinity absolute of x f of x d x is finite quantity in this case.

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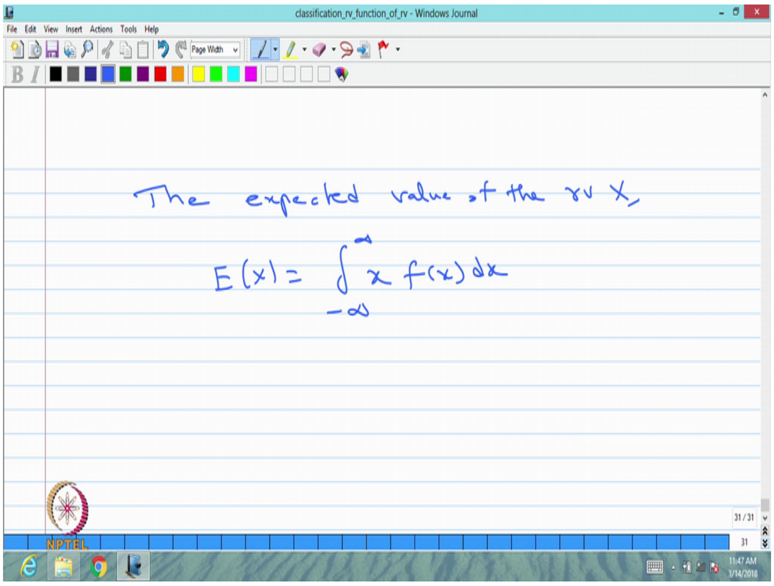

In this case, the expected value of the random variable X denoted by E of X is same as integration minus infinity to infinity x times f of x d x; note that integration minus infinity to infinity f of x d x is 1. Because f of x is a probability density function; whereas, here if the integration absolute x f of x d x is a finite quantity, then we can say that the expectation expected value of the random variable x exist and that value is same as e of x is a notation that is same as minus infinity to infinity x times f of x d x.

So, the provided condition is very important if the provided condition is not satisfied even you get the value of this integration without absolute that is not going to be call it as a expected value. It is very important to conclude in the absolute sense, the summation for a discrete type random variable integration in absolute sense for the continuous type random variable if it is finite quantity without absolute sense that is going to be the expected value of the random variable. I am going to give one example for random variable in which the expectation does not exist and 1 or 2 examples for expectation exist.

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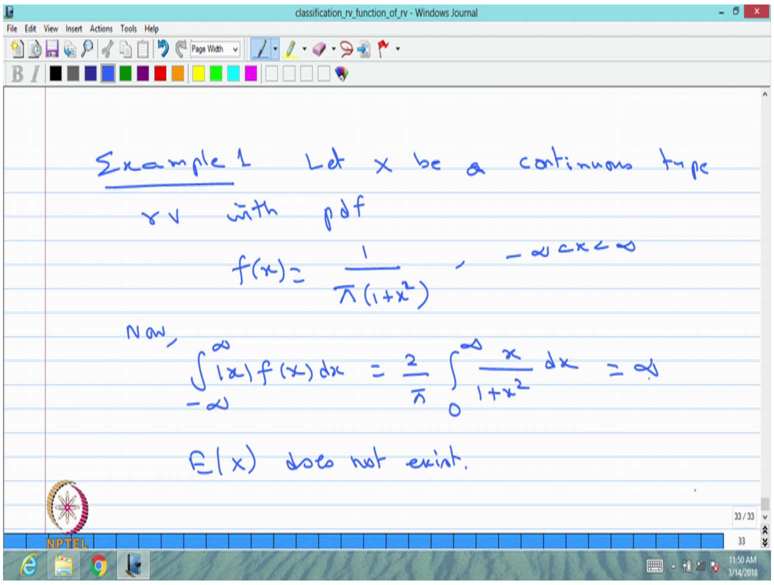

Example 1 let X be a continuous type random variable with probability density function f of x that takes a value 1 divided by pi times 1 plus x square, where x lies between minus infinity to plus infinity. The probability density function is 1 divided by pi 1 plus x square, where x lies between minus infinity to plus infinity. You can verify whether this is a correct probability density function by integrating a f of x from minus infinity to plus infinity, we will get the value 1. This is greater or equal to 0.

Therefore, it is a probability density function of continuous type random variable. Now you can compute the integration from minus infinity to infinity, absolute of x f of x d x. If you compute this quantity that is same as 2 divided by pi it is a even function. So, 2 divided by pi, 0 to infinity x divided by 1 plus x square d x. If you do the simple calculation, you will come to the conclusion this quantity is going to be infinity. You can make it as a homework computing this integration that is going to be infinity.

So, since this quantity is going to be infinity, you can conclude the expectation of X does not exist for this random variable. Even though, you will get the value from minus infinity to infinity x times f of x d x for this problem, but since in absolute sense this integration value is infinite. Therefore, the mean does not exist.

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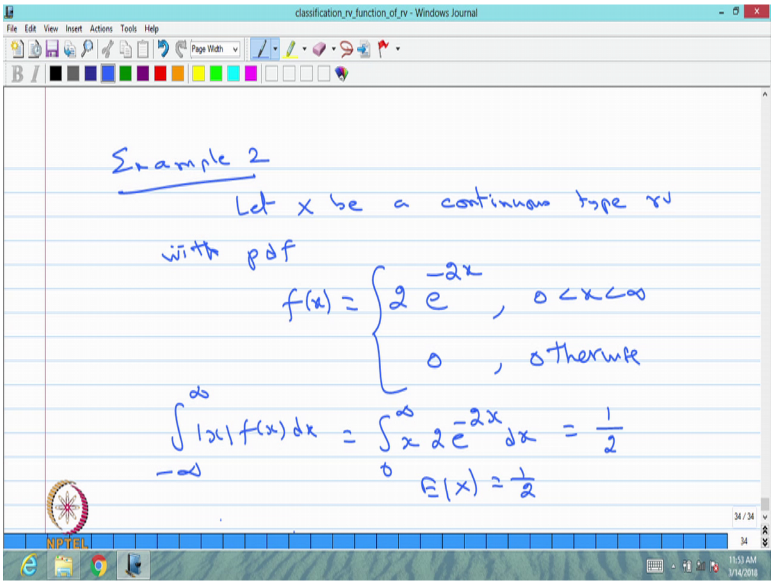

I will take a next example, where, expectation exist. Let, X be a continuous type random variable with probability density function that is f of x is 2 times e power minus 2 x, when x is lies between 0 to infinity, otherwise it is 0. You can verify this a probability density function or not. If you integrate you will get the value 1 and between the interval 0 to infinity because from minus infinity to 0, it is 0 and it is a positive function. Therefore, it is going to be a probability density function.

Since, the possible values of x is from 0 to infinity, we do not need to check whether this is going to be a finite quantity in absolute sense you can directly compute; if that is would be a finite quantity, then that is same as the expectation. Let us compute minus infinity to infinity absolute of x f of x d x that is same as since f of x is going to be 0 between minus infinity to infinity. Therefore, this is same as 0 to infinity of x times f of x is 2 times e power minus 2 x d x; that is same as if you do the simple calculation you will come to the answer that is 1 divided by.

So, since this is going to be a finite quantity. Therefore, we can conclude expectation x that is going to be 1 by 2 because the f of x is going to be greater than 0 between the 0 to infinity. Therefore, in the absolute sense answer is finite that is same as without absolute also. Therefore, the expectation of x is same as 0 to infinity of x times f of x d x that is going to be 1 by 2. So, in this case the expectation exist for this continuous type random variable.

Later, we are going to see this random variable is exponential distributed with the parameter 2; that we are going to discuss later. Therefore, the expectation of exponential distribution random variable is reciprocal of the parameter. So, here the parameter is 2 that is 1 divided by 2; it is a expectation for the random variable. We can go for one more example.

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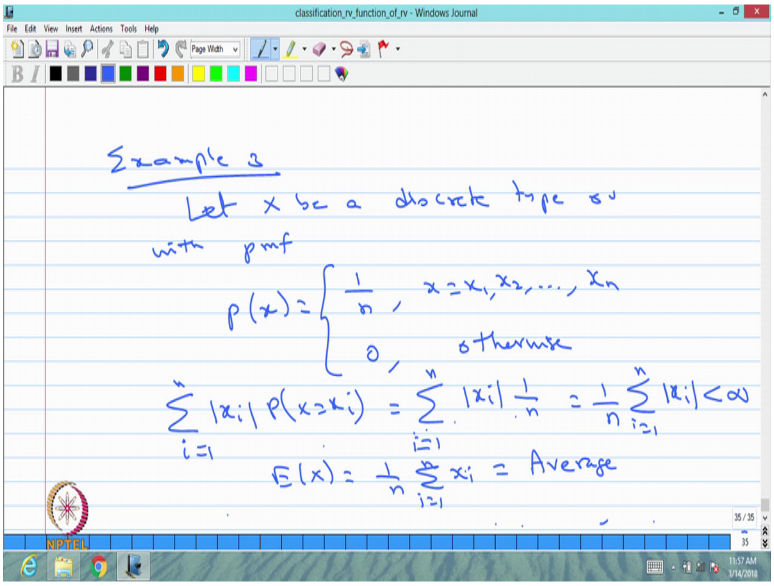

Let, X be a discrete type of random variable with probability mass function is given by P of x that is 1 divided by n, when x takes a value x1, x2 and xn; that is going to be 0 otherwise. So, this is a probability mass function of a discrete type random variable, with the possible values are x1, x2 and so on xn and this could be positive values or negative values or it could be 0 also ok.

Now, we will find out whether expectation exist for this one? So, if you compute in absolute sense of x i's probability of x equal to x i and summation over i. So, here it is a summation is i is equal to 1 to n. This is same as absolute of x i's. Probability of X equal to x i is 1 by n; i is equal to 1 to n. This is nothing but 1 divided by n summation of x i's. Since, we have taken all the possible values are from minus infinity to infinity, the summation of x i's that is also going to be finite quantity. .

Therefore, the expectation of x is going to be 1 divided by n summation of x i's. This is going to be the expected value of the random variable X, when X is a discrete type random variable with the probability mass function 1 by n for x i's are n values. But you know that 1 divided by n summation of x i's that is nothing but average. So, whenever the random variable has a equi-probable at a finite number of points in that case the mean or expectation that is same as average.

Later, we are going to conclude this random variable is called as discrete type uniform distribution. That means, for a discrete type uniform distribution the mean expectation that is same as average; that is what many times when you have a n values, you make a average out of it. So, the average is same as considering that as a random variable whose probabilities same as 1 divided by the total number of information or total number of values in that case a mean and or expectation that is same as the average.

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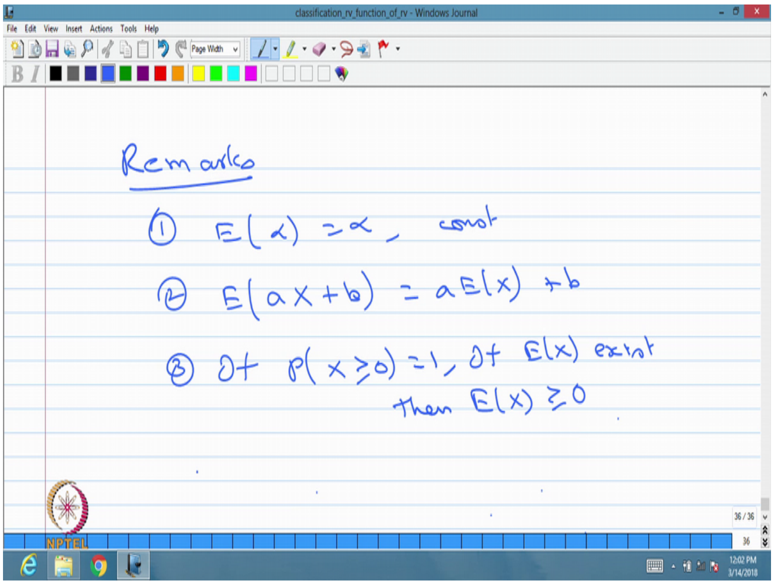

Now, we will discuss few important results for the mean as a remarks. The first remark the expectation of a constant is a constant. If you find the expectation or mean for a constant that is going to be a constant; this result can be proved easily. You can treat a constant as a random variable whose probability is 1 taking only that value. Therefore, expectation of constant is nothing but constant times probability of x takes a value constant that is 1. So, 1 into alpha; therefore, it is alpha. It can be proved easily.

As a second remark, suppose you have a some constant times random variable plus another constant here a and b are constants; in that case just now we made it expectation of a constant is a constant. But here, it is a linear combination and the constant can be taken out with the expectation of x plus expectation of a constant that is same as constant.

That means, a expectation of constant times random variable plus another constant is same as constant can be taken out. Third remark, if probability of X greater than or equal to 0 that is 1. The unit mass is distributed between the interval 0 to infinity. In that case, if expectation exist when the above 2 results also sorry above the second result also if the expectation exist then this is 2. So, if E of X exist, then the expectation of x also going to be greater or equal to 0.

This intuitively also you can say. If the probability of x is greater or equal to 0 is 1 and if expectation X exist, then the expected expectation or mean value of that random variable is also going to be greater or equal to 0. This also can be proved either from the definition the same way. Since whether you are going to use x is a discrete type or continuous type or mixed type, you can use the similar same definition for the existence in the absolute sense, then you can conclude if it is exist, then that value is going to be greater or equal to 0.

When I give the definition, I have given the definition for the expectation for a discrete type and the continuous type; I have not given the definition for mixed type. But the mixed type is the combination of a discrete and continuous. Therefore, wherever there is a mass you can do the summation; wherever there is a density you can do the integration. Therefore, the expectation formula is a combination of both discrete and continuous; therefore, I have not given the definition ok.

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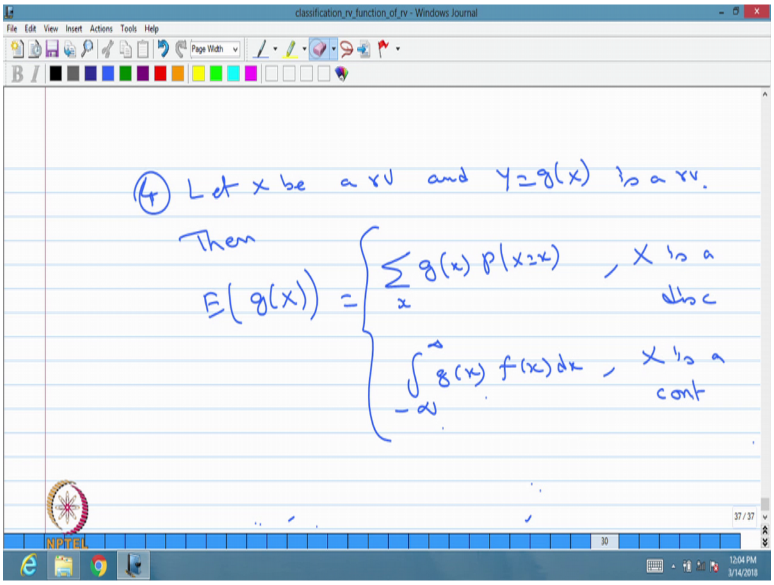

The fourth remark. Remark number 4, let X be a random variable and Y is equal to g of X is a random variable. This we discussed in the last model x is a random variable; g is a Borel measurable function in particular area, you can think of continuous r v. So, is continuous function?

Therefore, Y is also going to be a random variable. Then, you can find the expectation of g of X that is a same as summation of g of small x and the probability of a X takes a value x, summation over x.

If the random variable x is a discrete type. If x is a continuous type random variable minus infinity to infinity integration; g of x f of x d x when x is a continuous type random variable. This can be proved provided the expectation exist, provided the expectation exist you can find expectation value of a g of x with a summation if it is a discrete type with the integration if it is a continuous type.

Why this result is very important? Is you are not finding the distribution of g of x, but with the help of the distribution of x you are finding the expectation of g of x provided it exist. Therefore, this is a very important result in the sense you do not need to find the distribution of g of x whatever be the distribution of g of x as long as if it exist.

You can find the value of expectation of g of x with the help of the distribution of x means if it is a discrete type random variable, if you know the probability mass function;

if it is a continuous type random variable the probability density function of x. You can find the expectation by the summation or the integration without knowing the distribution of g of x.

So, therefore, we will be using this property again and again whenever you want to find out the expectation of a function of random variable; not only this definition similarly later we are going to introduce many random variables or random vector of size n with a n random variables. There also you can go for expectation of a functions of random variables in the similar way. Therefore, this is a very important result one can find it.