Chaotic Dynamical Systems
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Lecture – 06
Scrambled Sets

Today we will be looking into this topic of scramble sets which happens to be the first mathematical definition of chaos. We all know that chaos is something which deals with some kind of unpredictability, some kind of unstability. And we all also know that it was basically in late 18th century that Poincare observed that the planetary motion is an unstable system. So, within this observation there has been lot of unpredictability and lot of unstability observed in many natural systems, but there was no regress systematic mathematical definition on or any study of chaos. So, nobody looked into like putting it into an a geomantic form and studying it up.

We also know that many natural systems it was also observed that many natural systems they have some kind of a very random behavior, and with this random behavior there is also some kind of a pattern involved in it. So, this kind of pattern seen in a random behavior was something which many philosophers had termed as does god play dice.

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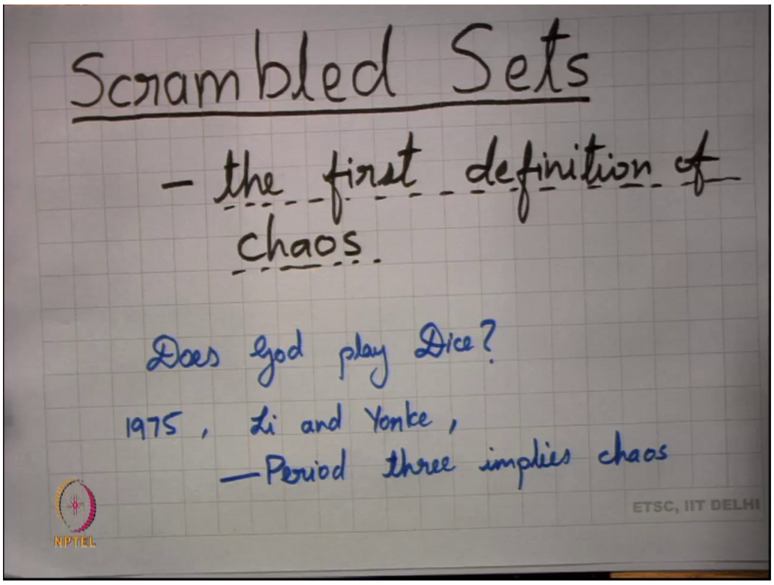

So, many of the natural systems that you see, you will find that there is a total random behavior, and with this random behavior you find that even though there is some kind of pattern involved in it. And today we shall be starting we which we want to look into we want to study these kinds of patterns, where you find regularity, you find some kind of predictability, you kind of sort of chaos.

So, we shall look into this aspect. Now it was for the very first time than 1975 that Li and Yonke gave this first mathematical definition of chaos in their epic paper. So, their epic paper is period 3 implies chaos. Now if we look into this definition, though when they define they have given this definition, this definition has given rise to many other definitions also, but as we know today this is a very weak. In fact, one of the weakest forms of definition of chaos.

So, there are other definitions of chaos also, which clearly imply this particular definition. So, first of all let us look into what this definition is.

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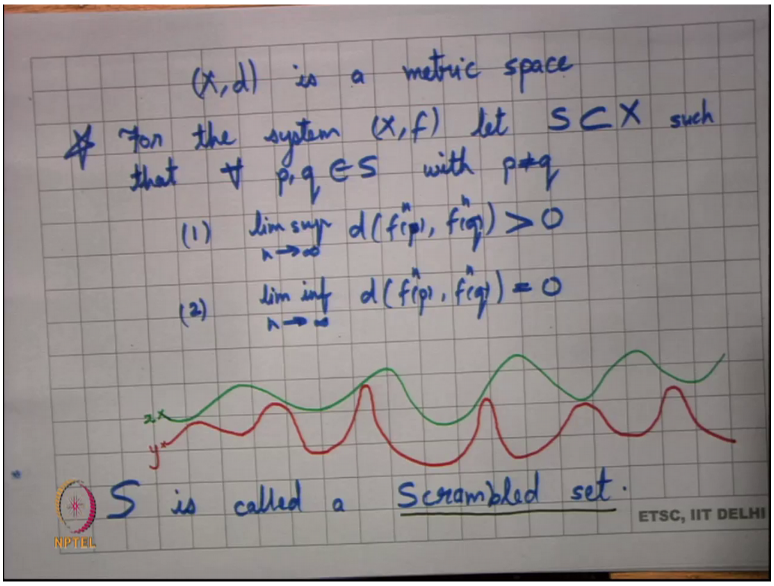

So, we all know for us we start with xd to be a metric space. And our system so, we start with this definition of a scramble set. So, for the system so, we have a dynamical system is x f, for the system x f, right, let S be a subset of x with the following properties, such that so, what we want is that for every p and q in S with p not equal, distinct p and q.

So, we have this first property that if I take the lim sup of the distance between this orbit of p and q, then this is always positive. And if I look into the lim imp, that is the infimum of this distance. So, we find that this orbit is sort of the points in all the set S is such that you take any 2 points over there, the orbits arbitrarily for infinitely often they will come arbitrarily close to each other, but then they will also diverge; that means, infinitely many times they will be very, very far away from each other.

So, if you want to look that maybe we can think of looking into this orbits. So, supposing this is my orbit of x. So, this is basically my orbit of x, I start with this point x. And my orbit of x goes y is another point here. So, in S then my orbit of y will be something like this. So, infinitely often it comes close to the orbit of x and infinitely often it is very, very far from the orbit of x. So, typically we are looking for a set S with these properties, then we say that if S has this property, we say that S is a scramble set. So, S is was scramble set.

So, we have this definition of scramble set here. We looking out basically for a scramble set. Now you can imagine that the unpredictability seen in this kind of system supposing there is a scramble set the unpredictability seen is too large. Because if you have these 2 if we come across these 2 orbits, right, you never know what is going to happen, because this will be basically away from each other, basically very close to each other. And we find that the definition of chaos basically is looking into this kind of unpredictability. So, of course, this is Li Yorke definition.

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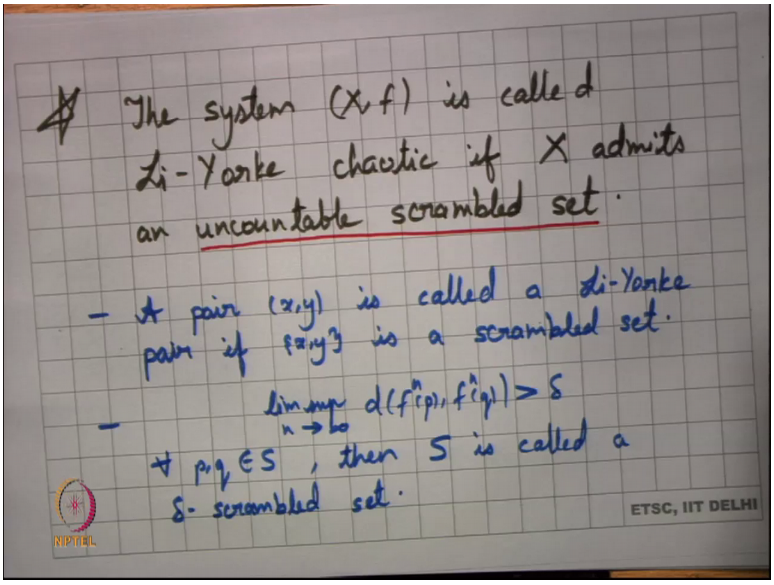

So, we come up with what is the definition. So, the system xf is called Li Yorke chaotic. If x admits an uncountable scramble set. So, that means, that now when I am looking out for my scrambled set, right. My system is said to be an chaotic if we have an uncountable set of points for which the orbits come arbitrarily close to each other, and also, they move apart right with some distance.

So, this is; what is basically the definition of chaos. The first definition of chaos proposed by li yorke. Now if you look into this definition of chaos, this definition of chaos as I said earlier happens to be very, very weak definition of chaos. This implies lot of other definitions. So, we first look into something more simpler here. Now we have something call and of course, we have the system call li yorke chaotic, but then now I am looking into a pair right. So, if I look into this pair. So, I take this pair x y, this is called a li yorke pair. If I take this 2 points at x and y this is a scrambled set.

Then there was a variation in this definition. And the variation in this definition came up that if for my scramble set instead of taking my lim sup to be or instead of taking my infimum sorry supremum to be greater than 0. If I have this condition, then S is called a delta scramble set. We have something call a li yorke pair, and it is very important many times to observe that the system has a li yorke pair; that means, at least we have 2 points, which are which constitute a scramble set.

So, based on this definition, the definition li yorke chaos goes up saying that there is li yorke chaos if it has an unaccountable scramble set. So, based on that there were other definitions of chaos proposed and these definitions are something called dense chaos.

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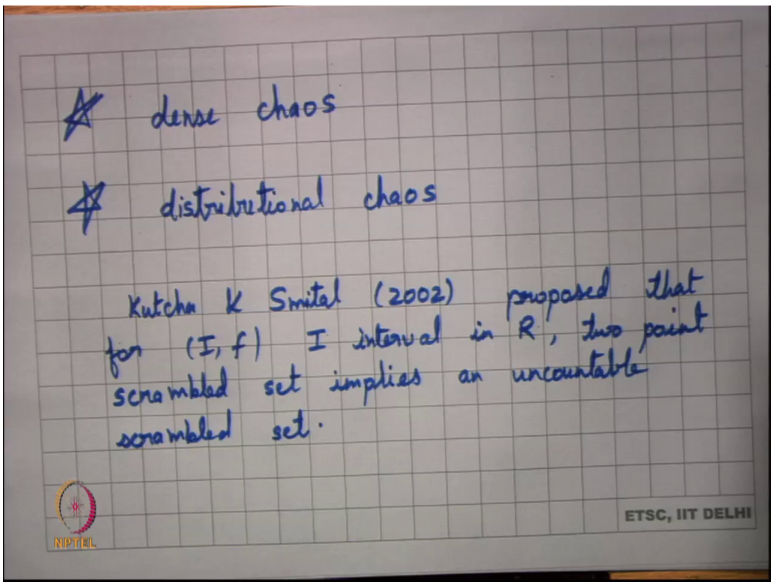

Dense chaos is simply saying that you have a dense set of scramble set. And there is another definition which is again used a lot quite a lot depends on the scramble set, which is called distributional chaos. I am not going into the details of distributional chaos, but again this is also something which is studied a lot and it is related.

Now, what li yorke proposed was that; if you have a periodic orbit of period 3, then basically that implies that there exists an uncountable scramble set. What happened was for the kutcha and smital sometimes in 2002, right, they proposed that for the system I have an interval I. And I have f where be my I S basically an interval in R, this 2 points scramble set implies an uncountable scramble set. So, for intervals it is enough to see whether we have a scramble set. We have at least, we have li yorke pair right 2 points scramble set means just a li yorke pair. So, for an interval it is enough to see that there is a le yorke pair. If there exist a li yorke pair then it means that it is li yorke chaotic because that would give us an uncountable scramble set.

Now, we go back to today we will be only discussing li and yorkes theorem of period 3 implying chaos. So, we look into this theorem. So, this is li yorke theorem.

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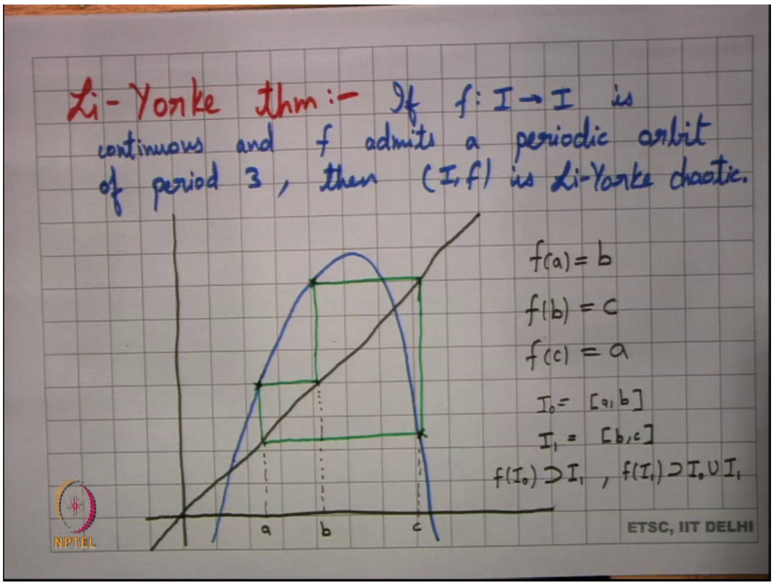

Now, again li yorke theorem is based on the interval because we were working on intervals. Last time in the last in the previous class we had seen that if we have a periodic point of period 3, right. That that implies that there exists periodic points of all periods. And if you try to recall the proof of that we had used this idea of looking into where does consecutive intervals map. So, we are again going to recall that that property right and we are going to look into this proof. So, what does li yorke theorem says that. It says that if f is a system from I to I right, some interval to interval is continuous; so, that means, I am I am working in the dynamical system x I f. And f admits a periodic orbit of period 3, then this system I f is li yorke chaotic.

So, li yorke theorem is very simple, you have a system on an interval, and on an interval if you if the system has periodic orbit of period 3, that implies that there is it is li yorke chaotic; that means, we are saying that there exists an uncountable scramble set.

Now, we will be looking only into the outline of this proof because a rigorous proof is very, very difficult to give at this particular stage. And of course, it would require lot of time. So, we will try to look into the outline which we can finish in this particular lecture. So now, what we want to do is; we want to look into this system again, and to look into the system again, what we need is we need a periodic orbit of period 3. So, that is what first of all I am trying to draw trying to draw the graph of the system, which has a periodic orbit of period 3. Now for this particular system, if I want a periodic orbit of period 3, then I know I should have something like this, I have an a here right. And I know that the value of a has to be b, right. And then I know that the value of b has to be say some c, and then I know very well that the value of c has to be equal to e.

So, I need to have a periodic point. So, basically this point has to be attained at a the value has to be equal to b, at b the value has to be equal to c, and at c the value has to be equal to a. So, I need a system right which has this periodic orbit of period 3. And we can simply draw the system not very difficult to draw the system. So, I think of the system maybe looking into the system right. So, it will basically it will be a curve which passes through these 3 points. We are very well aware that any kind of system it could be anything over here, lot of this depends because you have this orbit of period 3, lot of it depends on what is now you know that this has to be this graph at this particular point has to take up this kind of form right.

So, a lot of what we are going to discuss in this proof depends on, what will be basically the derivatives? Or basically what will be the slope what kind of curve we will have over here? So, what kind of curve can we trace which has this kind of property, and we find that there are most of the curves that we can trace with this particular property, will be basically of this kind. And then has it what our proof will basically depend on what kind of differentiable properties to this curve have right. So, what kind of slopes can you (Refer Time: 17:45) or what will be the kind of tangent plane that you can observe for this particular curve.

So, our proof depends basically on that aspect. And that is what we shall be discussing. Now we again recall something which we had already seen in case of our proof for circuses, case for basically the coronary of circuses case theorem. So, what as we going to look into we are going to look into this part. So, this is my a right. This is basically my b. And this is my c. And we observe that f of a is b, right. F of b is c. And f of c is a. Also, we recall what we had given as some kind of nomenclature that this interval a b, right I am writing it as I naught. And this interval bc is being written as I 1.

We also that what happens in this case is that if I take f of I naught, right. That particularly contains I 1 could be more we do not know, right. And if we take f of I 1 it contains the union of both I naught and I 1. So, this is something which we had already observed last time. There is something more which we had observed last time, and we can come back to that.

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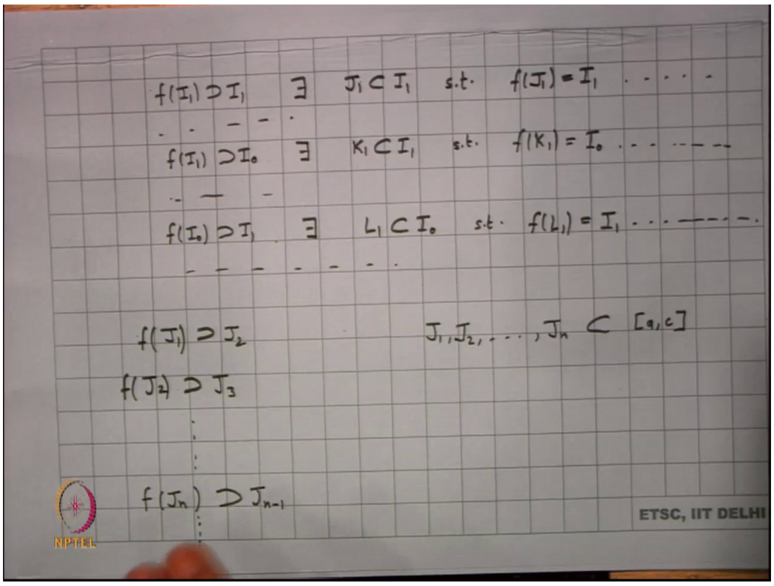

So, what we had observed last time was, that if I start with a small interval I know that f of I 1 contains I 1 right. So, we can start we have a small interval J 1 contain in I 1 right. So, there exist J 1 containing contained in I 1, such that f of J 1 right is equal to I 1, right.

And you can proceed in this manner I also know that f of I 1 contains I naught right. So, there can exist an interval K 1 contained in I 1 such that f of K 1 equal to I naught this is also something possible. And we had also observed that f of I naught contains I 1.

So, we can have another intervals L 1, right. Subset of I naught such that f of L 1 is equal to I 1 right. So, we have this observations with us, now with the help of this observation what we can do is we can simply start with some kind of intervals. So, let me start with an interval say J 1, I start with some interval J 1, now my J 1 is either a subset of I 1 or a subset of I naught I just start with J 1, right. And then I am looking into the fact that I have another interval J 2. And my J 2 is such that f of J 1 contains J 2.

Then given a J 2 so, once a J 2 is given, I can have taken interval J 3 such that f of J 2 contains J 3, right. Then again, I can keep on going in this manner. So, I can always have for an interval J n, right, I can always have another sub interval J n minus 1. Now these intervals will always be subset. So, I will have this J 1 J 2 J n, right. This will all be subsets of the interval a c, right. And whether they are a part of I naught or a part of I 1, right, will try to push some conditions into that, but we can always find such kind of intervals.

Now, what we try to do is for this particular kind of intervals, right we try to define a set of sequences. So now, I am interested in looking into a set of sequences. So, what do we start here? So, I take f to be so, I am taking this set of sequences.

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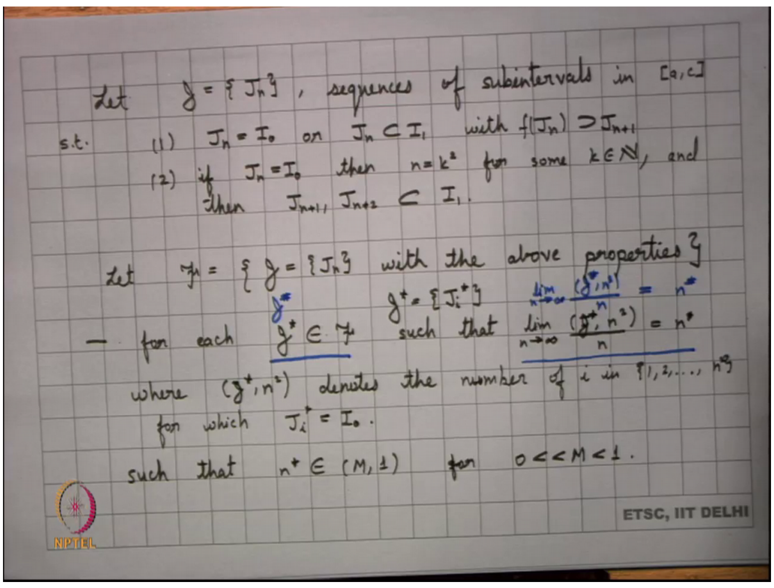

So, let J be the set of sequences J n right. So, these are basically sequences of sub intervals in a c, such that the first condition I want is that my J n is either I naught, right. Or J n is a subset of I 1, such that with my f of J n right containing J n plus 1. So, this is my very first condition, and the second condition that I put in this sequence of intervals is that. So, the second condition is that if my J n is I naught right.

So, at least I want that J n should be equal to I naught at some particular stage. Because I do not want to always I can always remain in J 1, but I do not want to do that. So, if J n is equal to I naught, right. Then n should be equal to k square for some k in n. So, it is when n equal to so, it can be I naught at many other places, but at least I want that whenever your n is equal to k square whenever n is a perfect square, right. Then for this perfect square my gn should be equal to I naught.

So, J n equal to I naught for n equal to k square. And then my J n plus 1 at least for 2 next to iterates J n plus 1 J n plus 2, they should always be a subset of I 1. So, I do not want to remain in I naught all the time. But at least 2 times I want to move out. And this is always going to be possible, because I cannot have 2 immediate perfect squares right with the difference of 2 that is always going to be possible. So, what we do is we take up a sequence of intervals, right with these property. Now one can say that I can have many such sequences of course, it is always possible. We start with whatever you start with your J 1, right you can start with J 1 to be anything, right. You can have many such possibilities. And for all these possibilities what we have is a collection.

So, let f be this collection. So, this is my collection of all J n with the above properties. So, I have this collection of all such sequence of intervals. Now, when I look into this sequence of intervals, right. I want to now associate a number with it. So, for each say I am calling it J star, right belonging to f. I want to now look into some kind of a density here right. So, such that I am looking into all those J star, such that I have some kind of density of numbers here. So, I take this limit as n tends to infinity. I am looking into a some number which I call as J star and n square divided by n. And I want to call it n star. Where what is first of all I need to define this quantity. So, what is my J star of n square?

So, this is basically this denotes the number. So, I am looking into the set 1 2 up to n. I am looking into the set, right sorry, n square I am looking into the set, right. 1 2 up to n square, right. All the number of I for which Ji start; that means, now I am looking into this J star right to be equal to basically the sequence of Ji star. So, my Ji star should be equal to I naught. I am looking into this number. So, I am looking into those number for which this is equal to I naught. And now I am looking into for those J star in f, right. Such that if I look into this particular density of n here what we find is that this average density of I for which this is equal to the number of I is equal to I naught. So, the average density of I naught in this particular sequence happens to be n star.

. So, we are trying to look into those we are collecting those kind of those J star in f. This is equal to n star. And so, this is one condition. So, what is this n star? So, I want this n star such that this n star I of course, I want it to be n star it would be n star for any n for whatever be the n star. But I want my n star with certain properties and I want this n star to belong to some for all this n star to belong to M 1, right. Such that belongs to M 1 for M is some integer which is very much larger than m, but it is less than 1. Now what is this M? This M basically depends on the derivative of f right in this particular interval ac. So, in the interval ac I am looking into the derivative of f as all the points and it depends basically this number I can pick up this number based on that particular derivative.

So, we look on to we look into the collection of all this J star for which this limit is n star. So, this J star is characterized by n star. And I am looking into all such J star for which my n star belongs to this uncountable set M 1. Where M is a number which is very, very close to one, but basically M is a number between 0 and 1, but it is very, very close to one compare to 0. So, it is very, very far away from 0, but it is very, very close to 1. So, I am looking into that particular problem. Now what more can we say about for this J star. Now once I had this J star right all we know is that for this particular J star. So, when I when I pick up this J star, right this J star is characterized by this particular n star.

So, if I change if I take another sequence here. So, for example, if I take another sequence may be J hash. So, if I take this another sequence J hash, it is going to give me another number n hash, right. Where what is this number? So, this number is basically limit as n tends to infinity, right. J hash n square right upon n. So, this number will be n hash. And we find that whenever n star is not equal to n hash, right. Then these 2 sequences will be distinct right. So, J star and J hash are distinct, whenever n star and n hash are distinct. And that is; what is our next observation that we take this collection.

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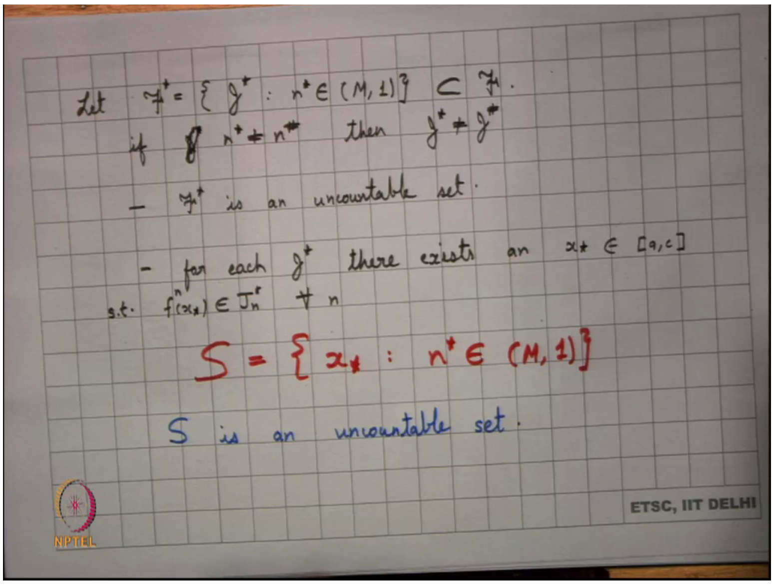

So, I am not taking this collection f star collection of this sequence is J star, such that n star belongs to M 1, right. I am looking into discussed sub collection. So, this is basically a sub collection of my f right.

So, we have this sub collection. Now one thing we can observe is that if or basically if my n star is not equal to n hash, right. Then my J star is not equal to J hash, right. We know that theyll will have different sequences here. And we are taking this collection for all n star belonging to M 1 right. So, basically, we are looking into all possible densities we are allowing all possible densities of the interval I naught coming into the sequence J right. So, since we are looking into that aspect. We can say that this f star is an uncountable set. So, this f star happens to be an uncountable set. The reason is for distinct we have distinct right. If n star is not equal to n hash, then we find that these 2 things are distinct. And since we are taking an uncountable stuff, we know that this f star is an uncountable set.

Now, for each f star, now for each J star we also observe that. Now where does x star belong to? So, x star can either belong to I 0 or I 1 depending on where we started our J 1 right. So, I can get an x star right belonging to this interval ac, right. Such that f n of x star belongs to J n right for every n. So, for every n we have f n of we have an x star such that fn of x star belongs to J n sorry, belongs to J n star right. So, we have this particular sequence in mind. So, this belongs to J n star for every n. And I take my S now specifically my S here is basically the collection of all x star such that of course, for every J star I have an x star and each J star is characterized by my n star right.

So, and I am looking into all those n star for which we can clearly observe, right? That S is an uncountable set. So, S happens to be an uncountable set. And what more can we observe about this S. So, so let me take p q in S.

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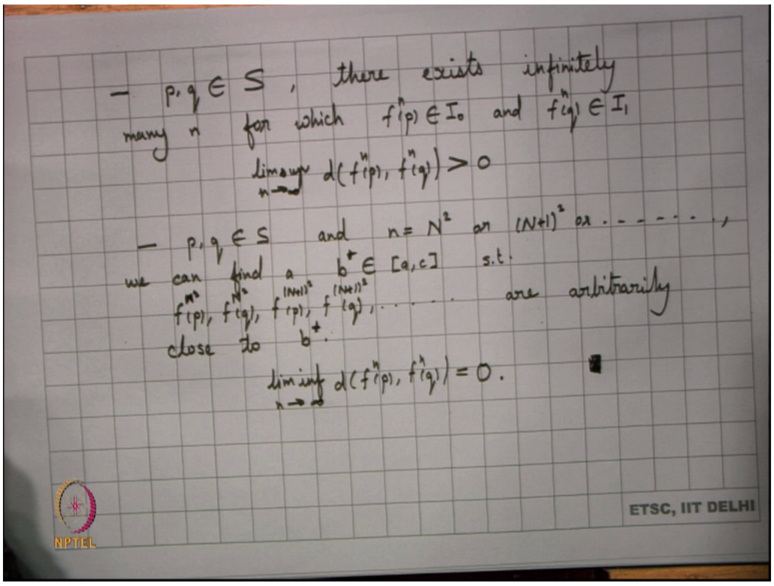

Now, I can find infinitely many n. So, there exist, because if I take pq in S, right each p and q will correspond to distinct sequences, right in my f stars. So, pq r in S and they correspond to distinct sequences in f star. And so, what we find out is that there exist infinitely many n for which if I look into fnp it will belong to I naught and if I look into fnq it will belong to I 1 right.

So, infinitely many of times we will find that fnp is in I naught and f n q is in I 1. And there will be infinitely many such n. So, what can we say about this particular scenario? Then we can say that the supremum here is positive always positive. Now what happens here when I am looking into p q and S and n being equal to n square or n plus 1 square? So, what happens? Because we had put some condition on the squares. So, what happens for the square right?

So, what happens here is; I am not getting into the details of the proof here, but we can always find I am calling it b star right belonging to this interval ac, right. Such that if I look into my f n square of p f n square of q right, f n plus 1 square of p f n plus 1 square of q right so on. So, if I find this thing, these will be all these are arbitrarily closed to b star. So, they will all come up very, very arbitrarily close to this point b star. And with this observation, all we can say is these are coming very close to one single point right. So, what can we say about the infimum of the distances between them right. So, the infimum of the distances between them will be going to 0 right.

So, if I look into the infimum here. So, that proves that you have period 3 on an interval. This period 3 gives you an uncountable scramble set. Although actually in practice it is very, very difficult to find out an uncountable scramble set. It is very, very difficult to say that fine this this is our scramble set and this is uncountable. Because again we cannot compute that computed always has some kind of precision, and it always calculates along with that precision. We end up here will see some more properties later.