Chaotic Dynamical Systems

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Lecture – 05
Periods of Periodic Points

Welcome to students. So, today we will be looking into Periods of Periodic Points. In the last class we had seen that we have different kinds of periodic point, and what can we predict about the dynamics of these periodic points. Today we shall be looking into what can be the possible periods of this periodic point. Like, can we guarantee that certain periodic point exists or not?

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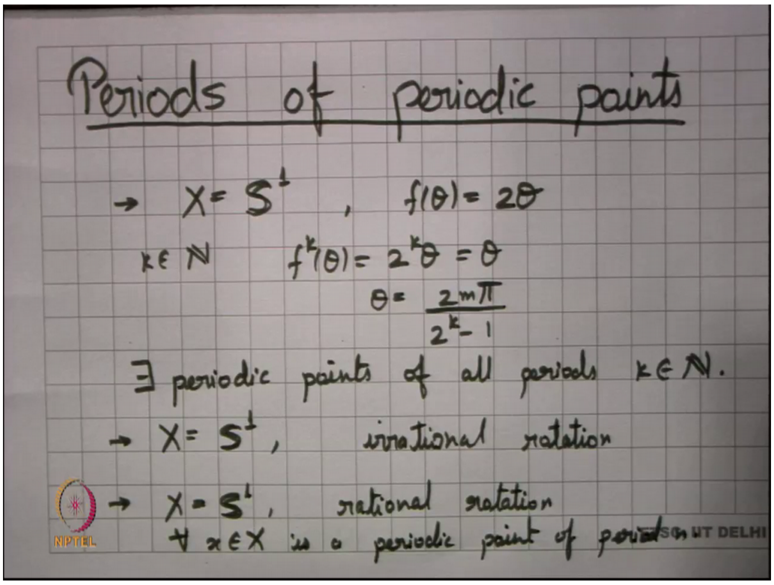

So, let us now look into this example of the circle which we had discussed last time. So, if we take the unit circle, we had considered this map f of theta going to twice theta. So, this is basically the doubling map that we had considered. And in this doubling map we had seen that, this has periodic points of all periods, right. Because given any k in N, we could find a point p right. So, basically, we are looking into fk theta, which is equal to 2 to the power k theta, and if this happens to be equal to theta we found out that theta comply something like 2 m pi, right upon 2 to the power k minus 1. And this points exists for every k. So, this has periodic points. So, I should say that there exist periodic points of all periods k.

Similarly, if for the same x, right. If we change our x is same S 1, and we are looking into the irrational rotation. So, if you look into the irrational rotation, then we could easily see that this does not have any periodic point for any period right. So, if I want to look into what are all the possible periods of periodic point? Well, there are no possible periods of periodic point, that possible period that happens to be an empty set.

But the same case when I looked into x equal to S 1, and now I am looking into the rational rotation, then we find that we are now looking into rational notation. So, our rotation the angle of rotation happens to be a rational multiple of 2 pi. And depending on what that rational is, we always get an n such that every x in X is a periodic point of period n. We have a fixed n here; we do not have any other periodic points. We just have one particular period here; that is n, and all points here happens to be periodic with period n

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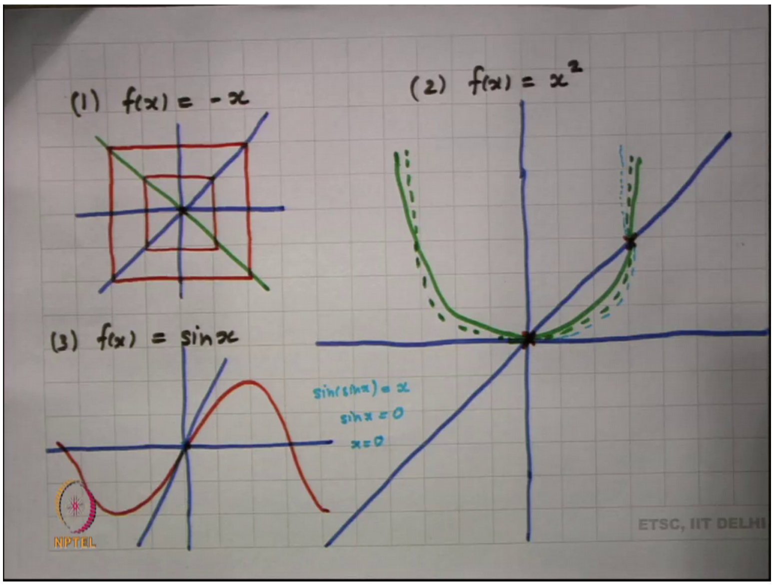

Now, let us look into some more examples here. So, I would like to go into this drawing here. So, we have this f x equal to minus x. Now if we try to look into if we try to analyze this graph fx equal to minus x, we find that this particular point 0 happens to be a fixed point it is a point of period 1 and if we look into all other periods right. So, if you look into all other points. So, all other points happened to be periodic points of period 2. So, fx equal to minus x has periodic points of period 2, and periodic point of period 1. So, there is only one periodic point of period 1.

On the other hand, if I look into this graph of fx equal to x square. So, if I consider the dynamics of x square, we find that it has 2 periodic 2 fixed points here a fixed point at 0 and a fixed point at 1. Now we have already discuss these examples earlier. So, it has a fixed point at 0 in a fixed point at 1. And what we find is that, does it have periodic point of other periods.

So, if we try to analyze that, maybe we can think of looking into the graph of f composite with f that would take x 2 x 2 the power 4. But if it takes x 2 x 2 the power 4, we find that these graph is again going to be something drawn along the dotted lines. And if we see this particular graph, right, then the only where the only place where it intersects the diagonal is 0 and 1. And we very well know that, the fixed points for x squared are also the fixed-point f square x equal to x are also the fixed points for f, right.

So, basically, they have the solution of fx equal to x is contained in the solution of f square x equal to x. So, we find that these are the only 2 points where it intersects. So, there is the possibility of a period 2 point here. What happens again if we try to look into f cube, right? Again, we would find something going along the dotted lines. Like, we will find something going along the dotted lines here, right. So, again here we find that this will not have periodic point of period 3. The only periodic points are periodic point of period 1. And 0 and 1 are the only 2 periodic points.

So, what happens in case of fx equal to sin x now; now a very simple graph of sin x, right. We know that sin x is always less than or equal to x here, and if we try to see what happens to sin of sin x equal to x, supposing we want to look into this part. So, here we have sin x equal to x has only one solution, right. That is x equal to 0. Now we try to see if it has periodic points of period 2. So, if to look into whether it has periodic points of period 2, we look into whether sin of sin x equal to x has some non-trivial solution.

But sin of sin x equal to x, since I know that sin x equal to x has only one solution x equal to 0. So, sin of sin x equal to x will also have only one solution, and that solution is sin of x equal to 0, but we know that we are now looking out for periodic points; in that case and sin x equal to 0 happens at into all say even multiples of 2 pi. And then in this case we can say that the only solution can be x equal to 0. So, we find here that, if I function is sin x and if I am looking into the dynamics of the function sin x. Again, I have only one periodic point of period 1 right. So, period 1 is the only possibility, right. We do not have periodic points of any other periods.

So, in general could we really claim something about the period of periodic points? Can we say that certain periods will come occur and certain periods will not occur? So, surprising result to that is something called Sharkovskii's theorem. So, I will look into what is Sharkovskii's theorem. I am just now this was a surprising work done by Sharkovskii's in 1964. In fact, this result came to be known to the rest of the world only after 1975. So, because this was published in a Russian journal, and during those days you know whatever research Russia did was used to be very kind of very kept very confidential. So, not many knew about this result and then suddenly after 1970. 1975 people tried to know or people came to know about this particular result.

Now, what is Sharkovskii's theorem? So, this was proved in 1964 by alexander Swarovski is a Ukrainian mathematician. And I think he is still doing some kind of dynamical work here dynamical. So, he is still working with some kind of stuff. So, you start with say f going from R to R. Now we are looking into interval maps. So, if f is continuous on the interval, and we consider the following ordering of natural numbers. So, we know that 3 now how do we order the natural numbers.

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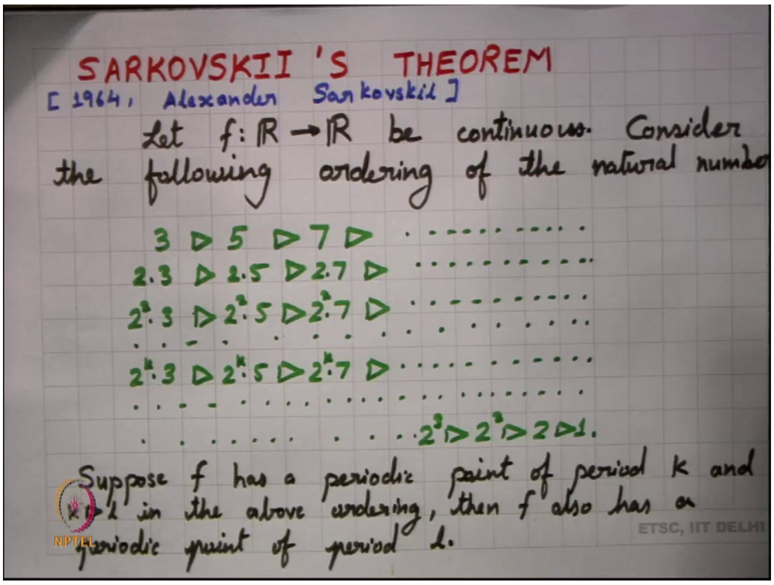

So, I am considering to 3 to be the greatest the largest number here. So now, we are ordering natural numbers 3 is the biggest one. Then 5 is less than 3, 7 is less than 5, and like that we are exhausting all the odd numbers here.

The next comes up what comes in the next order. So, once we have exhausted all the odd numbers. The next comes in to 2 multiplied by 3, right followed by 2 multiplied by 5 followed by 2 multiplied by 7. And so now, what we are exhausting is we are exhausting 2 multiplied by all odd numbers. What now remain is 4 multiplied by 3. So, next is 4 multiplied by 3, 4 multiplied by 5, 4 multiplied by 7 and so on, till we have exhausted the multiple of 4 with all odd numbers. And we proceed like that, we come up to say 2 to the power k multiplied by 3, followed by 2 to the power k multiplied by 5, followed by 2 to the power k multiplied by 7. And in that way, we exhaust all multiples of odds with 2 to the power k. And we continue this process. And then continuing this process we later on and up what is what remains is only the powers of 2. Because all multiples of odd numbers are exhausted, right. All that remains as powers of 2.

So, we look into the powers of 2, and then we order the powers of 2 now in the decreasing order. So, your 2 cube is followed by 2 square is followed by 2 is followed by 1. So, this is a different kind of ordering of the natural numbers, this is not the usual ordering that we take up. This is a different kind of ordering of natural numbers. And we find that in this particular ordering, 3 is the largest number and the smallest number is 1. And in fact, 4 is much less than 3, right.

So, this is the ordering of natural numbers, and what Sharkovskii's prove was a really wonderful idea it is something astonishing; that if f has a periodic point of period k, and k precedes l in this ordering, then f will also have a periodic point of period l. So, that means, supposing now, in any map from reals to reals, if I can find that there exist a periodic point say of period 4 into 7, then I will have periodic points of all these periods all these periods all these numbers which follow 4 into 7. There will be all these periods coming up. And then what we find here is maybe we may not find a periodic point of period 4 into 5, maybe we may not find a periodic point of 4 into 3, but we are guaranteed that all these periodic points will occur.

So, this was this is really something really unique, and it is unique about the interval maps only, because we are looking into the interval maps basically when you look into the real line it is a continuum and it is basically an ordered set. So, it is an ordered continuum we are just using the property of an ordered continuum. So, this result holds only for real line, but still this gives us a very nice, property of the dynamics of real numbers. So, this property of dynamics look into a periodic point you can guess, what are all the periodic points that would exist.

What we should try to see here is proof of Sharkovskii's theorem is a little bit involved it would take me again one more lecture to prove the Sharkovskii's theorem; it is not too difficult to understand. But we will not be doing Sharkovskii's we will not be doing the proof of Sharkovskii's theorem. There are many places where I mean now you can find out like if you just search on the web you can find out many different proofs of Sharkovskii's theorem. And you will find that each author claims that he has a better proof of Sharkovskii's theorem. So, you can look into any of the sources to look into the proof of Sharkovskii's theorem. What we shall discuss now is; what is the converse part.

Now, what do we mean by a converse part? As I said is that we know that if I have a continuous map, then for this continuous map we can say that you have if a periodic point of period k exists, and if l is following k in this ordering, then f will also have a periodic point of period l. The converse to Sharkovskii's theorem is something very different. It says that, if you fix an l can you say that if k is l is following k, then you have a periodic point of p, you have a function f such that it has a periodic point of period l, but it does not have a periodic point of period k.

So, this is what is this is what is known as the converse of Sharkovskii's theorem, and we will be discussing a few examples here. So, we look what happens conversely.

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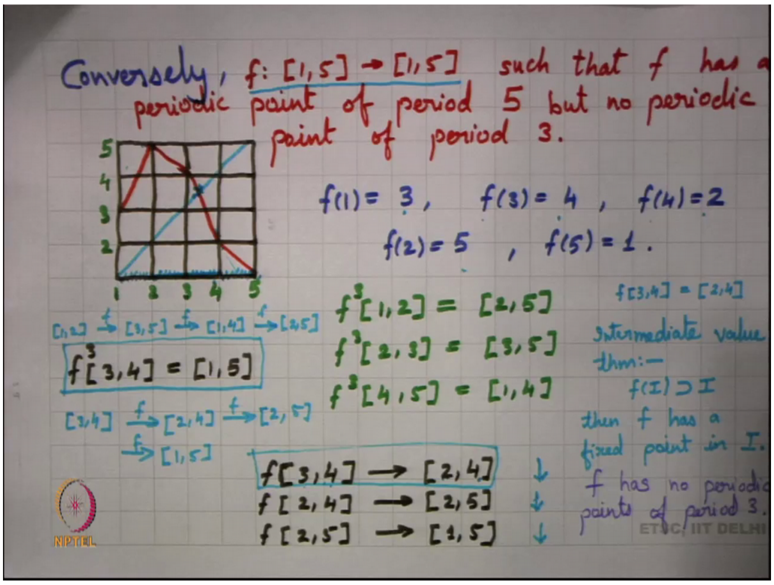

Now, I am just looking into one simple case. See my function f is going from the interval 1 5 to the interval 1 5. And f has a periodic point of period 5, we just going to show that it has no periodic point of period 3.

So, let me take this particular function. Now define this function in such a way that f of 1 is 3. F of 3 is 4. F of 4 is 2. F of 2 is 5. F of 5 is 1. And in between it is all linear. So, the function is linear in between it is a straight line with the graph is a straight line in between. So, it is a linear function in between, but this gives me now 1 3 4 2 5 gives me a periodic orbit of period 5.

So now we have this periodic orbit of period 5. We want to find out what happens. Does it have we want to show that actually it has no periodic point of period 3? Now remember in Sharkovskii's ordering 3 precedes 5 right. So, if I have a periodic point of period 5, it implies that I should have periodic point of all other periods. But I cannot guarantee about 3. And that is what this example shows us that this particular thing this particular function has a periodic point of period 5. And we will prove that it has no periodic point of period 3.

So now look into this fact I am looking into here. So, let me look into what happens here. So, when I see f of say I am looking into what happens to f of 3 4, then f of 3 4 is 2 4. Let us go back to our real analysis, right. And let us try to look into the intermediate value theorem. We just recall the intermediate value theorem, which says that if I have I am looking into a continuous map on an interval and if for my interval I, if f of I is containing I, right. Then f has a fixed point in I, right. Then f has a fixed point in I. Then f has a fixed point in I. So, the now if I am looking into this particular function, f of 3 4 happens to be equal to 2 4 right. So, f has definitely a fixed point in basically 3 4. So, there will be some x in 3 4 such that f of x is also equal to x.

Now, we want to look into we just want to show that it has no periodic points of period 3. So, we try to look into what happens to f cube. So now, we are just iterating this graph. We just iterating f cube. And what we find is that what is f cube of 1 2. So, if I look into 1 2. We basically find that this interval 1 2 is mapped into the interval 3 5. So, under one f it is map to 3 5 where is 3 5 map to. So, if I look into 3 5, right. So, this 3 5 is basically mapped into 2 5. So, under f 3 5 is mapped sorry 3 5 is mapped into 1, 1 4. 3 5 is mapped into 1 4 and where is 1 4 mapped into. So, I find that 1 4 is mapped into 2 5. So, under f cube my 1 2 is mapped into 2 5.

Similarly, if I look into my f cube of 2 3; now I am looking into now I know that where is this intervals I am looking into this interval. So, this interval is basically mapped into 2 5 right. So, 1 2 is mapped into 2 5. Now let me look what happens to 2 3. So, if I look into this particular interval 2 3. Where is 2 3 map to; so in the fury first instant, right. I can say that my 2 3 will be mapped into. My 2 3 is mapped into this part 4 5 right. So, 2 3 is mapped into 4 5 where is 4 5 mapped into. So, 4 5 is mapped into 1 2, right. And where is 1 2 mapped into. So, if I look into 1 2 it is mapped into 3 5. So, under f cube 2 3 is mapped into 3 5.

Now, what happens to now I am not considering 3 4, let us look into what happens to 4 5. So, so under f 4 5 is mapped into 1 2. 1 2 is mapped into 3 5. And where is 3 5 mapped into, it is mapped into 1 4 right. So, under f cube, 4 5 is mapped into 1 4; that means, if I look into these things, right. There is no x in 1 2 for which f cube x could be equal to x. Because these 2 are disjoint, right. The only place where there intersecting is 2, but 2 happens to be a periodic point of period 5, right

Similarly, if I look into the interval 2 3, right; there is no point in 2 3 for which f cube x could be equal to x. Because 2 3 is mapped under f cube 2 3 5, right. The only point of intersection is 3 which is a periodic point of period 5. And similarly, 4 5 is mapped under f cube to 1 2. So, if I look into 4 5, right. Then we find that this again this does not intersect this is the only place it intersect is 4 which is a periodic point of period 5.

So, there is no place where f cube could have mapped could take this particular interval. So, there is no x here which could be a periodic point of period 3. So, that means, there is no periodic point of period 3 in these 3 intervals. Now what happens to the interval 3 4? So, let us look into 3 4, we know that where is 3 4 being mapped to. So, 3 4 under f, right where is it map to? So, it is map to 2 4, where is 2 4 being map to under f?

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2 5.

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2 5 and where is 2 5 being map to under f.

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It is an map to 1 5. So, what we have is that f cube maps 3 4 to 1 5.

Now, since it maps 3 4 to 1 5, I had this intermediate value theorem which says that if f cube of 5 is containing I, right. Then there exist of f point here. So now, we have that f cube of 3 4, right is mapping into. So, this should contain a fixed point of it should contain a fix point for f cube. If I again look into this case, right. My f of 3 4 happens to be 2 4. So, I am looking into this case f of 3 4 happens to be equal to 2 4, right. Since this happens to be 2 4, I have f of 3 4 f of I also containing I. So, that means, that f should also have a fixed point in 3 4.

Now so, f has a fixed point in 3 4, now we want to say that since f is a fixed point in 3 4, we have f cube also has a fixed point in 3 4, right. We only want to claim that both x and f cube x both f fx and f cube x will have the same fixed point in 3 4.

So, already there is a fixed point of period 1 here, right. And I can simply draw that part right. So, I find that this is a fixed point here. So, already I know that f has a fixed point here. I want to claim that that f cube also has a fixed point. And that fixed point is same as this fixed point as the fixed point of f right. And so, there is no periodic point of period 3.

Now, what we what we judge here is that f of 3 by 4 is mapping to 2 4. Now when I look into 3 4, right it is mapping to 2 4 right. So, my f is decreasing in this interval right. So, the function f here is decreasing in this interval. Again, f of 2 4 is map to 2 5. So, if I look into f of 2 4. So, I am looking this whole interval, right. In f of 2 4 is mapped to 2 5 right. So, what it is map to it is map to 2 5, but again in this interval the function is completely decreasing. So, the function is decreasing in this interval also. And then f of 2 5 is mapped to 1 5; so if in again look into 2 5, right. It is map to 1 5, but again the function is decreasing in this interval.

Now, you have a monotonically decreasing function, right. You function here happens to be; so, if I look into f cube on 2 3 4, right. It is basically a monotonically decreasing function. Since it is monotonically decreasing function, it will basically cut the diagonal it will only at one point, or in other hands this if f cube has a fixed point, right. Basically, that fixed point will be same as the fixed point of fx. So, if x; so this f cube, right. Since it is monotonically decreasing here this f cube has just one fixed point here, and that fixed point is basically the fixed point of f. So, there is no periodic point of period 3.

Student: 3.

Right, so all we can conclude here is that f has no periodic points of period 3. I hope this is clear. So, let us try to look into another example here. So, let me try to push this up let me try to look into another example here. So, I took take this example.

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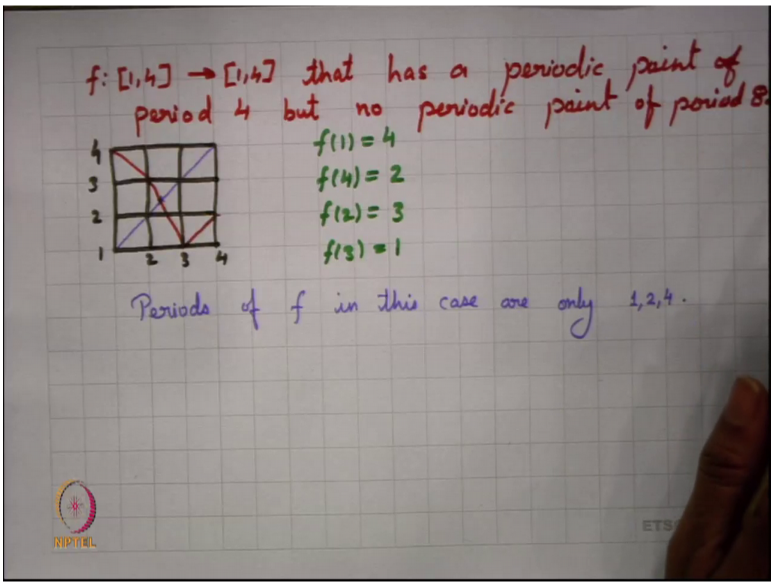

So, my f is from 1 4 to 1 4, and we find that it has a periodic point of period 4, but it has no periodic point of period 8. So, I am just trying to draw this particular stuff. So, f 1 is 4. So, I am looking into f 1 to be equal to 4 f 4 is 2. So, f 4 is 2 f 4 is 2 here, right. F 2 is 3 and f 3 is 1.

Now, this has a periodic point of period 4 we know that very well. But again, you can use similar methods to show that it will not have a periodic point of period 8, right. All you need to see is that the function happens to be completely decreasing it is monotonically decreasing in this interval 1 3, right. And hence it will not have a periodic point of period 3.

The only fixed point the fixed of period 8. The fixed point here is basically this particular point which lies here, right. We can think of we give we will have a periodic point of period 2. So, you can find that somewhere over here there will be a periodic point of period 2 right, but it will not have any periodic point of period 8.

So, the only periods that this f admits or 1 2 and 4; so it does not have any other periodic point. So, the periods of f here, in this case are only 1 2 and 4. And now we go back to our Sharkovskii's ordering, right. In our Sharkovskii's ordering, what happens to the end of the order is the powers of 2, right.

So now the that the number the natural number that piston at 4 was 8, right. Since it does not have a periodic point of period 8, right; we can think it will not have a periodic point of any other period we have not proved Sharkovskii's theorem, but it is a proved theorem right. So, it definitely holds.

So, here we find that this will have no periodic point of any other period. So, the only possible periods of f are 1 2 and 4. So, this case turns out to be really interesting case, and we would be interested in looking into further. At least we should be able to prove some part of it. And that is what we are going to prove here. Now, this is a surprising change of as I said that in 64 when Sharkovskii's proved that there is no period that there is a Sharkovskii's ordering, and all continuous interval maps basically satisfy the Sharkovskii's ordering in terms to of periods. This result was not known. Because it was published in a Russian journal nobody used to read Russian journal, and most of the research in Russia used to be kept confidential.

So, in 1975 when the first definition of course, this is very historically, very important because this was when the first definition of chaos was given. So, in 1975 there were 2 mathematician at university of Maryland. So, they gave this definition they give the first mathematical definition of chaos, and they proved something which can be called a corollary to Sharkovskii's theorem. And that proof is very independent because they did not know about Sharkovskii's theorem. It was only after this paper came up, and then people realize that oh this is a very simple affair and it is such an interesting affair, that then Sharkovskii's theorem was known.

So, we come back to now 75 and we come back to what is basically this was a theorem by Li and York, but we now as of as of now we understand it to be a corollary to Sharkovskii's theorem. So, we shall look at this corollary to Sharkovskii's theorem. This was proved by Li and York, and this corollary says that if my function f from R to R is continuous, and suppose f has a periodic point of period 3, then f should have a periodic point of all other periods, right. This was proved by in fact, as I said that this was the first paper or this was the first instant of a mathematical definition of chaos. We will definitely go into this paper further. Because we want to get into the mathematical definition of chaos, but we will first look into the proof of this corollary, right.

So, simple simply we want to say that, if your function f has a periodic point of period 3, then it has a periodic point of all other periods. So, how are we going to prove this particular theorem? So, we start with say I have drawn this figure here earlier. So, what we have here is that, I have 3 points a b and c. F of a is equal to b. F of b is equal to c. And f of c is equal to a.

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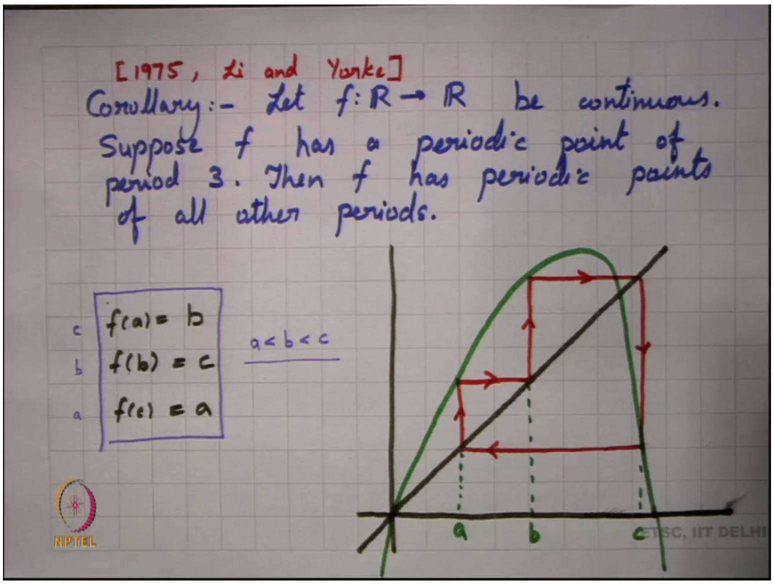

And I have this condition; that a is less than b is less than c. I am just looking into a periodic point of period 3. So, I am just looking into period 3 orbit here. So, period 3 orbit I could have had taken the other way round also. Say my f of a would have been equal to c, right. Then my f of b would have been equal to f of c would have been equal to b, right. And then f of b would have been equal to a, right. We could have at the other way round also.

But if you try to look into what happens in the other way round. So, supposing I had a see here, right. Then f of c would have been equal to b here, and then f of b would have been equal to a here. Supposing I have this particular kind of ordering here then in that case also what I would have had is a same figure. In fact, the mirror image of the same figures, right. No difference here would just have been a mirror image of the same figure, right.

So, what we are trying to say we will saw the other case vacuously follows from this particular case, right. The proof would be identical here even if we look into the other case. And hence we will try to prove it with this particular specific case that a is less than b less than c. F of a is b, f of b is c and f of c is equal to a. So, we look into this particular specific case. And we try to prove this particular corollary, or this particular Li and York's theorem, right in terms of this condition.

So, let us now look back into the proof. We will come back to this figure once again, but let us look into this proof. Now what are ideally we going to use. So, we are only going to use intermediate value theorem for this proof and nothing else.

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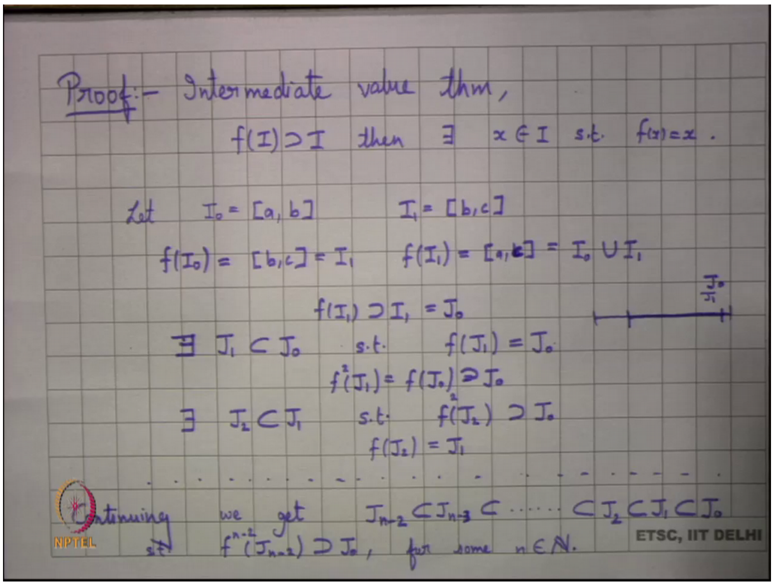

And as I had said earlier also, right let us start with this proof. So, we know that we have this intermediate value theorem, which says that if f of I contains I, then there exist x belongs to I such that f of x equal to x, right. We are just using this part. And now we will see how are we going to utilize this concept over here.

Now, what happens here is; let me look into this graph once again. So, we have this graph once again. Now I am only concentrating on 2 intervals now this function is from real line to real line. So, it is basically an infinite are is basically sort of an infinite interval, but we will concentrate only on 2 intervals. So, I am calling this interval as I 0. And I am calling this interval as I 1. So, I have this interval I 0 I have this interval I 1.

Now, what happens to I 0 and what happens to I 1. So, we know that so, let I 0 be equal to this interval a b, and I 1 be equal to this interval b c. Now what happens to I 0. So, what is f of I 0? So, this is basically b c which is equal to I 1. So, f of I naught is I 1 right, but what is f of I 1? So, this is like b c. Now b is map to cc is map to a right. So, this is basically going to be my interval a b, right. Which is basically nothing but it is I naught union I 1. It is ac and this is I naught union I 1.

Now, we have this. So now, what we have is that f of I 1 contains I 1. So, here my f of I 1 contains I 1. In fact, I 1 is a proper subset of f of I 1 right. So, let me call this I 1 to be this interval. Now I am looking into all intervals in terms of J s right. So, let me call this as my initial interval J naught.

So, this is my initial interval J naught. And we know that f of I 1 contains I 1. So, what we try to do is since f of I 1 contains J naught, right. Then that means that my I 1 will have a subset proper subset, a proper interval or a proper sub interval of I 1. Such that it is image will be the whole of I 1, right. I am just looking into the property of interval property of a continuous function which will take intervals to interval, right.

So, there exists there exists an interval J 1 which is basically a subset of J naught, right. Such that f of J 1 is equal to J naught. So now, I am looking into this part over here. So, let me look into this phase here. So, this is basically my bigger one, this is basically my J naught, right. And within J naught I get some subset here. I do not want to draw that part exactly, but I just want to say that I have a subset here right. So, J 1 such that J 1 is a sub set of J naught, but f of J 1 is equal to J naught.

Now, we look into this J 1 further. Now think of what happens to f of J 1 right. So, f of J 1 is J naught. So, what happens to f square of J 1? This is again f of J naught, which will again be equal which will again be containing J naught. So, this is again going to contain J naught. So, if I start with my J 1, I apply f 2 a 2 times successively. I find that that contains J naught. If it contains J naught it also contains J 1. Since it also contains J 1; that that means, that I should have a subset J 2 of J 1 right. So, basically there exist a subset a sub interval J 2, and now this is not and ordinary subset, I am looking into it in terms of interval. So, I get a sub interval J 2 containing J 1 such that if I look into f of J 2 f squared of J 2, right. That will basically be containing J naught and I can say that this basically maybe I can say that this f of J 2, right we will be equal to J 1, right. And hence f square of J 2 will contain J naught, right. We can think of it in this particular term. So, iteratively we are looking into this aspect.

So, if I continue in this manner, what happens if I continue in this manner? So, continuing what are we trying to do we are trying to prove a periodic point of all periods right. So, we get a J n minus 2 which is basically a subset of J n minus 1 which is again basically a subset of sorry, it is J n minus 3, it is basically a subset of J 2 subsets of J 1 subset of J naught such that f to the power n minus 2 of J n minus 2, right contains J naught.

Now, here we are having this for some n, right. We can iteratively do this for all possible. Hence, now let me look back in to this figure once again, right. Now this is possible for what kind of hence let us try to see here. So, let us look into this figure once again. Now, let us look into this part right. So, what we find here is that f of I 1 contains I 1, right. Since f of I 1 contains I 1 there is a periodic point there is a fixed point of f, right. Contained in I 1 and. In fact, one can easily see in this graph that this is the fixed point here. So, f definitely has a fixed point in I 1.

What happens to say when n equal to 2, what happens in that case? So, we go back here, right. We look into this aspect we are looking into the fact here, that we can make such kind of a continuation, just observation that we make such kind of continuation, right. For some m in N, and now let us go back to what happens for n equal to 1.

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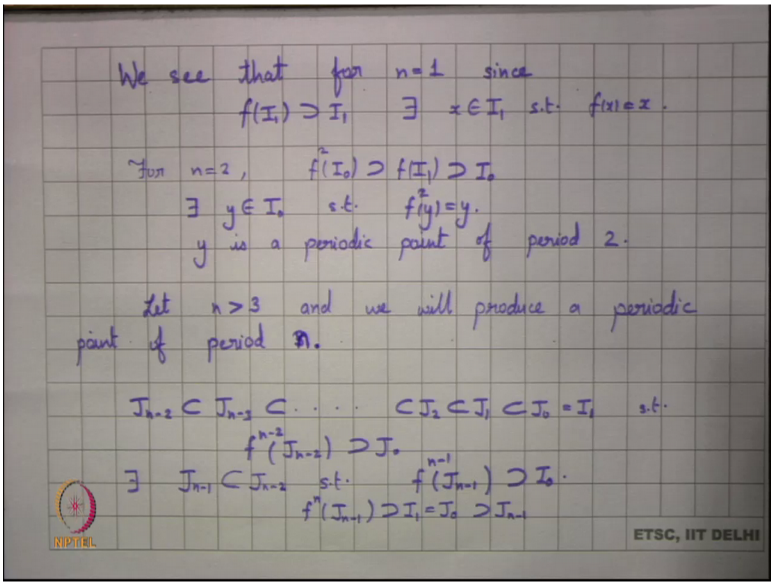

So, we see that for n equal to 1, right. Since my f of I, I 1 contains I 1, right. There exist x belonging to I 1, such that f of x equal to x. So, there is a fixed-point period 1 point contained in I 1. Now for n equal to 2 what happens in that case?

Now, let us look into this fact that f of I naught, right. Contains I 1. So, what happens to f square I naught? In fact, f of I naught is completely equal to I 1. So, what happens to f square I naught? So, that will contain f of I 1, right. And what happens to I 1 f of I 1? It contains both I naught and I 1 right. So, I can simply say that this contains I naught. Now my f square of I naught contains I naught; that means, that f square has a fixed point in I naught. So, that means, there exists y belonging to I naught, right such that f square y is equal to y. But then this y cannot be a periodic point of period 1 this y cannot be a periodic point of period 1 since f of I naught, right. Is basically I 1 right.

So, this y because this y belongs to I naught, this y cannot be a periodic point of period 1. And hence this y has to be a periodic point of period 2 right. So, this y is a periodic point, now we know that y is a periodic point of period 2. So, we have proved the existence of a periodic point of period 1, right a periodic point of period 2. Period 3 is already given to us right. So now, we can assume that let n be greater than 3, and we will produce a periodic point of period 3 of period sorry, n a periodic point of period n.

Now, we already know that we are getting this intervals. So, we will look into what we had done already. So, we have already discussed this part that we get this intervals J n minus 2, right contained in J n minus 3, right contained in J 2, right contained in J 1 contained in J naught which is equal to my I 1, right. Such that my x to the power n minus 2 of J n minus 2 contains J naught, right. We know that thought very clearly

Now, let me look into this parts is this contains J naught, right. I can find now what is f of J naught. F of J naught contains I naught also. So, I can find in interval. So, there exists J n minus 1, right a sub interval of J n minus 2, right. Such that f of f n minus 1, right of J n minus 1 this will contain I naught. So, what we tried to do was now we are trying to we just this is simple iteration what we did, right. Now I am come coming back to the figure again let us try to look into this part. The simple iteration was we started from here we started going into subintervals here. Every time we stayed here, right at the n minus 1th stage we are just moving here, because again this is been mapped here also. So, n minus 1th stage we are moving here, right. And then what happens here we are now again in the nth stage we are again we know that the image of this particular interval is here. So, the nth stage we are again going back here, right.

So, what we have here is f n minus 1 J n minus 1 contains I naught. And so, I can say that f of n of J n minus 1, right. We will be containing I 1 which is equal to J naught. And in fact, my J naught also contains J n minus 1, right. My J naught also contains J n minus 1;

so for f n, right. I find an interval J n minus 1 such that J n minus 1 f of fn of J n minus 1 is containing J n minus 1.

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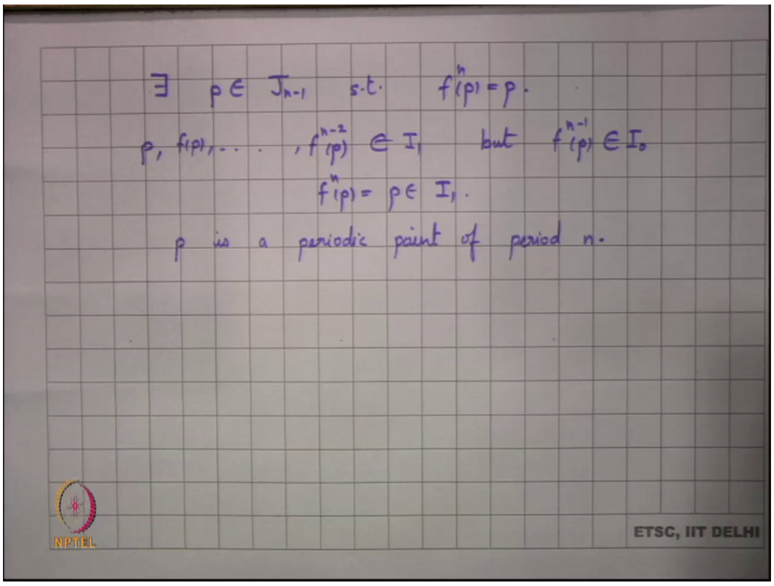

And so, there exist a point p right. So, there exists p or maybe let me push up in the next page. There exists p belonging to J n minus 1, such that fn of p is equal to p. Is this p a periodic point of period n? We said that our n is any number greater than 3, but what we observe is that; if I start with p my p belongs to J n minus 1. So, it is basically a point of I 1, right.

So, my p my fp, right up to fn minus 2 of p, right. This was all belonging to I 1, but my fn minus 1 of p was a I naught. And then again, my fn p is equal to p. So, that means, that this p this point p this since this point p satisfies fn p equal to p, right. This cannot be a periodic point of a period less than n. Because if this was a periodic point of period less than n, then the entire orbit would have been only in I 1, right. It would not have jump to I naught, right. The entire orbit would have been just in I 1, but it jumps to I naught.

And then coming comes back again to I 1. And hence we can say that p is a periodic point of period n. And so, we are proving the existence of a periodic point of period n for all possible n. So, for all-natural numbers periodic point exists, and what we have proved is we are basically proved Li and York's theorem. Or what one can say is the corollary to Sharkovskii's theorem. So, I push this again here what we have proved here and we end lecture today.