**Chaotic Dynamical Systems**
**Prof. Anima Nagar**
**Department of Mathematics**
**Indian Institute of Technology, Delhi**

**Lecture – 04**
**More on Orbits**

Welcome to students. So, today we will be looking more on Orbits. And since we are looking on orbits we will look into the general setting of metric spaces.

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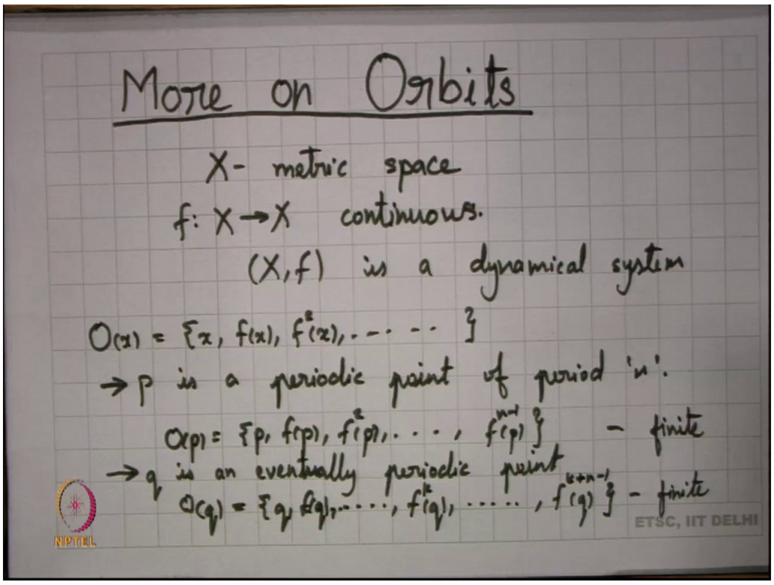


So, for us here x is a metric space, we do not always need it to be compact, but whenever we need it to be compact we will specify that part. And f happens to be a continuous map from x to x. For us, as you all know xf is a dynamical system. And I just want to recall that we have talked about orbit of x, which is basically the set x fx, right f square x and so on.

Now, what happens if my p is a periodic point? So, what happens in case p is a periodic point of period say n. In that case we find that the orbit of p is just p fp f square p fn minus 1 p. And then fnp is equal to p itself. So, we find that the orbit of a periodic point happens to be finite. Now what happens if my q is an eventually periodic point? Then if I look into my orbit of q, my orbit of q will be q f of q say and that goes up to some fkq, and then fkq is a periodic point. So, this goes up to some factor, and then we get k plus n minus 1 q maybe fkq is a periodic point of period n.

So, we get orbit of q to be equal to qfq fkq up to fk plus n minus 1 q. Now if you look into this orbit also, for an eventually periodic point also, the orbit happens to be finite. We can treat our periodic points also as eventually periodic. So, in that case we can simply say that we can classify periodic points or we can define eventually periodic points by those points whose orbit happen to be finite. So, these are having finite orbits. Now we want to look into what happens to this orbit closure.

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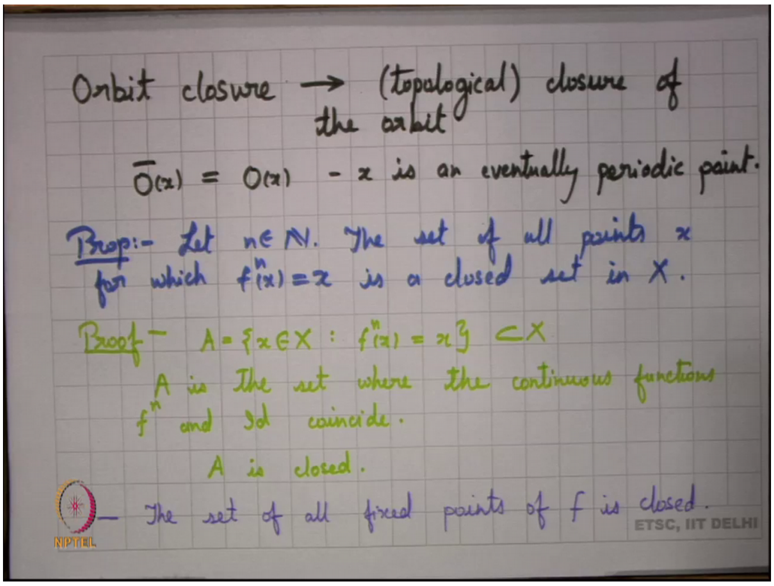


So, we are now looking into another aspect of the orbit. So, what do we mean by an orbit closure. So, if we talk of orbit closure. This is basically the topological closure of the orbit. So, I am talking of the topological closure of the orbit. So, I am looking into the closure of the orbit x. We know that if my x is a periodic point, then in that case since for a periodic point or for an eventually periodic point, the orbit is finite it will be it is own closure because we are now in the setting of metric spaces. So, we find that orbit of orbit closure of x is same as orbit x, in case x is an eventually periodic point.

We are now interested in looking into, what can what more can we say about this orbits. So, what we find here is; I am writing down a simple proposition here. So, let me take any integer any natural number n. So, n is N, then the set of all points for which fnx is same as x. So, the set of all points x for which fnx is same as x right is A close set close set in x. The proof of this is very simple. So, we want to look into the proof of this part.

What are all these points? I am looking into the set of all x in x. So, I am calling my a to be the set of all x in x such that fnx is equal to x.

So, what is my A? If I look into that part A is simply this A is a subset of x right. So, A is the set where the continuous functions, I have one continuous function which is fn that is composite of fn times and identity. So, this is my identity function. So, we are looking what is our a our A is the set where all continuous where the continuous functions fnn identity, right they coincide. When these sets coincide wherever 2 continuous functions coincide, we are now in the setting of metric spaces. So, we know that such a set will always be closed right.

So, A is closed. Now if I look into this aspect, right I can simply say that as a corollary that if I look into any set, right. The set of all fixed points is closed. So, the set of all fixed points of f is also closed, because it is just corollary to this aspect. We again look into this proposition once again. Our claim is we are looking into whole points where fnx is equal to x. What are all the points where fnx is equal to x; of course, periodic points of period n.

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But also, we are looking into periodic points, whose period is divisible or whose period basically divides in right.

So, we are basically not looking we are looking into a collage of periodic points, right. Not exactly periodic points of period n, we are looking into a collage of periodic points. All periodic points who is basically whose period divides n including n itself. So, what happens? If we are now exactly looking into periodic points of period n, will this proposition always hold? Do you think that this proposition will always hold? So, we look into a very simple example here.

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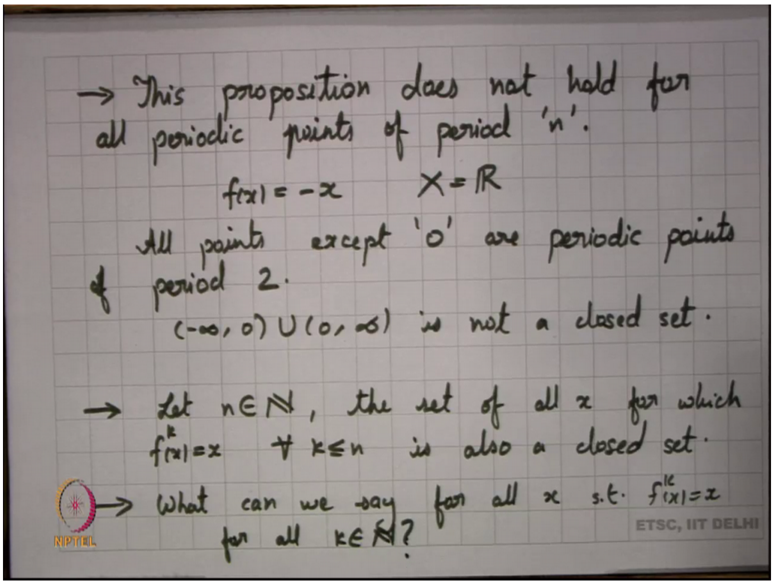


This proposition does not hold for all periodic points of period n. Simple example here I can take up my fx to be equal to minus x. Where I am working on the real line, right we have already seen this example earlier.

What happens here is that all points, right all points except 0 are periodic points of period 2. We have seen that earlier also. So, all points except 0 are periodic points of period 2, but what are all these points? So, these points are basically minus infinity to 0 union 0 to infinity all these points are periodic points of period 2, but we very well know that this is not A closed set right. So, this is not A closed set. I can also look into one more corollary here, supposing I again take up my n belonging to n.

The set of all x, for which fkx is same as x, for every k less than or equal to n, right this is also A closed set. So, basically, I am looking into again, we have this proposition saying that if fnx is equal to x for a particular n, right. The set of all such x satisfying that is A closed set. So, what we are trying to look into? We are just looking into finite union of closed sets. And so, the set will always be closed. But what happens now so, this was like finite union of closed set. What happens in the case of say union of all such points?

So, what can we say such that fkx equal to x, right for all k in the set of natural numbers. What can we say about this factor? Essentially, I am looking into the union of all periodic points. Because every point right which satisfies fkx equal to x is definitely a periodic point of some period. So, I am basically looking into all periodic points. So, what happens in case of all periodic points? I am looking into union of all periodic points. Does it form A closed set? So, let us try to look into some example here.

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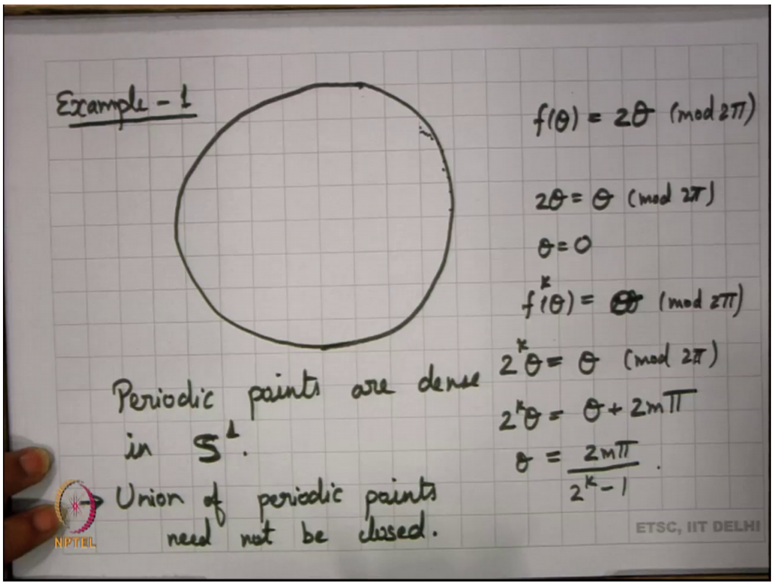


Now, I am again going back to my circle example. And I am looking into this function, we have already discussed it earlier f of theta is 2 theta we go back to this example once again. Now what is a fix point here? See if I want to look into what is a fixed point because this is like 2 theta mod 2 pi I am looking into a mod 2 pi functions right. So, this is 2 theta mod 2 pi, right I am saying that fixed point will be only those points for which 2 theta is equal to theta mod 2 pi. And the only point which satisfies this would be my theta equal to 0.

So, this is the only fixed point here, but does it have periodic points. So, if we want to look into periodic points say of period k, I am interested in looking into all points such that fk theta happens to be equal to twice theta. Of course, mod 2 pi sorry fk theta equal to theta mod 2 pi. These are all my periodic points of period k. Now when I am looking into fk theta equal to theta, basically means that I am looking into the equation 2 to the power k theta is equal to theta mod 2 pi. I am looking into mod 2 pi I can just remove mod 2 pi. And I can simply write the equation 2 to the power k theta equal to theta plus 2 m pi. M can be any integer right.

So, 2 m pi. Now if I look into this equation, if I want to solve this equation, this equation turns out to be theta equal to 2 m pi divided by 2 to the power k minus 1. Let us look into all theta which satisfies this particular form. So, we find that theta with this particular form because my k can be anything, right because k we just started with an arbitrary k. So, periodic we are looking into periodic points of period k. So, we start with any such arbitrary period. We find that for different values of m, right and different values of k, these points will be dense in the circle. Every interval will carry one such point, right. I take any arc here, it will always carry one such point.

So, these points so, we can say that for this map, right the periodic points are dense in our set S 1 right that is what my space was. So, the periodic points here are denseness one, and hence if I say that the union of periodic points, right the union of periodic points need not be closed. So, the union of periodic points, I can simply reduce here that the union of periodic points need not be closed. So, I hope this is clear to all of you, but here when we look into this part, union of periodic points, right is it always true that it is not closed.

Let us again look back to another example. So, this was maybe I can say that this was our example one. Let us look into another example here.

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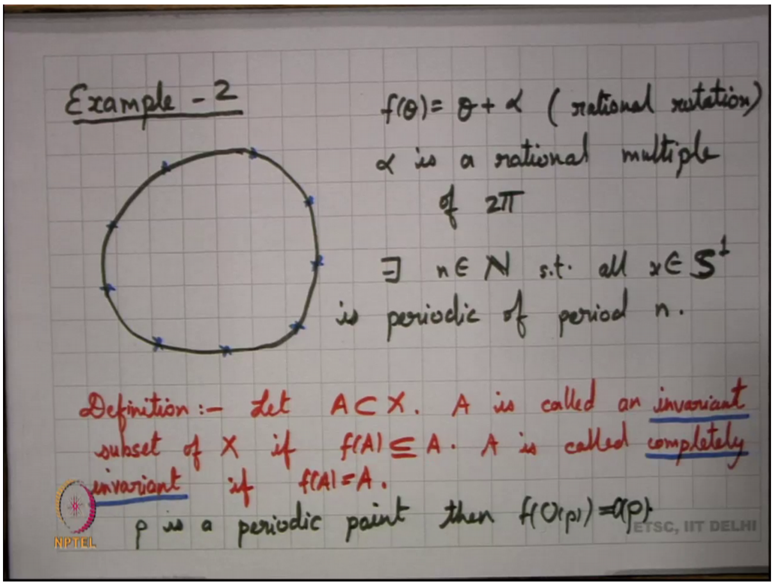


And again, I am going back to our circle. So, again let us look into our circle. And now I am looking into this mapping, I am looking into this isometry, which was f of theta equal to theta plus alpha where alpha is a rational multiple. So, basically this is a rational rotation. Now we are looking into a rational rotation now. And we have all seen this example earlier, that for this example all points are periodic there exists an n such that all points of periodic of period n.

So, here there exist an n belonging to n such that all x belonging to S 1, right is periodic of period n. So, all points so, if you look into the union of periodic points everything is a union of periodic points, and hence your union of periodic points happens to be A closed set here. In fact, all the points here this is a very special example where all the points are periodical. So, we want to now look into another aspect, where we look into some definition now here. So, we are looking into something else, now this is this example gives us some kind of motivation, that we can look into something more than periodic points.

So, what more can we think of periodic points. So, we start looking into this definition. And so, I am simply taking this as a definition. Let A be a subset of x. A is said to be invariant subset.

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A is called an invariant subset, if f of A is a subset of A. So now, I can think simply say that instead of now looking into fixed points of periodic points, now we are looking into fixed subsets or periodic subsets right. So, we say that we start with a very small case of we start with just an elementary case here, that we say that A is an invariant subset of x if f of A is a subset of A.

In fact, we say something more, A is called completely invariant. If f of A is same as A. Let us try to see what happens in our example here. So, we are looking into an invariant set here. And we are looking into what do we mean by a completely invariant set. In this particular example, right we have the whole set is invariant, in this fact x is always invariant because f of x is always a subset of x, right. In this case it is also completely invariant f of x is equal to this part, but can I have something more here.

So, supposing now I am looking into say alpha was we know that every orbit here is a periodic orbit, right every point is a periodic point. So, you can find some kind of a periodic orbit here right. So, we find this factor. And we find that this orbit, if I just look into this orbit, points from here just move along themselves right. So, here we find that a single orbit if I take a single orbit here, the single orbit turns out to be completely invariant. And hence we can simply say that fine this was a periodic orbit right. So, for a periodic orbit, we have this general case that if I have any p is a periodic point, right. If p is a periodic point is equal to orbit of p, f of orbit of p is equal to orbit of p this is completely invariant.

In fact, periodic points can also be characterized by this fact. Because let me try to take up an orbit of some simple point x.

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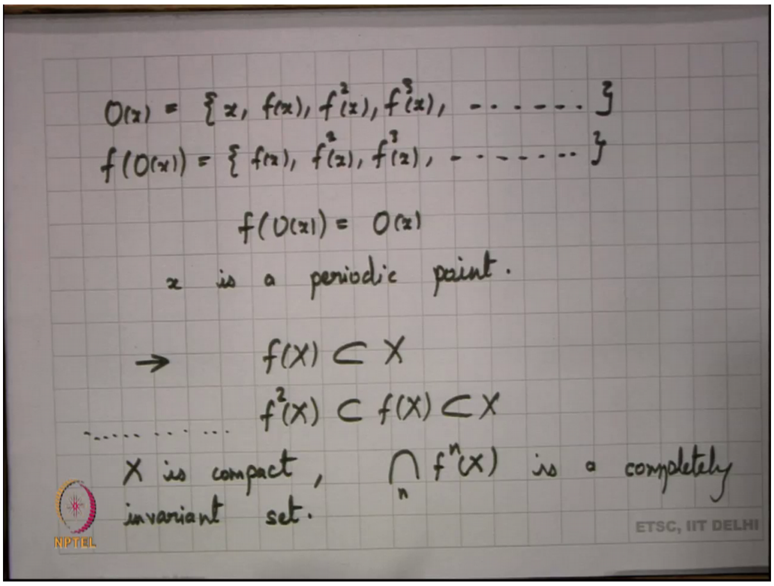


So, let me look into orbit of x here. So, how do how have we defined the orbit of x? It is x fx f square x f cube x, and so on what is f of orbit of x if I term. So, supposing my f of orbit of x is same as orbit of x; that means, this point x which was missing here, means somehow somewhere further come up here. If it is somehow further comes up here; that means, my x is the periodic point right.

So, that gives me that x is a periodic point. So, I can characterized my periodic point as those points for which their orbit happens to be completely invariant or before that let us look into another fact here. We know that f of x is always a subset of x, right. Let us look into this fact f of x is always a subset of x. So, in that sense I can say that, f square x will also be a subset of fx. And this is a subset of x. So, in that way I can continue this, right saying that fnx will always be a subset of x. So, if I look into now I am looking into this fact, that if my x is compact we had not taken this condition on x earlier.

So, if x is compact, then I can look into all these invariant subsets fnx, right. They form a chain so; the total intersection will be non-empty. So, if I take this intersection of fnx. This is always a completely invariant subset just the finite intersection property. So, this is a completely invariant. So, you start with say if your x is compact. There is always a guarantee that there will be completely invariant subsets. Of course, we cannot guarantee given any set or given any space right given any dynamical systems, we cannot guarantee that there will be periodic points, but for a compact metric space you can always guarantee that there will be completely invariant sets.

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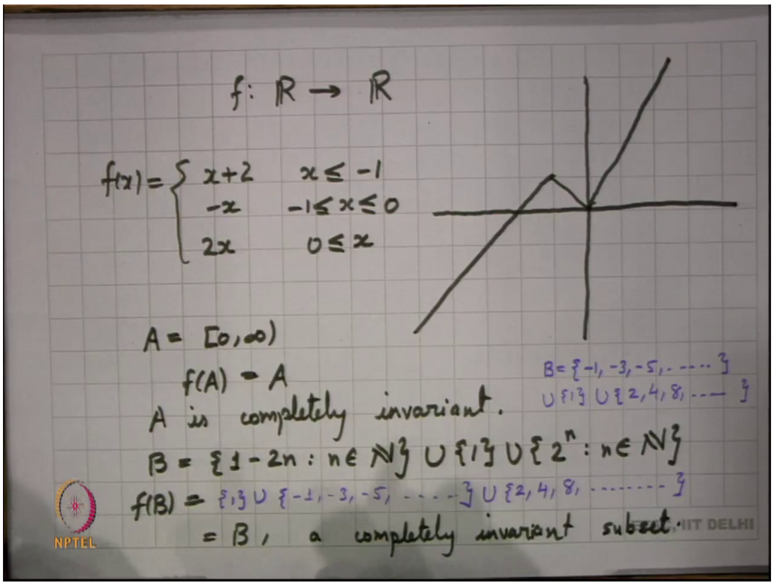


Let us look into another example this time. I am giving an example on the real line. So, my f is from R to R. And I am defining this f as say f of x equal to. So, this is x plus 2 for all x less than or equal to minus 1. I am defining this to be minus x for say x lying between 0 and we define this to be 2 x whenever. So, 0 is less than or equal to x. So, let us try to graphically understand this function.

So, what happens here is that; this becomes at x equal to minus 1, the value happens to be equal to 1. So, this is x plus 2, whenever x is less than or equal to minus 1. It is minus x whenever it is between minus 1 and 0. And it is 2 x, right whenever it is greater than equal to 0. What are the periodic points for the set? We find that this set has 0 as a fixed point right 0 is a fixed point. Do we expect anymore periodic points here? I leave it for you to think about it, now I want to look into something else. So, let me take the set A to be equal to 0 infinity. Now this is A close set. What can you say about this f of A. What is f of A is same as A right.

So, your A is completely invariant. Let us now look into this set B. Now this set B is very typical set. I am defining this set B as 1 minus 2 times n for all n in N union. I want this point one very special point for me, union the set of all 2 to the power n for all n in N. I want you to tell me what is f of B. I should understand what B is maybe I will write it down typically here what is my B. So, my B happens to be if I am trying to write it down, this happens to be 1 minus 2 n for all n in N right. So, this is like when n equal to 1 this becomes minus 1, right then I have minus 3 I have minus 5 and so on.

So, basically, I am looking into the negative of odds right union I have 1, right union I am looking into this point again 2 to the power n. So, what I have is I have 2, I have 4, I have 8, and I have all the powers of 2. This is my set B. Now I want to look into what is f of B. So, I look into f of B by looking into the first part. Now the first part consists of all points which are less than or equal to minus 1. And my function there happens to be equal to x plus 2. So, if I look into my first part, right where is my first part being mapped into. So, my first part here will be mapped into minus 1 is mapped to 1 right.

So, I have this one here union. I am looking into the first part, where is the first part being mapped to. So, minus 3 is mapped to minus 1, right. Minus 5 is mapped to minus 3 minus 7 is mapped to minus 5 and so on. So, my second part will be basically mapped to this set minus 1 minus 3 minus 5. So, on what happens to 1, right where is 1 mapped to. So, we know that when I am looking into 1 it is lying greater than 0. So, greater than 0 I have the map to be equal to 2 x here. So, this gets mapped to 2, right 2 is mapped to 4, 4 is mapped to 8. And what I get is I retrieve all the powers of 2 here.

So, ideally what is B? What is f of B? It is same as B. So, B is also a completely invariant subset. Now this is typical analysis which we can try to do. What happens if I am looking into an invariant subset?

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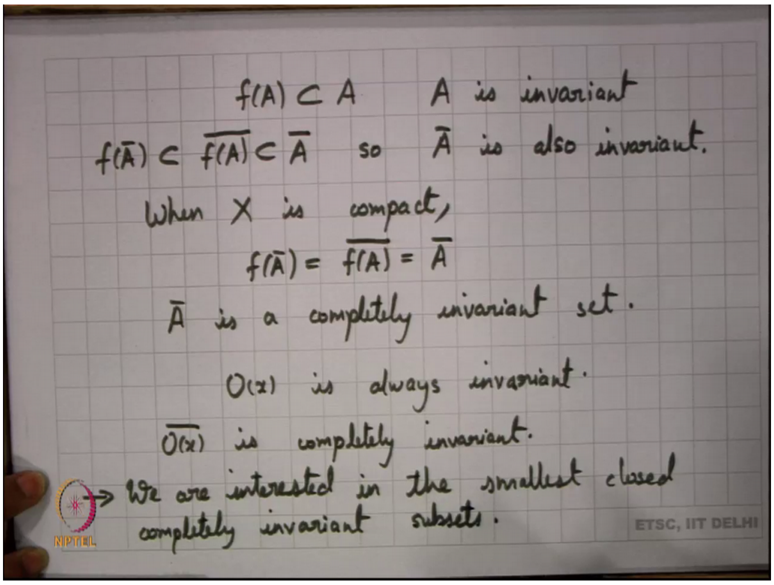


So, if I take f of A to be subset of a when we say that A is invariant, right what happens to the closure of A? So, if I look into what is f of A closure, now since my mapping is continuous, right this will be basically a subset of f of A hole closure, but what is f of A hole closure, that will always be a subset of A closure.

If I have a to be an invariant set, it is closure is also it is topological closure is also invariant. So, A closure what happens when x is compact? Now when my x is compact I know that A closure is also compact. So, I am looking into this condition f of A closure, right that will be also be a compact set. Since being a compact set it is also closed. So, f of A closure is a subset of fa whole closure, but f of A whole closure is closed. It is A closed subset of A closed set, what happens in that case? F of A closure is equal to f of A whole closure, right and f of A whole closure is basically equal to A closure.

So, my A closure is a completely invariant set. So, one thing is sure here, that when I talk of any orbit x, right my orbit x is supposing I am in a compact space, or in that sense if I am not in a compact space if I only assume my a to be compact, right. Even in that case your A closure happens to be a completely invariant set right. So, for compacting A closure is a completely invariant set in particular I am looking into my orbit of x. So, my orbit of x is always invariant in a compact metric space ordered x closure is completely invariant. And in case my x happens to be a periodic point orbit x closure happens to be a compact set right.

Basically, it is a completely invariant set, right. We can just reduce from this part. So, orbit x closure is completely invariant. So now, we are trying to look into all subsets of x, right such that we these are the smallest closed invariant subsets. So, we are interested in what do we say about this set. So, how do we define this set? So, I know that my x is definitely invariant right. In fact, if my f is on to I can say that f is completely invariant. So, we know that x is invariant we can always find an invariant subset of x, right. If I take an invariant subset supposing say my A is invariant.

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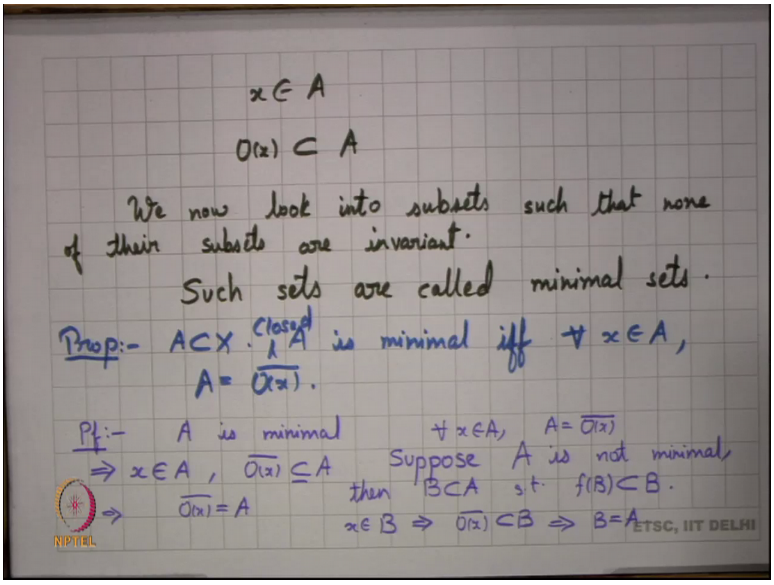


And if I take any x belongs to A, then I know that orbit of x will always be a subset of A because A is invariant.

Orbit of x is a invariant subset which is always contained in the invariant subset A. Now I am interested in looking into those subsets such that none of that none of their subsets happen to be invariant. So, we are looking into subsets, we would to look into that part this is the smallest possible invariant set, in the sense we want not we can not use the word smallest in possible. You can say that this sets how with this sets will have the property that they are minimal with respect to be invariant. Nothing smaller than them can be invariant, right they are minimal with respect to being invariant.

And so, we call such subsets as invariant. So, such sets are called minimal sets, what do we mean by minimal sets is again they are having mean they are minimal with respect to being invariant. No proper subset can be invariant, they are minimal with respect to be invariant. So, we start with the preposition here, let A be a subset of x, then we say that A is minimal if and only if for every x belonging to A, A is the same as orbit closure of x. In particular I can say that if x itself is minimal, it means I do not have any proper subset of x, which is invariant, then x will be equal to the orbit closure of each and all of every of it is point.

And the proof of this is very, very simple here. So, if I try to look into the proof of this fact right. So, what happens now? If I assume that A is minimal, what does that imply? If x belongs to A, right of course, I want here something else x A is a subset of x my A is closed here right. So, I am just looking into this fact closed a right. You looking into closed subsets. So, x is in a right; that means, my orbit closure of x is also a subset of A. Now my orbit closure is always a complete invariant set, but A is minimal. So, orbit of closure of x has to be equal to x right. So, this is a subset of A, but then this has to be equal to x because A is minimal.

So, this implies that orbit closure of x is equal to a it cannot be smaller than a it has to be equal to A. On the other hand, if for every x in A, A is equal to orbit closure of x, what does that give you? Supposing I am assuming that A is not minimal, right. Then that means, that suppose A is not minimal, then I have a B subset of A. Which is minimal or B subset of a which is invariant. So, a B such that f of B is a subset of B, but f of B is a subset of B what does that mean? I take any point in B x belongs to B right. So, if x belongs to B that would imply that, orbit of x closure when I am taking B to be again A closed subset of A, orbit of B closure is contained in B. But what is the orbit of A closure? It is same as n right. Then B we have already taken to be subset of A. So, this implies that B is nothing but equal to A. So, that means, that there is no subset of a proper subset of A which can have this property of being invariant. And so, A is minimal right.

So, we can prove that A is minimal. Let us try to take here an example of a minimal system. So, we take us example where the full x is minimal. In fact, from here you can always guess that, if I look into periodic points if I look into periodic orbits, right they are all minimal subsets. Because if I look into one periodic orbit, right. There is no proper subset of it which is invariant. So, periodic orbits are always minimal subsets. And now we look into the case we look into the example where the full space x happens to be minimum. I am not going further I am not looking into any example which we have not discussed earlier.

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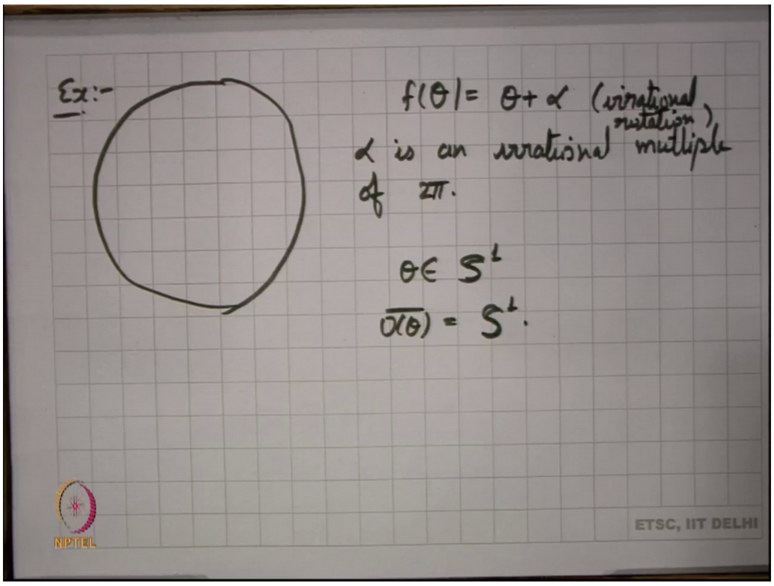


So, let us try to look into backend of a circle. So, I lost count of what number example was it. So, I am just saying that this is my example. So, you look into this example, where my theta f of theta happens to be equal to theta plus alpha. Where alpha is an irrational multiple. So, I have this theta plus alpha, alpha is an irrational multiple of 2 pi. So, this is basically my irrational notation. Now I am having an irrational rotation here. And for this irrational rotation, we have already seen this example earlier; that for this rotation every point of the orbit of every point is dense right.

So, for all theta belonging to S 1 we have seen that orbit of theta closure is whole of S 1. We have seen this fact only orbit of theta is dense here. So, this is typically an example of a minimal system. So, this system is minimal.

So, we stop today here. If you have any difficulties or something maybe we can discuss that.