Chaotic Dynamical Systems
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Lecture – 34
Chaos in Toral Automorphisms

Welcome to students. So, in the previous class we had looked into toral automorphisms. We had seen some properties of toral automorphisms like what are the basic periodic points, and also that they are topologically transitive. So, as such toral automorphisms tend to be chaotic systems. But today we will be looking into more aspects of chaos of toral atomorphisms. Now since this is some kind of a homeomorphism in a 2-dimensional space. So, this is this forms a very nice example of a chaotic system in 2 dimensions.

So, let us now recall toral automorphisms. And we can look into the general definition of toral automorphisms.

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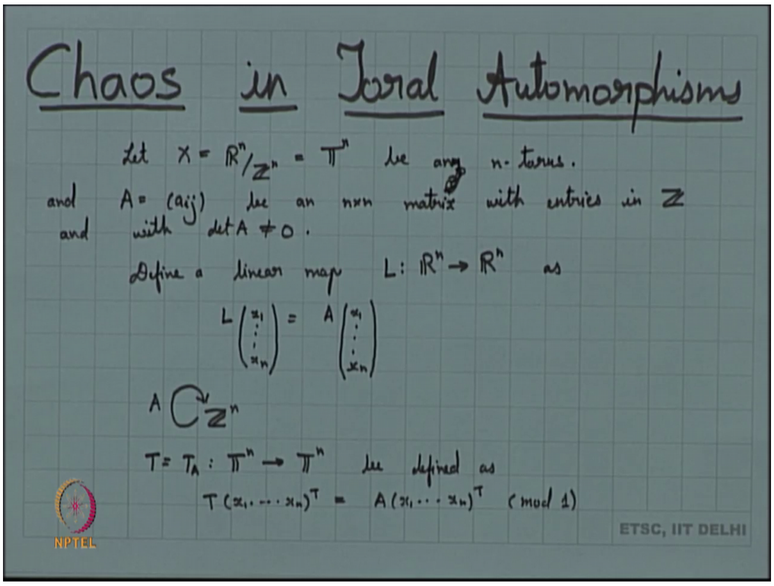

So, let x be say R n upon Z n. So, it is basically T n denoted as T n be any n torus or be an n torus, because there is only one n torus. So, this is an n torus for any n you can think of any n in z a n greater than 1. And let A equal to A ij be an n cross n matrix with integers with entries in z, and with determinant of A not equal to 0. So, this is an invertible matrix that we are looking into with integer entries.

Now, we can define for such an A n for R n we know that any matrix will give us a linear transformation on R n. So, define a linear map say l from R n to R n as l of you have this vector x 1 x 2 xn is basically a of x 1 x 2 xn. Now you know that very well any linear transformation in a final dimensional space can be given in terms of a matrix. Now since a is an integer matrix it maps z onto itself. So, A maps I should say that if I have Z n right, A maps Z n to itself, and since A map Z n to itself, we can think of A as allowing us to define a linear transformer allowing us to define a transformation on the n torus.

Let T equal to T A, right from T n to T n, we defined as T of say I have x 1 x 2 xn transpose is same as a of x 1 x 2 xn transpose mod 1. So, this gives us so, we look into this particular mapping and we first check that this mapping is well defined.

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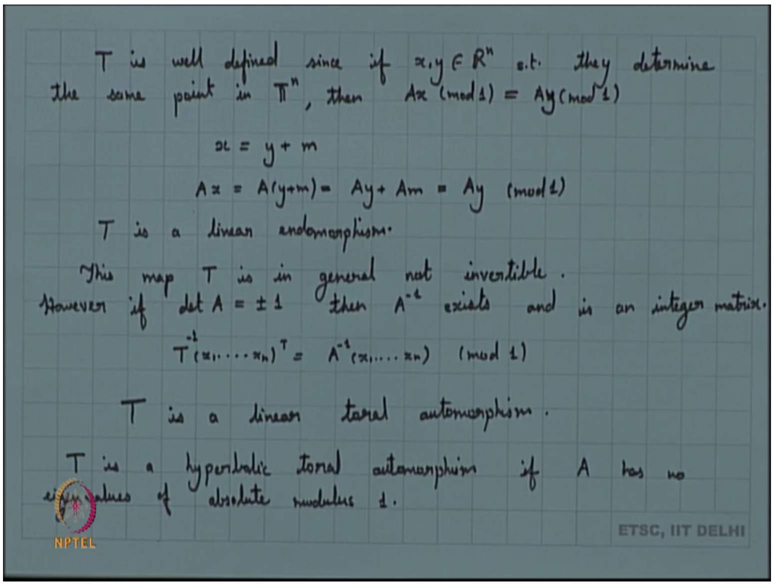

So, T is indeed well defined. Since if i have x and y belonging to R n such that they determine the same point in T n, then A x mod 1 will be same as A x A y mod 1. The reason here is supposing x and a x and y are 2 points in R n which determine the same point in T n; that means, I can write my x as say y plus some integer m, and hence they will be basically since these they differ by an integer right.

So, they give you the same point in T n. And then you apply a on both the sides, you get A x equal to a of y plus m, which basically happens to be equal to A y plus A m and if you look into am this is all having integer coefficients right. So, if I try to take mod 1 right. So, this becomes A y mod 1 right. So, this disappears in that. So, if 2 points determine in R n determine the same thing. So, this basically this mapping T is well defined. So, this mapping T is a linear endomorphism. So, T is now a linear endomorphism, but if we look into general T right, T need not be invertible. The reason is the way we have defined T right, T heavily depends on A and the entries of a are all integers right. That is what is actually helping us in saying that this a right is basically mapping points of T n into T n.

But if we try to look into T inverse, right T inverse need not be i mean T need not be invertible, because T inverse would basically come up from some matrix A right, if you are looking into the linear transformation. It would come up some some matrix A which need not have integer coefficients, but the scenario completely different if we take determinant of A to be plus or minus 1, right. In that case your A inverse will also have integer coefficients or sorry integer entries. So, we see that this T this map T is in general not invertible; however, if your determinant of A is plus or minus 1, then A inverse exists and is an integer matrix.

And what can you say about T inverse? Then my T inverse x 1 x 2 xn can be written as A inverse. So now, we have an invertible endomorphism right. So, we can say that our T is a linear toral automorphisms. So, T is a linear toral automorphism. And we now want to look into whether this is a hyperbolic toral automorphism or not, and we have seen this earlier. So, we say that T is a hyperbolic toral automorphism, if A has no eigenvalues of absolute modulus value one. So, in that sense we say that T happens to be a hyperbolic toral automorphisms. And as we had seen in the previous lecture, that a hyperbolic toral automorphism is devaney chaotic.

What we will try to see is again we will try to look into the chaos of a hyperbolic toral automorphisms, what we will try to determine is what is the topological entropy for a hyperbolic toral automorphism. In fact, what we are not going to look into is that a hyperbolic toral automorphism can also be conjugated with say if i have an n torus then a hyperbolic toral automorphism can be conjugated with the symbolic system on in n symbols, and that gives you like lot of properties because you can think of lot of properties over there because again we know that our symbolic system is some kind of A invertible system, right. And if you are looking into a sub shift of finite type we know that it has various other properties.

So, we can think of conjugating this to in n dimension is in in a symbolic system in n words in an a let, sorry alphabet of set n. But we are not going to get so much into details here. All we will look into what is the topologically entropy of a toral automorphism. Now as such for any dimension we can calculate this part, but it becomes easier to demonstrate it in case of 2 dimension. Our eyes are more used to seeing 2 dimension.

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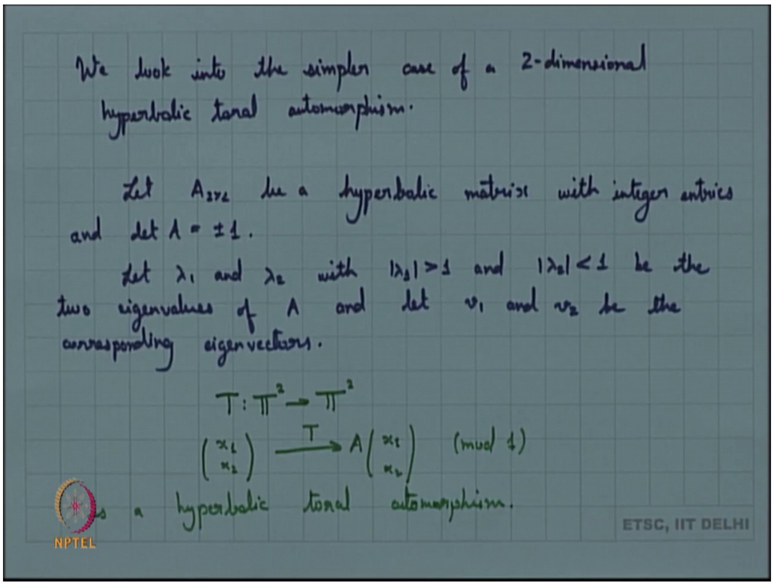

So, what we will do is we will look into the simpler case of a 2-dimensional toral automorphism. So, we start with a simpler case. So, we look into the simpler case hyperbolic toral automorphism. For that we need some let us build up some notations. So, let A cross A sorry a 2 cross 2 B of matrix.

So, let it be a hyperbolic matrix with integer's entries. And determinant of A is plus or minus 1. Let lambda 1 and lambda 2 with mod of lambda 1 greater than 1 and mod of lambda 2 less than 1 be the 2 eigenvalues of A. And let v 1 and v 2 be the corresponding eigen vectors. So now, as we have seen what happens here is that you have T from T square to T square defined as say you have x 1 and x 2 and under T they are mapped to A times x 1 and x 2 mod 1.

So, this is a hyperbolic, toral automorphism. Now we will try to see a theorem here, and that is what we are going to do in this particular class.

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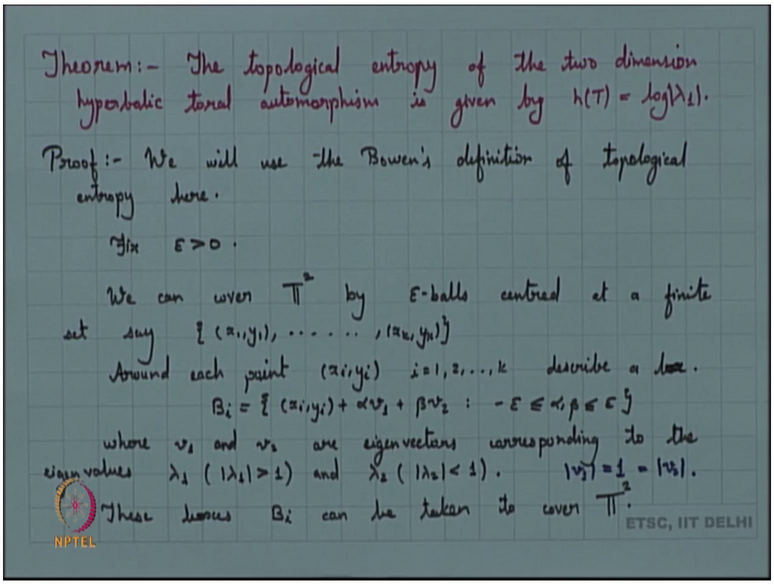

So, we look into the theorem, the topological entropy of the 2-dimensional hyperbolic toral automorphism is given by log of lambda 1 or I should say log of modulus of lambda 1. So, the topological entropy of a toral automorphism is log of mod lambda 1. And since lambda 1 happens to be the larger eigenvalue of a toral automorphism. We know that the topological entropy is positive and hence the hyperbolic toral automorphism is chaotic.

So, let us now look into the proof of this part. Now we know that we can start with our topological entropy we can start the definition with either of the 2 definitions we can either use the Adler's definition, since this is a torus is a compact metric space, we can either use the Adler's definition or we can use a Bowen's definition. But here it will be easy for us to use the Bowen's definition. So, we will use the Bowen's definition of topological entropy.

So now for Bowen's definition, all we need to find out is some n epsilon separated sets. So, let us start working it out. So, we first fix an epsilon. Now we can cover T square by epsilon balls centered at a finite set, say my finite set is x 1 y 1 xk yk. Now when we are looking into epsilon balls this is our T square is gone factor i. So, it will have a finite epsilon net, but when I am looking into covering it by epsilon balls. I am defining my balls not using any matrix. So, as a different kind of metric that we are trying to use over here or I would say that the definition of balls here is slightly different from what you could see as an ordinary say ball of radius epsilon centered at this points.

We are taking a slightly variation we are taking some variation in the definition. So, how do we define a ball here is; see, around each point x 1 y 1, say around each point xi yi for each of this I we define a box. So, my box is Bi which is basically xi yi plus alpha times v 1 plus beta times v 2, such that minus epsilon is less than or equal to alpha beta is less than or equal to epsilon. So, this is a kind of box that we are using. So, one can think of this in terms of say, we now have at each point we have 2 directions, right. We have this linearly independent directions v 1 and v 2, and along is each of these directions what we are trying to do is we are trying to take minus epsilon; epsilon around one direction minus epsilon; epsilon on another direction, that gives us a kind of a box.

And we can think of very well we can think of this to be defining our open spheres, right. And as we know that our space is compact, there are finitely many of them which cover the whole torus. So, here my v 1 and v 2 are eigenvectors corresponding to the eigenvalues. Now we keep in mind that when I am talking of lambda 1 my mod lambda 1 is greater than 1 and lambda 2. So, here in my mod lambda 2 is less than 1. We know that these boxes right. So, we can think of this as an epsilon net x 1 x 1 y 1 xk yk. And so, these boxes Bi can be taken to cover T square, right.

So, how many number of boxes you have? You have number of boxes your boxes are covering here T square they are covering your torus. So, let us try to see how a typical box is.

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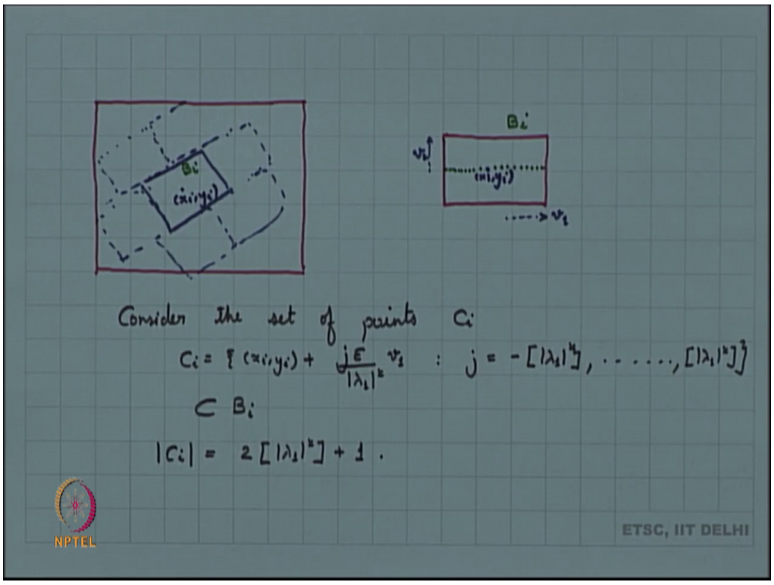

So, say we are looking into so, for example, this is our unit square. And in this unit square we are typically having supposing, this is my point xi yi, then I have 2 directions and in both these directions. I am looking into minus epsilon epsilon. So, this is basically giving my box Bi. So, this is my box Bi and then along any every such point, right. I would be having some such boxes right covering the whole of torus, right. We can think of such boxes covering the whole of torus.

So, these boxes they cover the whole of torus. And we have a typical box Bi where you have one direction. So, you have this longer direction basically in the direction of v 1, right. The shorter direction is basically in the direction of v 2. So, typically I can write my Bi. So, supposing this is the box Bi. So, this is basically the box Bi, and we have this point x 1 y 1 lying over here xi yi lying over here. So, you have xi yi lying over here, and we know that this is basically the direction of v 1. And this is basically going in the direction of v 2, right. And we may not have seen this, but what we will try to do here is, when we take our very eigenvectors v 1 and v 2.

Let us also assume that the magnitude of v 1 is 1. And so, is the magnitude of v 2. So, basically let us take our eigenvectors to be unit vectors right. So, we can start with our eigenvectors to be unit vectors. So, that we do not have to worry about whatever length it is right this. So, these are unit vectors; however, for us what is important is the direction. So, we start with them our eigenvectors to be unit vectors, we have this boxes Bi and now let us consider the set of points Ci. So, consider the set of points Ci, where i am defining my Ci to be the set of all points of the form xi yi plus j times epsilon upon mod of lambda 1 to the power k in the direction of v 1, such that your j lies so, j takes all the values between you have minus now you are looking into. So, you have mod of lambda 1 to the power k. You are looking into the integer value of mod of lambda 1 to the power k. So, from minus integral part of mod of lambda 1 to the power k to the integral part of mod of lambda 1 to the power k, you have these many points. So, these are basically 2 k plus 1 points, right. So, you are looking into these points here. Now sorry, not 2 k plus 1 this would be exactly 2 times mod of integral part of lambda 1 to the power k plus 1.

So, you are looking into these points we are considering these points Ci. Now what are this points Ci over here if I want to see that. So, basically in the direction of v 1 right we have this points here right. So, these are the points in the direction of v 1. So, basically, we are looking into these points. So, our Ci basically typically is a subset of Bi it is in the box. So, all this points Ci are from the box Bi and what is the cardinality of Ci. So, looking to the cardinality of Ci. The cardinality of Ci turns out to be twice the integer value of mod lambda 1 to the power k, right plus 1.

So, is basically the cardinality of each Ci. Now we want to claim something.

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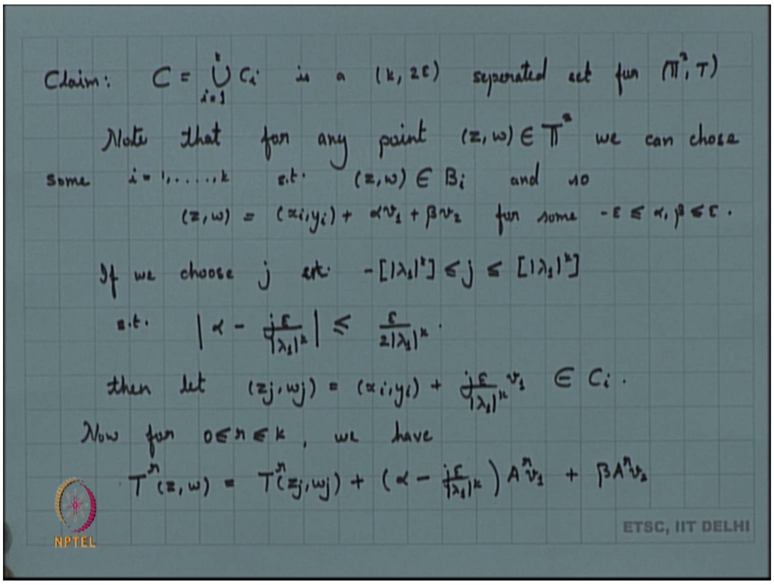

So, what we claim here is take C to be the union of all such Ci. So, looking into Ci with i going from one to k. So, C is a k twice epsilon separated set for T square T. So, want to be claimed and this happens to be a k twice epsilon separated set to see to we first prove this claim. So, we note that for any point z w and T square we can choose some, i say we know that we have this k boxes right. So, basically our z w will be lying in one of these boxes right.

So, we can choose some i such that your z w lies in this box Bi because Bi is cover the torus. And so, since the property of vi is it contains all those points which are at a distance alpha times v 1 plus beta times v 2 from xi yi. So, we can say that your z w is xi yi plus alpha times v 1 plus beta times v 2 for some alpha beta which lies between minus epsilon and epsilon. So now, we are fixing our alpha and beta also, depending on this point z w. If we choose our j such that we are choosing your j between minus lambda 1 to the power k integer value of mod lambda 1 to the power k less than or equal to j less than or equal to integer value of lambda 1 to the power k.

If you are choosing your j between this part right, or I should say in in this range right, such that your mod of because you have already fixed an alpha here. So, your mod of alpha minus j times epsilon upon mod of lambda 1 to the power k, if I am looking into this modulus. So, this modulus is less than or equal to epsilon by 2 mod of lambda 1 to the power k. Supposing, I want to choose this j such that this inequality is satisfied. Now if you look back into our figure, this you can always find such a j, right. Because your z w lies somewhere over here right.

So, you have these points j. So, these are basically a points j right. And since they will be lying somewhere over here, I can choose that particular value of j for which this inequality is satisfied. So, this is possible. And then we will let now I am taking my point z j wj to be equal to this xi yi plus j times epsilon upon mod lambda 1 to the power k times v 1. So, if I chose this part, then this point with this particular coefficient this j here, right. This will be basically a point of Ci. What remains to see is that this happens to be a separated set. So, take any z w in Bi, right you have some j right for a any z w in Bi you have Aj for which this inequality satisfied. And in that case, you can always choose a point zj wj right in Ci which is basically having, which has basically this equation right. So, it is basically your point zjwj can be fixed, right for a given j, once you know what is your z w.

So now given a z w basically given as z w we are choosing a zj wj in Ci, or in C. Now since your z w could be in any Bi, right for any z w in the torus, you have a point here fixing a point of C. Now for what happens is 0 less than or equal to r less than or equal to k, what happens here is we have Tr of z w is basically Tr of zj wj, right plus I have alpha minus j times epsilon upon mod lambda 1 to the power k times a to the power r v 1 plus beta times a to the power r v 2. So, this is basically my Tr z w. And hence if I am looking into this equation. So, I want to say that this is basically this equation.

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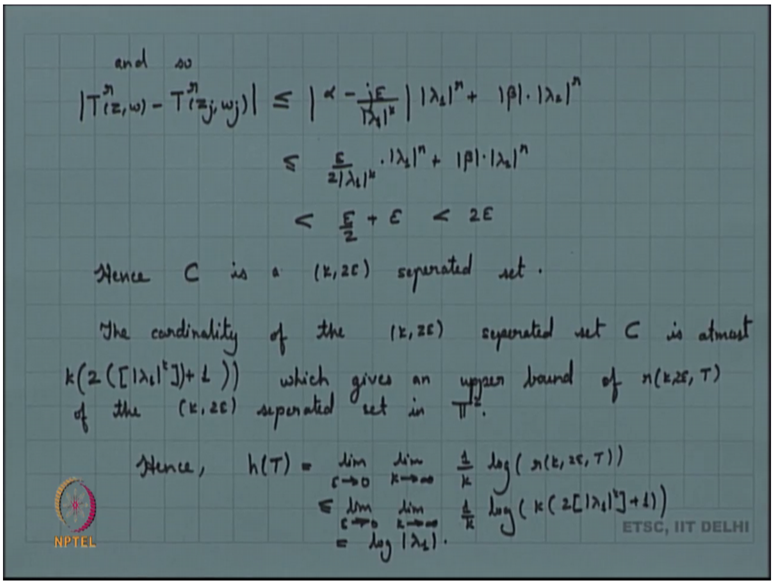

And according to this equation what is my mod of Tr z w minus T to the power r zj wj? This modulus would be less than or equal to now we again come back to this equation right.

So, I am taking this on the other side what is this modulus. So, this modulus will be less than or equal to modulus of this part. But now if I look into my v 1 and v 2, these are both are linearly independent right. So, the modulus of this vector will be basically less than or equal to I can take the modulus of this and take the modulus of that. Again, your v 1 happens to be a eigenvector for A r right. So, A r v 1 would be same as lambda 1 to the power r v 1. And A r v 2 would be same as lambda 2 to the power r v 2.

So, what happens here is what is this modulus, this modulus is less than or equal to we have this modulus of alpha minus j times epsilon upon mod lambda 1 to the power k times lambda 1 mod lambda 1 to the power r, plus mod beta into mod of lambda 2 to the power r we have already taken over v 1 and v 2 to be unit vectors. So, the modulus of them will be 1. So, what is this less than equal to i can say that this would be less than or equal to we have already taken what is this part, right we have already assumed that we are choosing our j. So, that this particular modulus is less than epsilon by 2 mod lambda 1 to the power k.

So, what we have here is this is less than or equal to say epsilon by 2 mod lambda 1 to the power k, into mod lambda 1 to the power r plus you have mod of beta right to the power lambda mod lambda 2 to the power r into mod lambda 2 to the power r. Now think of this factor, your lambda 1 to the power r upon lambda 1 to the power k, right this quantity is less than 1. This is basically mod lambda 1 to the power or I should say 1 upon mod lambda 1 to the power k minus 1. This quantity is less than 1. So, I can say that this would be less than epsilon by 2, what happens to this quantity? We know what is our bit where is our beta lying it is lying between minus epsilon to epsilon.

So, mod beta at the most you can have value epsilon. And if I look into mod lambda 2 to the power n right mod lambda 2 is less than 1. So, mod lambda 2 to the power r is also going to be less than 1 right. So, this is less than epsilon right and in all we can say that this would be less than twice epsilon. So, what we have here is that our C happens to be hence C is a k twice epsilon separated set. And now what is the cardinality of C? So, if you look into the cardinality of C. So, the cardinality of is at most twice right we have seen what is the cardinality of Ci.

So, the cardinality of Ci happens to be at most integer value of lambda 1 to the power k plus 1, right. This was the cardinality of Ci. But sorry, but if i look into what is the cardinality of C there will be k such Ci right. So, the cardinality of C happens to be this factor, which gives an upper bound of we know that we want to look into the maximum cardinality of an k twice epsilon separated set. So, this gives an upper bound for that maximum cardinality. So, k epsilon T, right of the k twice epsilon separated set in T square right.

So, in the torus, and hence what we have is the topological entropy happens to be equal to limit as epsilon tends to 0, right limit as k tends to infinity, right I have 1 upon k log of r k, sorry this is twice epsilon epsilon T. And this turns out to be less than or equal to limit as epsilon tends to 0 limit as k tends to infinity you have 1 upon k times log of, because this quantity is having this as an upper bound right. So, I have log of k times 2 twice integer value of lambda 1 to the power k plus 1. And if I want to look into that factor right if I try to take this limit as epsilon tends to 0, and this limit as k tends to infinity, this turns out to be equal to log of mod lambda 1.

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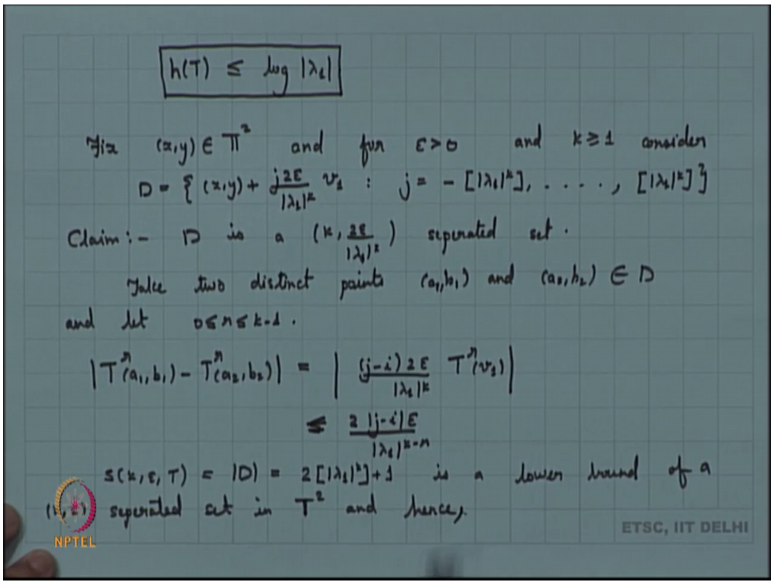

And so, the result that we have here is that your entropy of T is less than or equal to log mod lambda 1. Now this is one way right, as we have seen this is just in a single direction. We will try to look into this inequality the other way round also. What we had done is that we had found out the upper bound for n n 2 epsilon for a k 2 epsilon separated set, we can find similarly we can find a lower bound also. So, that would give us the reverse inequality. So, we fix an xy in T square, and for epsilon greater than 0 and k greater than equal to 1, we consider D to be equal to the set of all xy plus say j times 2 epsilon upon mod of lambda 1 to the power k times v 1, right. Such that your j varies from minus integer part of mod lambda 1 to the power k to the integer part of mod lambda 1 to the power k.

Let us look into this set d. Now our claim here is D is a k 2 epsilon upon mod lambda 1 to the power k separated set. So, take any 2 distinct points a 1 and b 1. So, take 2 distinct points say a 1 b 1 and a 2 a 2 and let 0 less than or equal to r less than or equal to k minus 1. So, what we have here is mod of T to the power r a 1 b 1 minus T to the power r a 2 a 2 this is basically a mod of I am looking into this factor it is basically mod of j. So, I have this a 1 b 1 and I can write my a 1 b 1 as this factor right some j here.

So, this is basically j times 2 epsilon upon lambda 1 to the power k a 1 b 1. And I can take my a 2 b 2 to be basically this plus i times 2 epsilon upon lambda 1 to the power k. So, this would be some j minus i times 2 epsilon right upon mod lambda 1 to the power k, right into T to the power r of v 1. And this we can very well see that this would basically be should say it would be less than or equal to or basically this will exactly be equal to this is twice, now I had twice mod j minus i epsilon, right I have lambda 1 to the power k here.

So, and I have this is again lambda 1 to the power r here. So, this would be upon mod lambda 1 to the power upon k minus r. So, I am see thinking of this part. So, this would exactly be equal to this factor. So, if I take my set s. So, if I look into my s k epsilon T, this happens to be the cardinality of D. And the cardinality of D is twice, right again I have integer value of lambda 1 to the power k plus 1, right is a lower bound of a k epsilon separated set. I can think of because this was just a constant k epsilon separated set in T square.

And hence what do we have here now?

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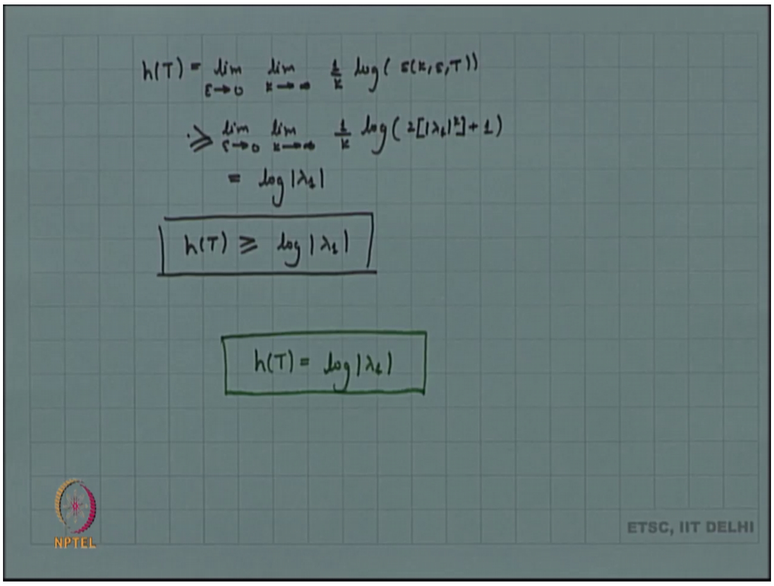

So, the entropy of T is basically limit as epsilon tends to 0 limit as k tends to infinity 1 upon k log of s k epsilon T can think of that part. And that becomes greater than equal to because I just know a lower bound for this term. So, this becomes greater than equal to limit as epsilon tends to 0, limit as k tends to infinity, I have 1 upon k times log of twice integer value of lambda 1 to the power k, right plus 1. And this is basically our log mod lambda 1.

And hence from this equation we get that the topological entropy of T is greater than or equal to log mod lambda 1. You look into this fact the first time we proved that topological entropy of T is less than or equal to log mod lambda 1. Now we have a result saying that topological entropy of T should be greater than equal to log mod lambda 1. And that proves that topological entropy of T is basically equal to log mod lambda 1. Where lambda 1 happens to be the eigenvalue with modulus greater than 1 for the hyperbolic matrix A. And I hope this is clear to all of you. So, your hyperbolic toral automorphism is a chaotic system. We look into more chaos in the next class.