Chaotic Dynamical Systems
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Lecture – 33
Hyperbolic Toral Automorphisms

Welcome to students. So, today we will be discussing about a chaotic system in 2 dimensions. And for that we will look into an example on a torus.

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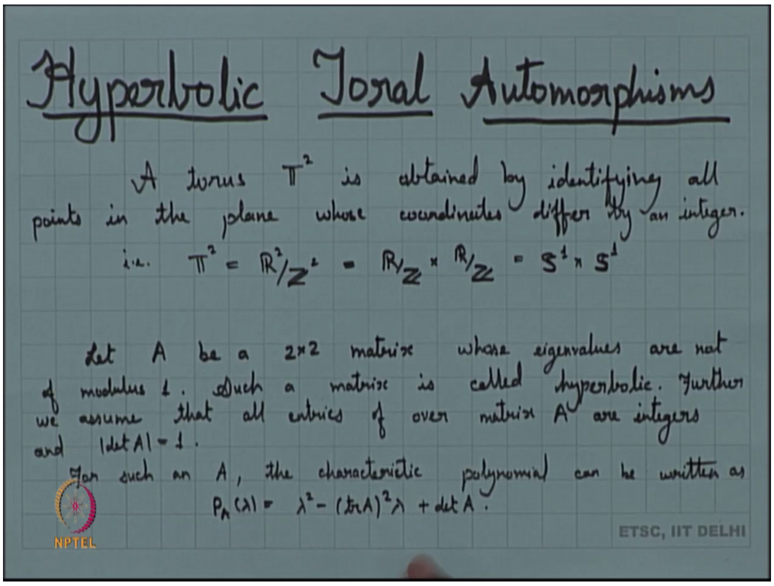

So, we will be looking into toral automorphisms, and especially the hyperbolic toral automorphisms that give us chaos. So, we recall that a torus which we denote as T square is obtained by identifying all points in the plane whose coordinates differ by an integer.

So, you can say that a 2-dimensional torus is basically R square by Z square. You can also say that this is the product of R by Z with R by Z and we know that R by Z is a circle unit circle. So, this is basically S 1 cross S 1. You can also geometrically think of a torus as taking a unit square, right. Then identify the 2 opposite ends. First up identify the 2 horizontal ends, then that would give you some kind of a cylinder. And then you are identifying the 2 sides of the cylinder. So, that gives you a torus. So, geometrically a torus can also be obtained, but basically it happens to be one circle. So, on one hand you have one circle, on the other hand you have the other circle right. So, you have a circle cross circle which gives you a torus.

Now, what are what is the action that we are going to take on this torus? So, let a be a 2 cross 2 matrix, whose eigenvalues are not of absolute modulus 1. We basically want a matrix whose eigenvalues right their absolute value either is greater than 1. Or is less than 1 such a matrix is called hyperbolic. So, we call such a matrix is hyperbolics. Further we assume that all entries of over matrix A, these are all integers.

So, you are basically looking out for a integer matrix, and we want the determinant of A to be equal to 1. So, one can think of a specially near group here right on integers. So, we are basically looking into such a matrix. So, this is our hyperbolic matrix that we are going to consider. If it try to look into such a matrix, we know that the characteristic equation here. So, for such a for such an a the characteristic polynomial can be written as pa lambda, which is your lambda square minus trace of a whole square times lambda plus determinant of A.

So, what you have here is that, your characteristic polynomial is written in this particular term. Now we know that the constant term if you have a polynomial equation, the constant term happens to be the multiplication of the roots right.

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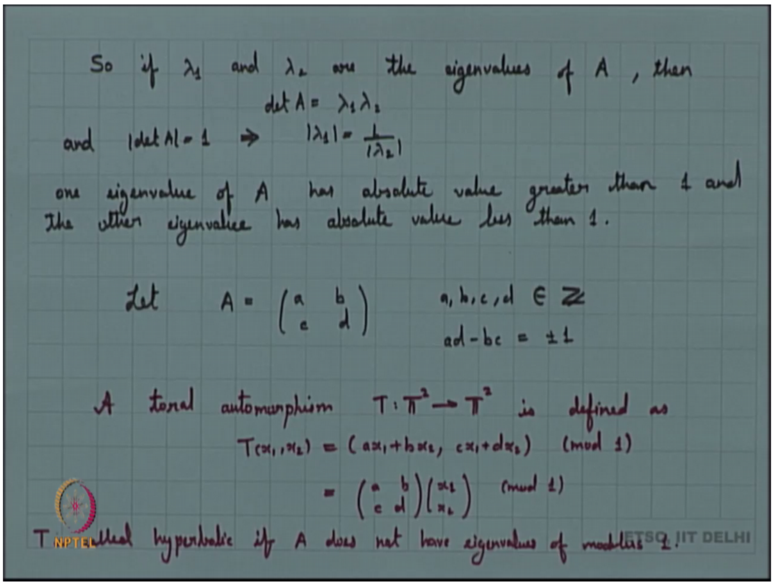

So, if lambda 1 and lambda 2 are 2 solutions are the eigenvalues of A, then determinant A lambda 1 times lambda 2. And so, I am looking into the determinant A to be equal to 1 modulus of the determinant A is equal to 1. So, this implies that modulus of lambda 1 happens to be 1 upon modulus of lambda 2. So, what does that mean, right? That means, that one eigenvalue of A has absolute value greater than 1, and the other eigenvalue has absolute value less than 1.

So, in general the spectral radius is greater than 1, but we are in the condition where or we are in the situation where your absolute value is so, one of them is greater than 1 the other one is less than 1. Let us look into our matrix. So, let us assume that our matrix is given as abcd, where the condition here is that your abcd, these are element subset. And your ad minus bc this is either plus or minus 1.

So, the determinant is plus or minus 1. So, we are looking into this aspect. Now a toral automorphism; so, maybe I can push this this is the definition here. So, toral automorphism we call it T from T square to T square is defined as T of x 1 x 2 basically you have ax 1 plus bx 2 cx 1 plus dx 2 times mod 1. So, I can also write this as I have this matrix abcd right. And this is operated on x 1 and x 2, and we are looking into the result modulo 1.

So, this is a toral automorphism and then this toral automorphism is called hyperbolic. So, T is hyperbolic if a does not have eigenvalues of modulus 1. So, a hyperbolic toral automorphism is basically a mapping a mod 1 mapping on the torus, which is given by multiplication of the matrix with the vector x 1 x 2. So, we will try to see what is basically the action on of the toral automorphism.

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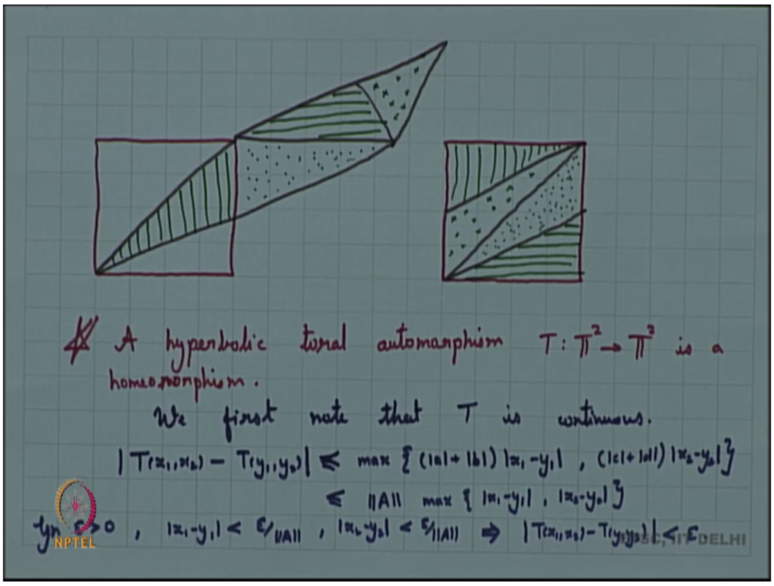

So, I will draw a picture. Here think of that we had this unit square here. Now what exactly happens to this unit square is that you are multiplying by a hyperbolic matrix. So, there is a slight change, right this basically gets deviated what do you have here is so, you have some such structure here. So, this is what is your action on the matrix you have the units square torus is just a unit square. So, on the unit square this is what how the matrix x. And then we are looking into modulo 1. So, we are bringing it back to the unit square. So, when we bring it back to the unit square.

So, when we bring it to the back to the unit square, we have some criteria we have something like, this this is what it appears to after bringing it back to the unit square. So, this is basically how the toral automorphism it acts on the torus, we first want to look into some properties of toral automorphism. And the very first property that we can see for the toral automorphism here is that a hyperbolic toral automorphism calling it T from T square to T square is a homeomorphism we will try to prove this part that this is indeed a homeomorphism. And to first prove this part we will note that this is continuous.

So, we first note that T is continuous. So, let us look into what is mod of T x 1. So, if you look into this part, we know that this would be basically, I am now looking into modulus of say we know that T x 1 x 2 will be a times x 1 the first coordinate will be a times x 1 plus b times x 2. The second coordinate will be c times x 1 plus d times x 2 and you can replace x 1 and y 1 here.

So, if you look into this part, we can simply say that this would be less than or equal to, I am now looking into the maximum of I have mod of a plus mod of b times mod of x 1 minus y 1. And mod of c plus mod of d times mod of x 2 minus y 2. So, the resultant will be less than or equal to maximum of this quantity. And if we try to see just going back to a little bit of operator algebra. That you have a matrix A we have a norm of a matrix A, and you can always think of a norm of a matrix A to be something like it is the maximum of the 4 entries that it has right.

So, looking into this aspect, we can say that this would be less than or equal to the norm of a right into the maximum of mod of x 1 minus y 1 x 2 minus y 2. Now we are taking the maximum matrix on R square right. So, basically this is giving you the distance between the 2 points x 1 and x 2. And we have norm a which is since A is fixed this is a fixed quantity.

So, given epsilon positive, we can say that whenever you are mod of x 1 minus y 1 is less than epsilon upon norm a mod of x 2 minus y 2, this is less than epsilon upon norm a then this would imply that mod of T x 1 x 2 minus T y 1 y 2 is less than epsilon. So, that proves that your mapping T is continuous.

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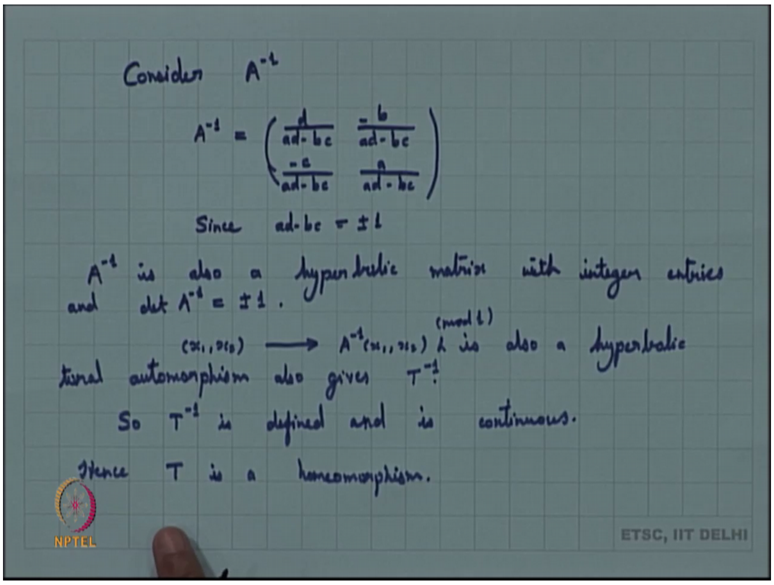

Now, consider A inverse. I can easily write my A inverse to be equal to this is very simple to compute. Now we know that our determinant is either plus 1 or minus 1. So, since your ad minus bc is plus or minus 1, either plus 1 or minus 1 you are A inverse is also hyperbolic matrix with integer entries, and your determinant of A minus A inverse is also plus or minus 1. And so, if we look into this part right if we look into x 1 x 2 being mapped to A inverse of x 1 x 2. This is also hyperbolic toral automorphism. And this gives your A inverse not only this is a hyperbolic toral automorphism, but it also gives a T inverse.

So, also gives T inverse. So, your T inverse is not only exists, right it is continuous right. Of course, I want a mod 1 here and is continuous. So, your T is a homeomorphism. Your torus can also be realized as a group right. And so, this is basically an automorphism on that group. So, topological torus is a topological group, right. T happens to be an automorphism on the topological group, right. Hence, we call it a toral automorphism.

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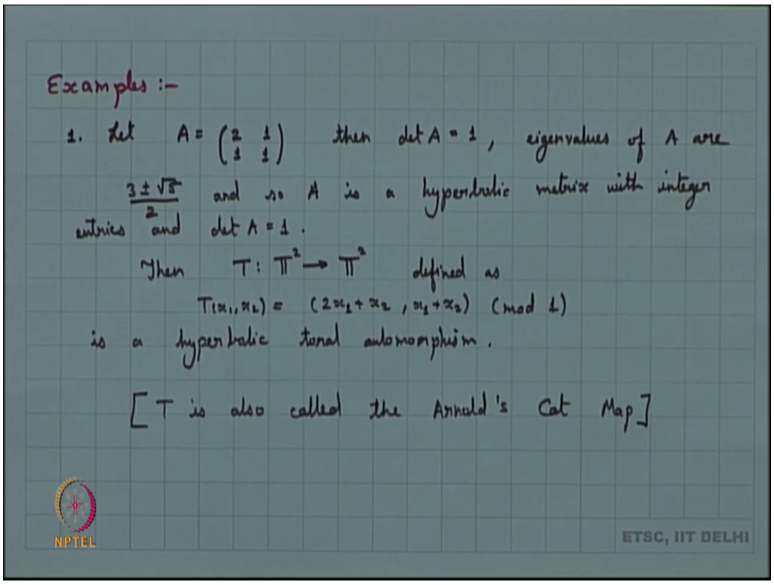

So, let us look into some examples here. So, one is the first example here. Let us consider the matrix 2 1 1 1. Yes, can you compute determinant of A here, what is the determinant of A the determinant is 1. What are the eigenvalues of A? What are the eigenvalues here? 3 plus minus root 5 by 2. So, the eigenvalues here are 3 plus minus root 5 by 2. And so, we see that the eigenvalues do not have absolute value one right. And so, A is a hyperbolic matrix with integer entries, and determinant of A is 1.

So now we define T on a torus defining this action 2 x 1 plus x 2 x 1 plus x 2 mod 1. So, this T is a hyperbolic toral automorphism. The picture I had drawn was basically showing this part this skewing the part, right. It is elongating one direction on the angle of 2. You can also compare that in this factor, right. Your first if you look into the first part, right, this is basically the angle doubling on the first part, right. The other part remains as it is. So, and there is some kind of a skew here, because you are mixing the coordinates x 1 and x 2. So, there is some kind of skewing here. So, this is a very nice example and in fact, this is one of the most famous examples of hyperbolic toral automorphisms. This is also called the arnolds cat map. So many places you will find the reference to this example as the cat map.

Let us look into one more example here.

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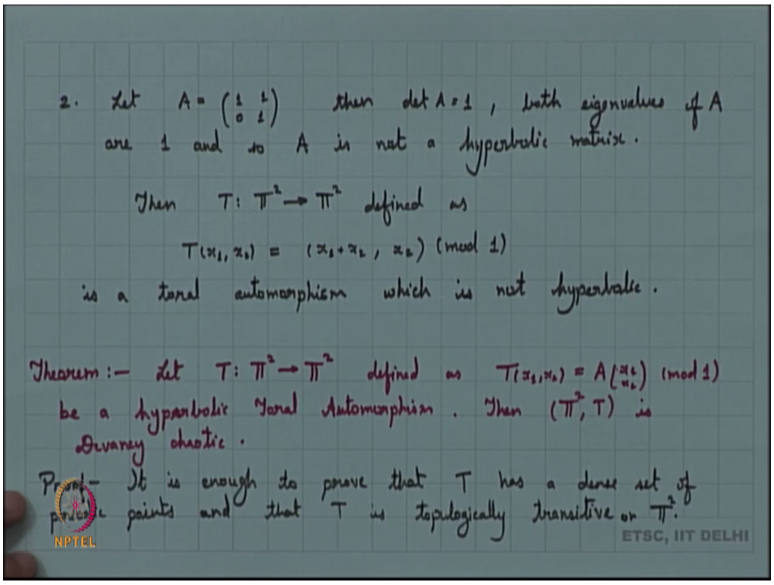

So, here let a be equal to needless to say here determinant A is 1. What are the eigenvalues here? 1, right? Both eigenvalues here will be 1. A is not a hyperbolic matrix. So, the entries of A are integers, right. The determinant of A is 1, but since the eigenvalues a value 1 right this is not a hyperbolic matrix.

And hence you take T the torus to torus defined as T of x 1 x 2 is just x 1 plus x 2 and the second one is x 2 mod 1. This T is a toral automorphism which is not hyperbolic. So, this is a toral automorphism, but this is not hyperbolic. So, we will try to see what are the characteristics of the toral automorphism. And as I will earlier mention we are looking into an example of a chaotic system in 2 dimension and this is one of our examples. So, will sure that the toral automorphism irrespective of whatever the entries of the matrix are and hyperbolic toral automorphism is always devaney chaotic right. So, will look into devaney chaotic today.

let T be T defined as a multiplied by x 1 x 2 mod 1 be a hyperbolic toral automorphism, then the system taking on the torus T this is devaney chaotic. And we shall look into the proof here. If we try to look into the proof here. We know very well that if we have transitivity and if we have dense periodic points then it implies sensitive dependence on initial condition. So, all we need to show is that the periodic points here are dense and this is topologically transitive right.

So, it is enough to prove that T has a dense set of periodic points and that T is topologically transitive on the torus. So, we get into the proof here. So, first of one our claim is T has a dense set of periodic points in the torus.

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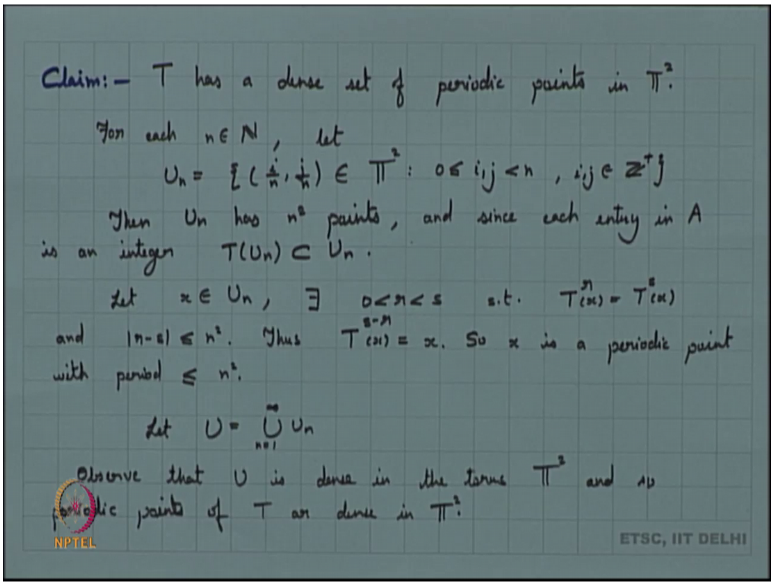

Now the idea here is that we would be showing that every rational number here, happens to be a periodic point. And then with the observation that when you are looking into a torus right when you are looking into all points in the torus, right. Those points in the torus who have both the entries right whose coordinates are rationals right such a the set of such points is dense in the torus. And so, the set of periodic points will be dense.

So, all we need to observe here is that each and every point with rational things happens to be a periodic point. So, for each n, in N let U n be the set of all points with rational coordinates denominator is n right. Since we are working this on a torus, we know that 0 is less than or equal to ij is less than n. And your ij will be elements of Z plus right. So, we know that we are looking into integers between 0 and n we are looking into all such points U n.

Now, what is the cardinality of U n, right? You have n possibilities here, n possibilities here right. So, the cardinalities n square. So, U n has n square points. And since each entry in A is an integer when you are multiplying this points, right by a fine since every entry of A is integer you are you are always going to get an entry the resultant mod 1 will always be an entry of this form.

So, and since each entry of A is an integer, T of U n right will be a subset of U n. So, T of U n is a subset of U n. So, try to take any x in U n. So, let x belongs to U n take any element of U n here. There will exist 0 less than R less than S such that T to the power R of x is same as T to the power S of x. And your mod of R minus S will be bounded by n square. So, what you find here is that you start with any point in U n, right. You are multiplying T again and again, again and again. You find that there will be some entries R and S whose difference is bounded by n square such that T to the power R of x is same as T to the power S of x and what does that mean T of S minus R of x will be same as x. So, what do you find is these are going these are going together right. And of course, once they go together after that steps there always going to be go together.

So, we can always assume that T of R minus S is x. So, T of R minus x happens to be x. And so, x is a periodic point, with period less than n square. Or I can say that it is less than or equal to n square. So, what we find is that x happens to be a periodic point period less than or equal to n square. How many such periodic points are there? You find that every point in U n see such a periodic point. So, what we find ultimately is that all these points with rational entries right all points with rational entries are periodic point.

So, we take this set u to be the union of U n where n goes from one to infinity, we are looking into all rational because this contains now this contains a set of all rational points. All of them are periodic points, right? And if I look into this set u this set u happens to be dense in the torus, right. Then we observe that u is dense in the torus. And so, periodic points of T are dense in T square. I hope this proof is quite clear.

So, we have our next claim is to show that T is topologically transitive on T square. Now in order to show that basically it requires a little bit of more of some theory which we have skipped. So, instead of saying that we are giving a proof of that, right I would just say that we are giving a outline of the proof for this one, right. Because this requires a little bit more theory. And to be very frank we have not enough time to cover up that aspect of the theory also.

So, this is some aspect of differentiable dynamics. That one can think and these are just basics. So, it is always easy to get through that. So, without getting into details there without naming any property there, I will just get into the outline of the proof here.

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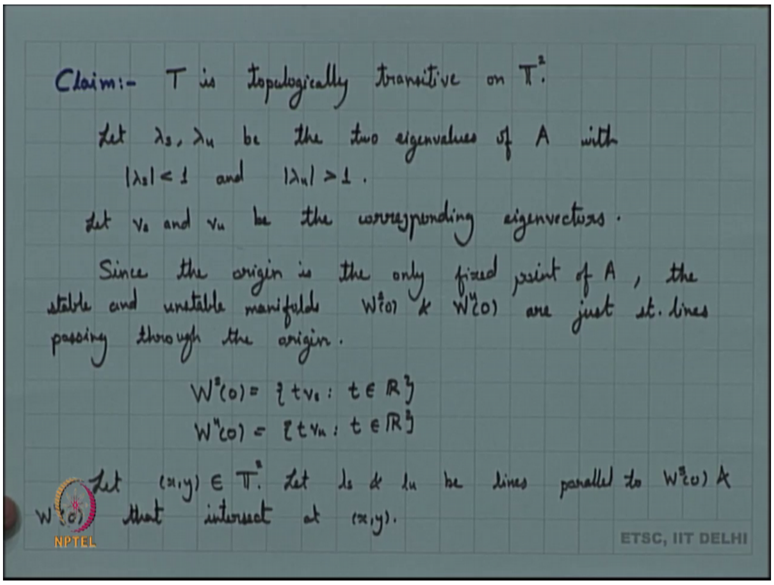

So, our next claim is T is transitive on T square, topologically transitive. So, how do we prove this part? First start with 2 eigenvalues of A. So, let lambda S and lambda u be the 2 eigenvalues. I am naming them also we know that the 2 eigenvalues, one of them will have modulus greater than 1, one of them will have modulus less than 1.

So, we want that modulus of lambda S is less than 1, and modulus of lambda u is greater than 1. Now since we have eigenvalues we should also have eigenvectors. So, let vs and vu be the corresponding eigenvectors. We have seen this property of hyperbolic matrix, origin happens to be the only fixed pointer. And hence if we look into what is your vs and vu. So, we are looking into the directions of eigenvectors, right. We can think that they will be basically straight lines passing through the origin. And if we look into what is the action of a on this eigenvectors it is contracting in on one of the factors, and it is expanding on the other vector.

So, if you look into the stable manifold, and the unstable manifold right of the origin, then the stable manifold and unstable manifold are just straight lines passing through the origin. So, since the origin is the only fixed point of A. The stable and unstable manifolds of the origin A are just straight lines passing through the origin. So, I can say that my stable manifold happens to be equal to the set of all T times vs such that t belongs to R. And my unstable manifold happens to be T times vu such that t belongs to R, right. These are the stable manifolds and unstable manifolds.

Now, we start with the point in on the torus. So, let my xy be a point on the torus. Now when you are looking into stable manifold an unstable manifold these are lines passing through the origin. What happens here is that I can always translate them, both the lines can be translated. So, that your center happens to be xy right. So, you can always translate ts and tu you can all on always translate vs and vu right in such a manner that the origin is basically translated to xy right.

So, let your ls and lu be lines parallel to that intersect at the point xy. So, what happens then? So now, we know that these are like I am looking into points xy, right though the point is on the torus. I am looking into the point xy in R square. And now I have ls and lu looking over there. So, I am taking the projection of ls and lu on T square right.

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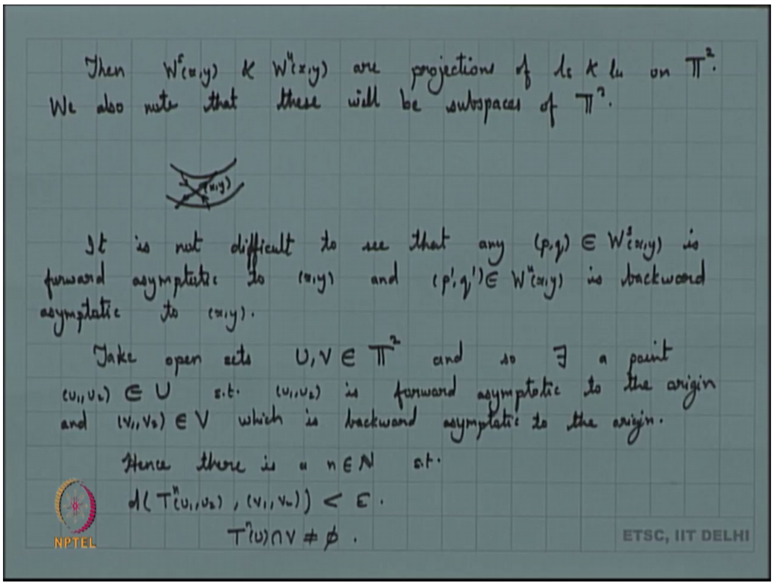

So, W s xy and W u xy are projections of ls and lu on T square. We also note that this will be subspaces of T square.

So, what happens here is you have a torus here, right. This is your point xy on the torus. And then you have something like the stable and unstable manifold. So, what happens here is the stable manifold this direction, this mapping into this direction, it is mapping on the other direction right. So, what happens here is you have this kind of a structure here on the torus. And these are subspaces over here. So, they will be forming in the whole torus right these are subspaces. And you find that these points intersected xy.

So, it is not difficult. Think of that on one direction it is going further over from xy the direction is coming further closer to xy. And if you look into this points these are just projections of some kind of translation of the stable and unstable manifold of the origin. So, what happens is it is not very difficult to see that for any pq belonging to W s xy is forward asymptotic to xy. And take the point p prime q prime, then this point in W u xy is backward asymptotic to xy. So, this is how the picture looks. Take any open in T square.

So, there exists a point now think of that this is happening for any point xy, right. For any given point xy and T square such a thing is happening. We find that you can find lot of such points over on T square. So, we start with any points you to start with any open set u and v. It will contain such a point xy right. And so, you find a point. So, there exist a point u 1 u 2 in u such that this u 1 u 2 is forward asymptotic to the origin. And there is a v 1 v 2 in v which is backward asymptotic to the origin.

So, what we have? You take any uv because the structure is something which is sort of a copy of a structure what happens near the origin. So, if it start with any uv, you find one point in u which is forward asymptotic to the origin, you find one point in v which is backward asymptotic to the origin. And hence you get an n in N. So, hence there exist an n in N such that your d of T n u 1 u 2 and v 1 v 2. So now, you can take any distance on that. So, this distance is less than epsilon. So, find that one is moving one is moving towards backward it is asymptotic, right. One is forward asymptotic right.

So, somewhere it is near the origin right. So, you find an n such that these 2 points are quite close by. And that gives me that T n of u intersection v is non-empty. And hence your T happens to be topologically transitive. As I said that we have used some kind of a theory which we have not discussed earlier, right. It is some kind of basics which one should have discussed, but since we have limited time. We were not able to get into details on this aspect, but this is the basic general idea of why the toral automorphism happens to be topologically transitive. So, I hope this is clear to all of you and for today we end here.