Chaotic Dynamical Systems

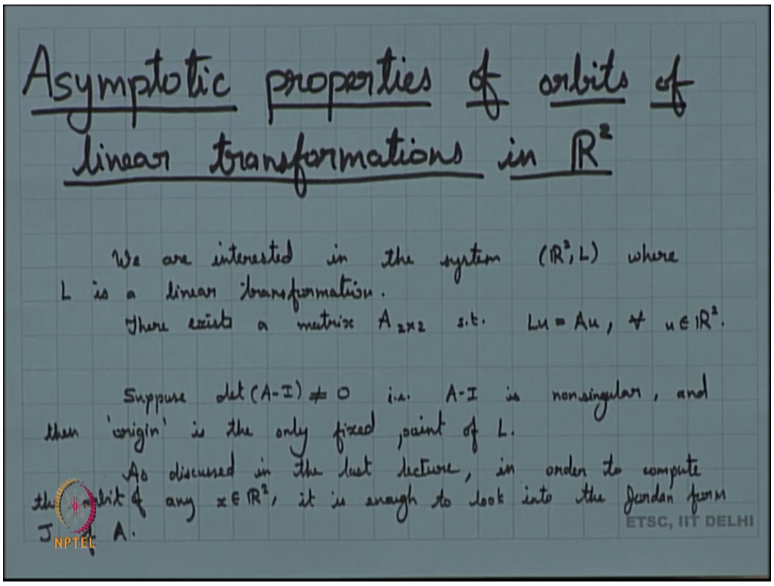
Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture – 32

Asymptotic Properties of Orbits of Linear Transformations in IR^2

Welcome to Students. So in the previous class we had seen that, we can look into the dynamics of linear systems. Especially they are interested in two dimensions we are not going for higher dimensions now, by getting into its Jordan form. So, today we will be continuing with that discussion by looking into the asymptotic properties of linear transformation in R square.

(Refer Slide Time: 00:54)

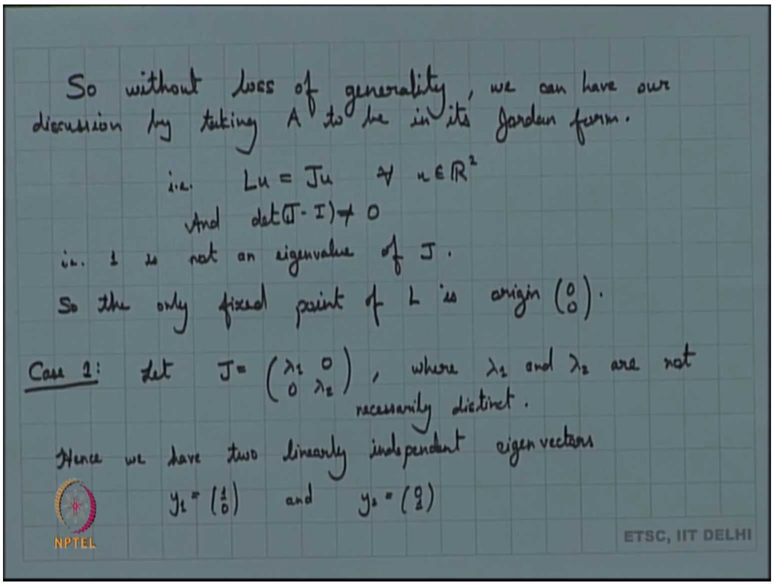


So, we are looking into the system. So, basically we are interested in the system R square L where L is a linear transformation. Now, since L is a linear transformation there exists matrix; a matrix A which is basically 2 cross 2 matrix, such that L of u is basically Au. And this is true for every u in R square.

Let us look into the condition. So, we are looking into this condition that determinant. So, we suppose that determinant of A minus i the identity matrix is not equal to 0. To say that the determinant of A minus i is not equal to 0 means that A minus i is invertible right, its nonsingular. And A minus i is nonsingular means that, this will not have any. So, basically it means that one is not an eigenvalue right. So, this means that A minus i is nonsingular. And what happens in that case? Origin is the only fixed point of A. So, origin happens to be the only fixed point for our linear transmission L.

Now, as discussed in the last lecture in order to compute the orbit of any x in R square it is enough to look into the Jordan form J of A. All we need we know that when we a computing a on orbit all we need is just to look into what is the Jordan form and the Jordan form helps us in computing the orbit.

(Refer Slide Time: 04:25)



So without loss of generality, we can take our a to be in its Jordan form, right. So, we can have our discussion. So, I can just take A to the in its Jordan form. And hence, what we have here is; so our Lu is basically Ju; where J is a Jordan matrix for every u in R square. And other than that we are putting up the condition that determinant of say J minus I is not equal to 0; that means J minus I is invertible and that means that 1 is not an eigenvalue. So, 1 is not an eigenvalue. And since this happens we know that origin is the only fixed point of L. So, the only fixed point and we know origin means our vector 0 0.

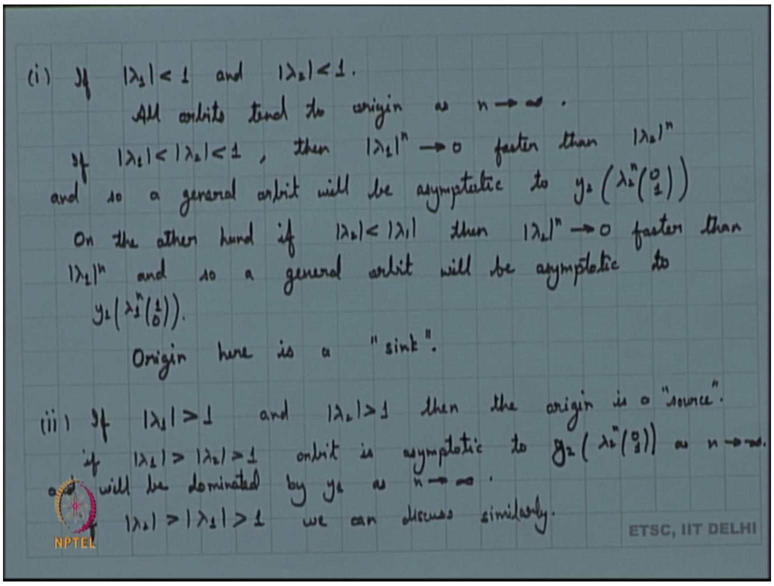
We are now interested in looking into the asymptotic properties of all the points of the orbits of all points in the in R square. So, what are they behaving? We know that we have one fixed point which is the origin. And then we also know that since we are working with eigenvectors, we will have some kind of a splitting my eigenvectors. So, basically they will be spanning the set; either the eigenvectors or they generalized eigenvectors will be spanning the space. So, we have a structure, we have a nice geometric structure given by these eigenvectors and that is what helps us in determining the dynamics here.

So, we look into the first case: now since you are working with just a Jordan form right, enough to specify what the Jordan form is and work with it. So, the first case will let our J to be of the form lambda 1 0 0 lambda 2; where lambda 1 and lambda 2 are not necessarily distinct. So, our cases when we have our Jordan form in a diagonal form what happens in that case. Now when the Jordan farm is in a diagonal form nice form we know that, lambda 1 will have an eigenvector lambda 2 will also have an eigenvector.

So, in that case what happens? In this particular case we have two linearly independent eigenvectors. Let us name this eigenvectors. So, this I am calling it y 1 which is my 1 0, right; this is an eigenvector for lambda 1. And we have y 2; 0 1 which is an eigenvector for lambda 2. So, we have this two linearly independent eigenvectors y 1 and y 2. Now even in this case we can have several sub cases here as a sub cases depend on what is the value of lambda 1 and lambda 2.

So, let us try to look into the first sub case here. So, what is this first sub case here?

(Refer Slide Time: 09:00)



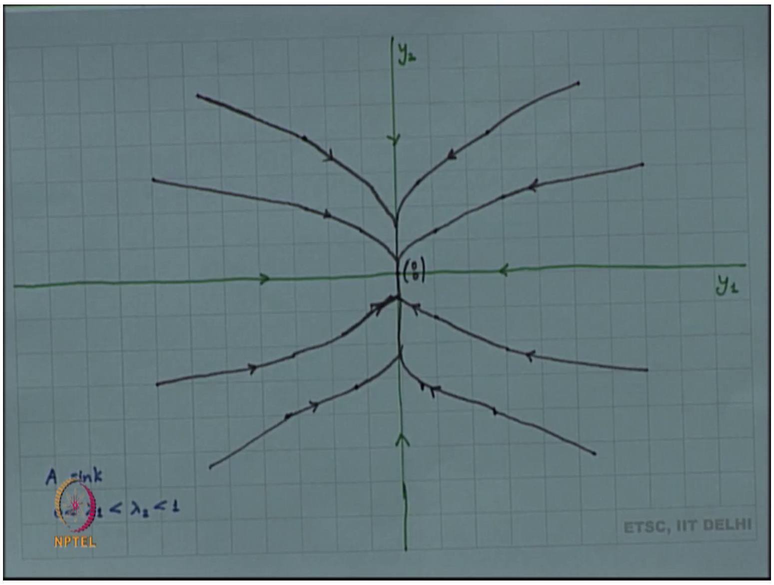
So, the first sub case here is that if my mod of lambda 1; now we are taking up mod because the way it tends either it is alternately, so it is basically either what to say roaming around 0. So, how it tends to 0 it depends on what is the value of lambda 1, so it may switch over from one place to the other and then tend to whatever properties it is tending to. So, we will look into what happens here when lambda 1 is less than 1; mod of lambda 1 is less than 1 and mod of lambda 2 is also less than 1. We are looking into this particular sub case.

Now if this happens, what does it mean is that? Basically, we know that every vector can be written as a linear combination of eigenvectors. And because of that it is basically reaction on eigenvectors that is determining the action on every point. And when my lambda 1 is less than 1 lambda 2 is also less than 1 that means there is some kind of a contraction observed in the direction of eigenvectors. And hence, overall since every point is a linear combination of eigenvectors there is an overall contraction observed. And so, the origin here becomes a sink right; if every orbit will be asymptotically tending towards the origin.

So, here what happens in this case is that; all orbits tend to the origin as n tends to infinity, but there is some different still that you can observe here. So, if we have say mod of lambda 1 is less than mod of lambda 2 this is less than 1. What happens in that case? Then you see mod of lambda 1 to the power n it tends to 0 faster, because it is less. It tends to 0 faster than mod of lambda 2 to the power n. And hence, the general orbit will be asymptotic to y 2 or I should say it would be lambda 2 to the power n 0 1.

So, general orbit basically this will be a dominating factor, because this is tending to 0 faster this is a dominating factor. So, the general orbit will be asymptotic to this y 2. And again we know that also y 2 is also tending to 0, so ultimately everything will tend to 0.

(Refer Slide Time: 12:08)

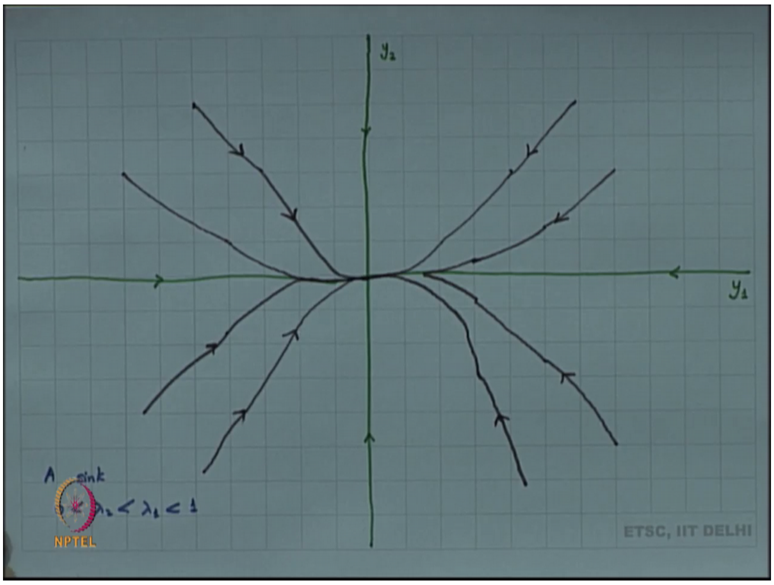


On the other hand, so let us try to look into the picture here. So we have this picture here, what happens here when your; lambda 1 is less than lambda 2 is less than 1. So, you have this direction of y 1, you have this direction of y 2, now in the direction of y 1 and y 2 also since the eigenvalue is less than 1 it is converging to in this direction it is converging to 0. But now, what happens here is your lambda 1 is tending to 0 faster than lambda 2. So, every orbit from wherever you start every orbit will be first converging towards y 2, so they will become asymptotic to y 2. And then, as y 2 moves around to 0 this is also moving along to 0.

What happens on the other hand? So, we have that the origin here which is basically the point 0 0, the origin here happens to be a sink in this case, but the behavior is a little bit different here because everything is first tending to y 2; first of all bring asymptotic y 2 and then converging to 0.

Now, what happens on the other hand? So, let us look on the other hand. If my mode of lambda 2 is less than mod of lambda 1 then we know that mode of lambda 2 to the power n will be converging to 0 faster than mod of lambda 1 to the power n. And so, a general orbit will be asymptotic to y 1. In other hand it should be basically lambda 1 to the power n, right 1 0. So, it is asymptotic to y 1. And then again since my y 1 is converging to 0 this will converge to 0. So, we can see the picture over here.

(Refer Slide Time: 14:13)



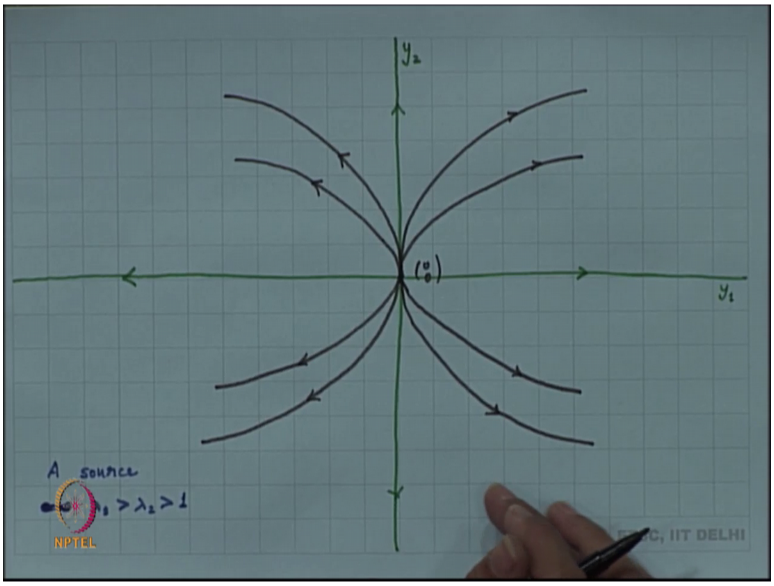
So, we try to see the picture over here, we have this direction of y 1 direction of y 2. And since my modulus of lambda 2 is less than modulus of lambda 1; this is tending to 0 faster and so you take any general orbit it is basically being asymptotic to y 1 and then it is tending to 0. So, here also your origin happens to be a sink, but the behaviors are different from the previous case. So, this is basically the behavior when you have your both eigenvalues less than 1, but then the difference in behavior happens when 1 is less than the other.

So, all we can conclude here the origin here is a sink whatever be the case, the origin is a sink; the origin is only fixed point and the origin here is a sink. Let us look into the second case. If mod of lambda 1 is greater than 1 and mod of lambda 2 is also greater than 1 then the origin is a source. Your original source because everything is basically asymptotic to the origin in the negative direction as n tends to minus negative it is asymptotic to the origin. But what happens here again? I can still have some difference between these two. So, what happens if mod of lambda 1 is greater than mod of lambda 2 is greater than 1; what happens in that case? So, the orbit will be first asymptotic to lambda 2 y 2 sorry; to y 2 which is basically lambda 2 to the power n 0 1 right, as n tends to minus infinity and it will be dominated by y 1 as n tends to infinity.

So basically, as your time increases it is basically going to be dominated by y 1 because the action of y 1 lambda 1 is greater there, right so the action of y 1 happens to be having a larger impact there. And if we look into what happens as you decrease time, right as time tends to minus infinity it will be basically dominated by the other vector because your lambda 2 happens to be smaller.

So, what happens here is; you have this origin to be a source, so let us look into the figure here.

(Refer Slide Time: 17:18)

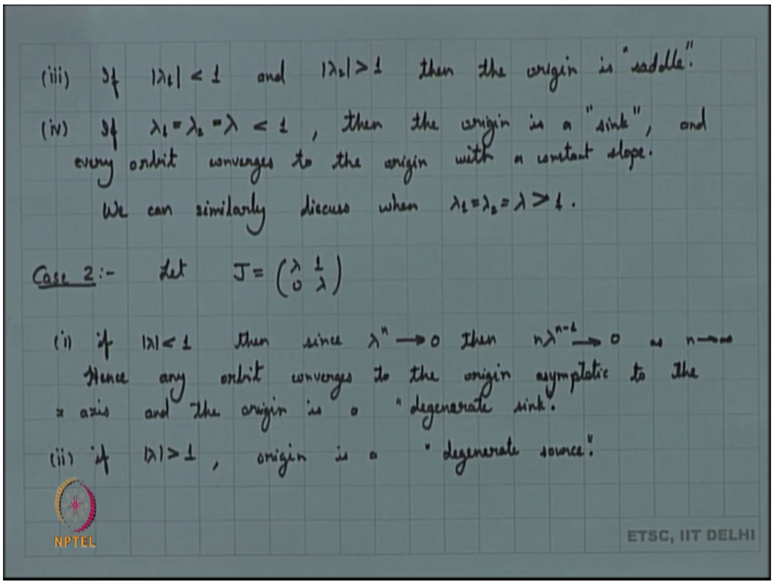


So, again you have this y 1 direction y 2 direction; your origin happens to be a source here so this is basically your origin which is a source here. And what happens here is that it is basically being see, in the negative direction its being asymptotic to y 2. So, the orbits are asymptotic to y 2 as n tends to minus infinity and then they are dominated by y 1 because its more increasing towards a y 1 direction as n tends to infinity.

What happens now if mod of lambda 2 is greater than mod of lambda 1; is greater than 1 right we can discuss similarly. So, what happens in that particular case? Again we can have similar concept here, but then here what would happen here is that I would have something like this, right. This is my y 1 this is my y 2 and everything is coming up in this particular direction. So it is just the same idea here.

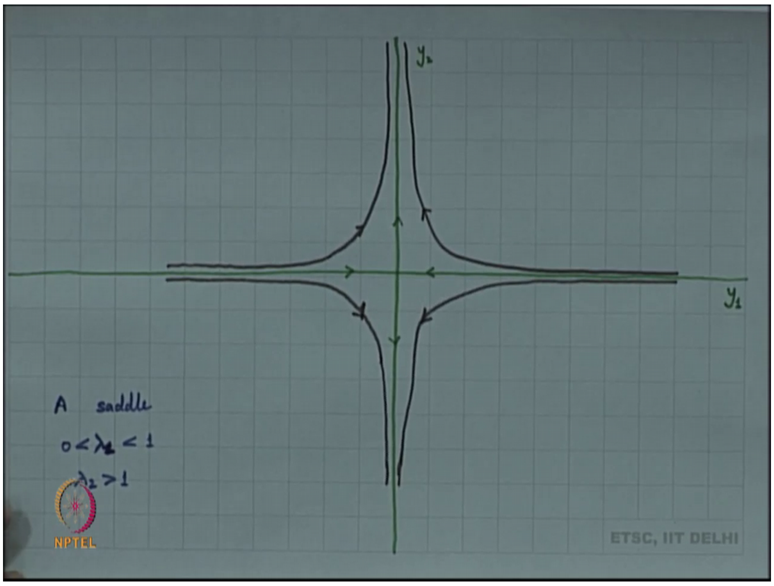
Now there is a third criteria here where one of them can be greater than 1 and one of them less than 1, right. So, we have a third case here; third subcase here.

(Refer Slide Time: 18:51)



And that say that if my mod lambda 1 is less than 1, and mod lambda 2 is greater than 1; what happens in that case. So, it is being contracted in one direction and it has been expanded in the other direction. So, your origin basically becomes a saddle. It is neither a sink nor a source, but it is a saddle. So, the origin is saddle. So, I can say something like a it is basically something like unstable because it is a saddle; nothing is converging to the origin, but it is attracting also from one direction and it is repelling also from the other direction. So, the origin happens to be a saddle.

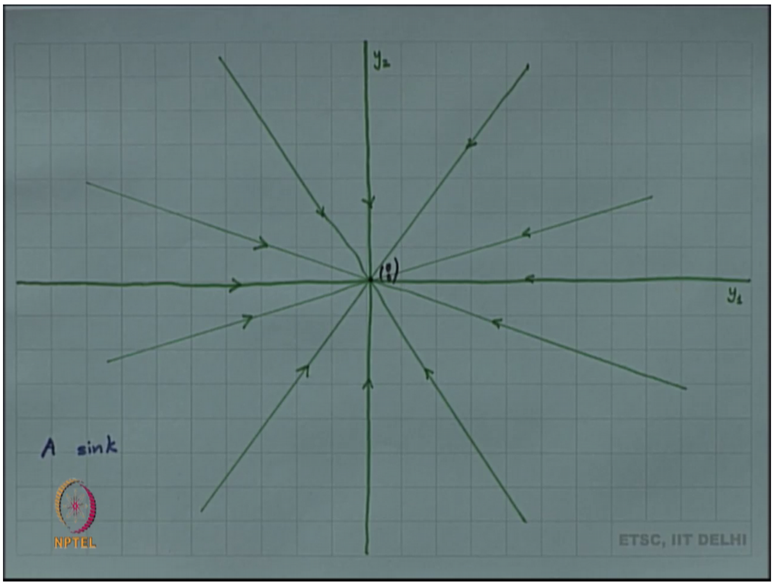
(Refer Slide Time: 19:43)



And you can see this part where your lambda 1 is between 0 and 1. So, basically it is converging towards the lambda 1 direction, but it is diverging towards the lambda 2 direction, so basically in the y 2 direction it is diverging. So, this behaves like a saddle. So origin behaves, in this case origin behaves like a saddle.

And we are left with the last sub case, because we are working with a diagonal Jordan form the last sub cases when what happens when both the eigen values are the same, right. So, the last sub cases if your lambda 1 is same as lambda 2, is same as lambda and let us assume that this is less than 1. Then the origin will be a sink, but we have something else observed here. And every orbit converges to the origin with a constant slope. Look into that. We are starting with an orbit; every point is a linear combination of the two eigenvectors. But now what happens to both the eigenvectors, because we know that both the eigenvectors have the same eigenvalue; the both are contracting, so they are equally contracting. And since they are equally contracting, what happens you start with any orbit any place; it will be basically converging to converge to the origin, but it is going to converge to the origin with the constant slope.

(Refer Slide Time: 21:45)



Everything converges to the origin with a constant slope. So, again here the origin is a sink. So, this is the origin and the origin here is a sink. So, there it is a sink, but we have a difference here that every orbit is converging to the origin with a constant slope.

Now, we can similarly discuss what happens if your lambda 1 is same as lambda 2 and that is greater than 1, right. Then basically everything will be moving apart, but how does it move apart. So, basically everything will be asymptotic to the origin, as n tends to minus infinity, but else it was a move going to move apart. And it will move apart also it will move similarly, right with constant slope. We will move apart also in the same manner.

So, we can similarly discuss; in the diagonal case when the Jordan form is a diagonal right, is a diagonal matrix we have four sub cases. The behavior of the orbit ultimately it is the same, right. Everything is converging to the origin depending on of course either it is sink our source and it is saddle we have these three cases, but even when it is a sink or even when it is a source the behavior is a little bit distinct depending on what is your value of lambda 1 and lambda 2 either they are greater than 1 not less than 1.

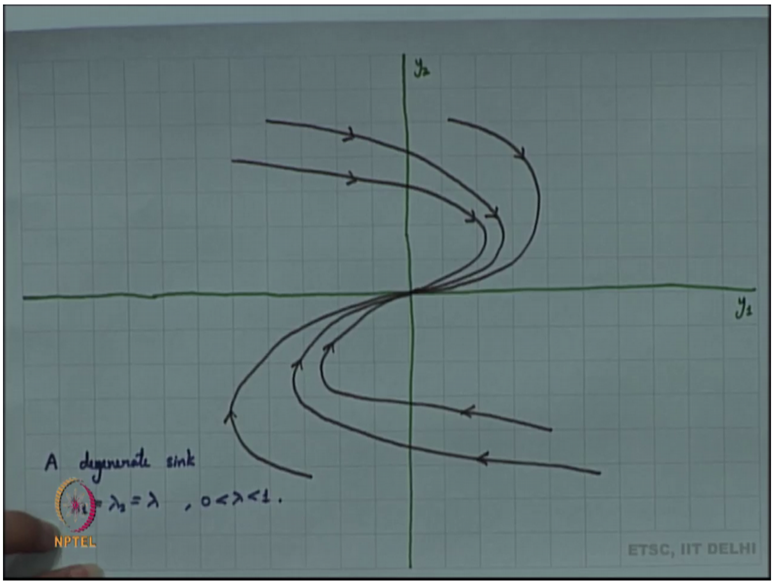
But ultimately the behavior automatically turns out to be something similar. Physically there may be a little bit difference, geometrically there may be a little bit difference, but analytically we have the same what is the behavior here. So, we are interested and looking into; what happens now if your Jordan form, because we know that there are three types of Jordan form. So, we are looking into the case second case. So, we look into the case 2: where my Jordan form is of the form lambda 1 0 lambda. So, basically I have an eigenvector and I have a generalized eigenvector.

What happens if more of lambda is less than 1? So, let us look into the first sub case here if mod of lambda is less than 1 what happens in that case. Then, since my lambda to the power n is converging to 0, right now lambda to the power n converges to 0 it can either converge monotonically or it can switch over and converge to 0 both the things are possible. So, lambda n is converging to 0. And lambda n is converging to 0 we see lapicus rule here, look into the derivative there. That means, that n times lambda to the power n minus 1 is also converging to 0. And what does that mean as n tends this happens as n tends to infinity.

What happens in this case now? If you look into Jn it was basically lambda to the power n times lambda to power minus 1 lambda to the power n and this is basically converging to 0. So, if you start with any orbit right, any orbit is going to converges to 0. So, any orbit converges to 0 so basically can say that- hence any orbit converges to the origin asymptotic sometimes this is a dominating factor here it is origin; asymptotic to the x axis because we have only one eigenvector here and our eigenvector is basically the x axis, the second one is a generalized one, right.

So, it is basically converging to the origin which is asymptotic to the x axis, because the x axis is dominating there. So, it is basically it is asymptotic to the x axis and the origin is a degenerate sink. So, what do we mean by saying that it is a degenerate sink; let us look into the picture.

(Refer Slide Time: 26:32)



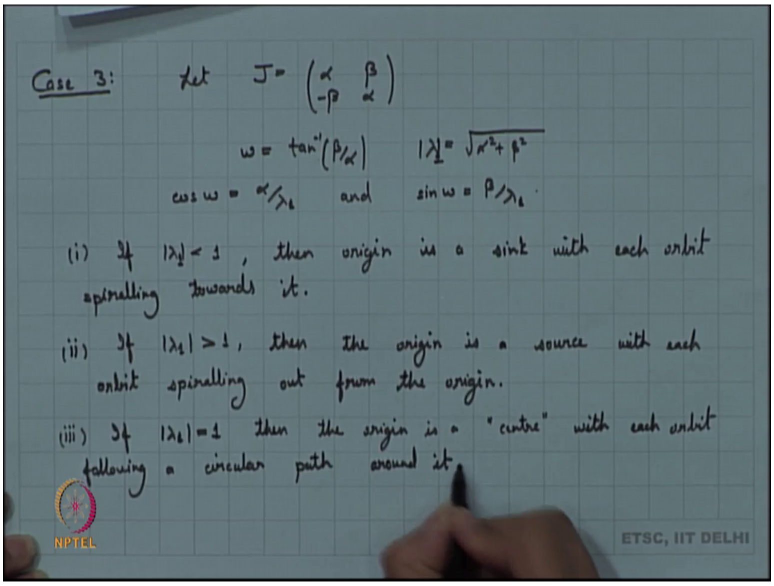
So look into the picture here: I have this y 1 here, now y 2 is no longer up eigenvalue, right we have a generalized eigenvalue which depends on y 1. So, this is one of the eigenvectors y 2 is no longer an eigenvector y 1 is an eigenvector and we have a generalized eigenvector depending on this part, so this is y 1. And if you try to see what happens to the orbit. So, the orbit is first be basically, any orbit is first being asymptotic to y 1 and then it is converging to 0. So, start with any orbit it is basically going to be converging to the x axis and then it is converging to the origin. So, this is basically a degenerate sink.

What happens in the case when mod of lambda is greater than 1? Now we know that mode of lambda is greater than 1 lambda to the power n is going to tend to infinity. So, in that is case our origin is a degenerate source. Again we can see the figure here, what happens in this particular case. So, we have the origin. So we have this eigenvector y 1, y

2 is no longer an eigenvector here, but then this is greater than 1; the eigenvalue is greater than 1 so everything is basically moving away from the origin. This is moving away from the origin it is basically moving away from the origin in a degenerate form, not a direct form its degenerate form.

So, this is what happens when your Jordan form is of the form lambda 1 0 lambda, and you have this two values. We are not looking into the case when mod lambda to be equal to 1; we are not looking into 1 to be an eigenvalue here. So, basically you can think of that to be a hyperbolic case.

(Refer Slide Time: 28:52)



So, let us look into the third case now. So alpha beta minus beta alpha, this is what your J is. Now we have already seen how do we work out with such J, right. So, we take our omega to be tan inverse beta by alpha and we let lambda b equal to root of alpha square plus beta square. And then we know that we have cos omega to be equal to alpha by lambda, and we have sine omega to be equal to beta by lambda. We had send this particular construction.

So, if you try to look into now any orbit here, so the orbit would be of the form so you start with an x the orbit would be of the form Jx, J square x, J cube x, and so on. And in trying to compute J to the power n we had seen that we used we have converted that alpha and beta into omega. And then we had tried to look into; so you have omega lambda and then we have try to look into Jn can be computed in terms of cos and sine.

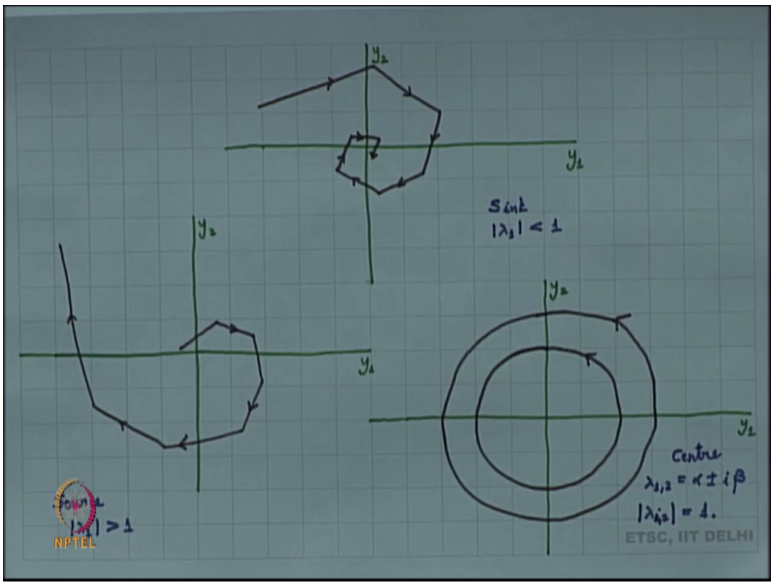
Now since Jn can be computed in terms of cos and sine when we had take up the orbit of any point this will have, the orbit will have a presence of cos omega and sign. I should say cos n omega and sine n omega, so cos omega sine omega definitely they are present there.

So, if you try to look into this particular point we can never have this completely tending towards 0 in some sense, because you have cos omega sine omega. So, exactly it is not going to tend towards 0, because if your cos omega tend to be 0 your sine omega is going to take care of that part.

So, what happens in this case? We have a very nice picture here. So, let us look into the first sub case here. So the first sub case here is: if mod of lambda, because we now have this lambda which is basically the modulus of alpha plus of the eigenvalue. So, the eigenvalue is alpha plus beta i right, so we are looking into the modulus of eigenvalue. So, what happens here is if mod lambda is greater than 1, basically this is actually my mod lambda. So my mod lambda is greater than 1, sorry is less than 1 then the origin is the sink with each orbit spiraling towards it.

And what do we mean by each orbit spiraling towards it; let us try to see the picture here.

(Refer Slide Time: 32:11)



My mod lambda or I should say mod lambda 1 here because this is what we had started with lambda 1, so this was our lambda 1 because we had only one eigenvalue here. So, we look into this part, we look into this factor. So, what we have here is mod lambda 1 is less than 1, each orbit is basically spiraling towards the origin untimely defiantly it is going to converge to the origin because your modulus the magnitude is basically lambda n is basically going to tend to 0, right. So, this is like when you are looking into Jn the nth inter iterate right it will have something like mod lambda 1 to the power n into some rotation, right. So, that is basically a rotation matrix. So, since your rotation matrix is rotating but ultimately your lambda 1 to the power n is tending to 0, so it will ultimately converge to the origin. So, origin here is definitely a sink, but how is it going it is basically just spiraling around because of the presence of the rotation matrix.

So, what happens here in the second case? So, we look into the second case here. So, the second case here is a if mod of lambda 1 is greater than 1. Again what happens in this case, again you have a rotation matrix and you have your lambda and your lambda will basically lambda greater than 1 means it is basically tending to infinity. So, your original is a source now, but the orbits are spiraling around it. So, the origin is a source with each orbit spiraling out from the origin. Or, I could say that each orbit is spiraling towards the origin as n tends to minus infinity.

So, this is what happens to our source. So, let us look into this picture here once again. So, they have this particular source when mod lambda is greater than 1. So, you have that n it nearby orbit is basically going to spiral around the origin and then it is going to move apart. So, origin tends to be something like a source.

And then we have the third case here, because it is quite possible or one is not an eigenvalue that is definitely true, but it is quite possible that you may have mod of lambda 1 to be equal to 1. So, our third case here is if lambda 1 is 1; what happens in that particular case.

Now, we try a look into mod lambda equal to 1 think of that what is your Jn now; it is mod lambda 1 to the power n into a rotation matrix. So, mod lambda 1 itself is 1 so that means there is nothing changing there all it is doing this rotation. So, what is your Jn doing its taking a vector and it is just rotating the vector by some angle. So, since it is just rotating the vector what you would find is that you take any orbit you start with any orbit and basically the orbit follows a circular path, center at the origin; it will just form a circular path center at the origin modulus will remain the same; it just takes a circular path center at the origin.

And in that case what would you call the origin? It is neither a sink nor a source, nor it is some kind of a unstable also that it behaves in some manner in some direction. So, we gave a special name to the origin here we says that the region happens to be a center. The origin is a center here and rest everything is just moving around the origin, right all the orbits just moving around the origin. So, if mode lambda 1 is equal to 1 then the origin is a stable center, I would call it center with each orbit following a circular path around it.

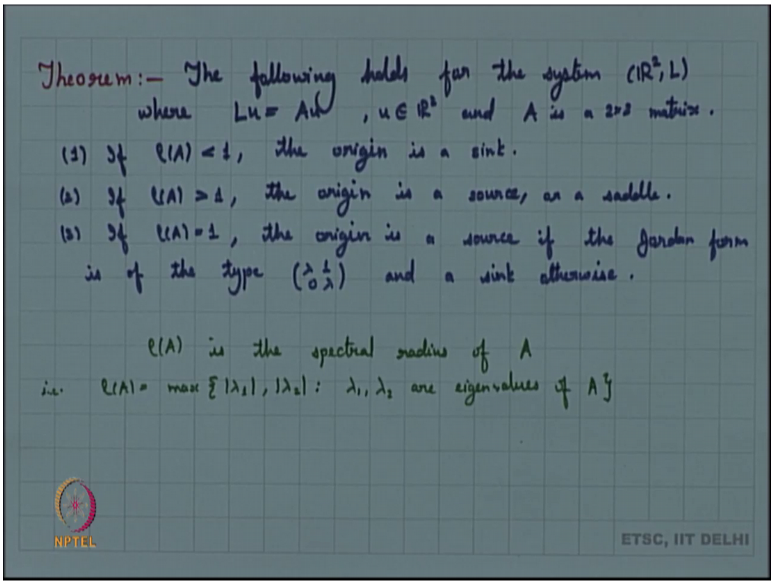
So, we get back to our picture. So, what does our picture say here is- that I have these are my eigenvalues here, alpha plus or minus I i times beta and the modulus as 1. And if you think of every orbit right, every orbit is just rotating around the center. So, origin is a center and every orbit is just rotating around the center the circulate path, right. The radius remains the same, it is just rotating around the circle.

We know that what happens to a general orbit in R square. This is basically we are looking into when 1 is not the eigenvalue; that means, they are basically looking into the case of hyperbolic. So of course, I will define hyperbolic later, but we look into the case when 1 is not the eigenvalue. And when 1 is not the eigenvalue we find that there is only one fixed point that is the origin. So, either everything is tending to the origin or everything is moving away from the origin or everything is rotating around the origin. So, these are the only possibilities that you can see. Or you can think of what (Refer Time: 38:20) to be a saddle, but then in that case also everything is moving out of the origin, right. So, these are the only possibilities. So, the dynamics is happens to be very very simple in that particular case, right.

So can you summarize all of this, because this is like we started with a Jordan form and we started discussing with the Jordan form, but as we have already seen that the eigenvalues remain the same if you consider similar matrices. Every matrix can be written in a Jordan form, right. So, ultimately if we look into what happens to the picture also, the dynamics ultimately depends on the eigenvalues; does not depend on anything else it just depends on the eigenvalues. Of course, we could see we could draw think simply, because when you look into the Jordan form you have your eigenvectors you can take here basically you can take your y 1 and y 2 to be eigenvectors in that particular case. So, you have your unit vectors here.

But what happens here is in general, if you look into any matrix A. We start with any matrix A your matrix A also tends to be having the same behavior. There will be just a change of some vectors right, so just a linear transformation. But otherwise the behavior will be the same. So, we can summarize that in form of a theorem, maybe I am writing theorem right now.

(Refer Slide Time: 39:45)



So, the following holds for the system R square L, where my Lu happens to be Au for u belongs to R square and A is a 2 cross 2 matrix. So the theorem says something like this, the following holds. So, the first thing that holds us: if rho A is less than 1, the origin is a sink; if rho A is greater than 1, the origin is a source or a saddle; the third case is if rho A is equal to 1 then the origin is a source if the Jordan from is of the form; the type I should is of the type lambda 1 0 lambda and a sink otherwise.

Now I written rho A, but I have not defined what is rho A. So, basically rho A is the spectral radius of A. That means, I can say that rho A happens to be the maximum of mod lambda 1 mod lambda 2; maximum of one of these two numbers where lambda 1 and lambda 2 or eigenvalues of A.

If we have a linear system in R square then if your spectral radius is less than 1; spectral radius is less than 1 means both your eigenvalues are less than 1. Both your eigenvalues are less than 1 the origin is definitely going to be a sink. This spectral radius is greater than 1, right. There are two possibilities here the spectral radius is greater than 1, then what happens to the second eigenvalue the second eigenvalue could be greater than 1 or it could be less than 1 the both the things are possible. And in case both of them are greater than the 1 the origin happens to be a source. In case one of them is less than 1, so the other one is less than 1 right what happens here is it becomes a saddle. So, if the spectral radius is greater than 1 the origin is a source or a saddle.

What happens if the spectral radius is equal to 1? Now this is something which we need to think because we have not thought about that aspect, we have looked into this case only when you are looking into the complex eigenvalues. So, what happens if it is equal to 1, right? Think of that. You have, if it is equal to 1 in the Jordan form is of the form say the diagonal form then basically a Jordan form is an identity matrix. What happens in that case?

Student: Identity.

(Refer Time: 44:01) going to identity nothing happens there, right. So basically that case is not interesting it all or you can think that fine reason happens to be sink whatever is there remains there itself, it say the identity case. So, it is just some kind of an identity that you observe nothing much changes there. What happens if it is of the Jordan form is the form lambda 1 0 lambda. Now you have like 1 1 0 1 and if you try to take the iterations basically you would keep multiplying that with itself, you are going to tend to some matrix which has value greater than 1, right. So, it is just start with multiply that with any vector there.

So, what happens in that particular case; is you are tending away from the origin. And when you are tending away from the origin your origin would be a source in that particular case. So, the origin happens to be a source here. So, when the Jordan form is of this particular kind for A then this spectral radius is equal to 1; we mean that this is going to happen to be a source. In the other case again the spectral radius is 1; that means, the other thing can be less than 1; the Jordan form is other kind either it is equal to 1 both of them then it is identity nothing changes. If the other eigenvalue is less than 1 then again it is going tend to the; because it is not changing but on the other side it is definitely converging, it is contacting on the other side. So, this is definitely converging to the origin.

And we have already seen what happens in the case when we have complex eigenvalues, right. Then in that case the spectral radius is 1, gives you that the origin happens to be happens to be a center, right the origin happens to be a center. So, the origin is a center I am right, I am written a sink maybe I should write it sink or center otherwise; sink or center otherwise, because that combines the other case also when our J happens to be the identity matrix. So, it is a sink or center otherwise.

So we have this. Basically the structure of the dynamics of the real systems on R square tends to be of this particular form. But the case becomes more interesting when we are trying to nothing to something like if it; look into R square of you looking into two dimensional differentiable manifold. So, in that case we are not looking our derivative right, is also in form of a matrix. And then we are trying to look into what are the eigenvalues there we try to look into that part.

So, what we are going to do now is basically we will first of all look into what happens if in case we have some kind of fine maps, because these are linear maps. We can similarly think of what happens in the case of fine maps. So, we try to see what happens in case of fine maps, or if you are not working in R square but we are trying to work into some differentiable manifold, right.

So, maybe that is what we would try to look into now this lecture, but today we stop here.