Chaotic Dynamical Systems

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Lecture – 31
Linear Systems in Two Dimensions

Welcome to students; so, we have looked into dynamical systems on the real line and we have looked into dynamical systems in general. What we have observed is that in the real line; chaos basically comes up because of the non-linearity of the system. So, what happens in case of linear systems? Can we really experience some chaos kind of properties for linear systems?

So, to know the answer you can move into one dimension higher of course, you can study linear system in any dimension, but we will be specifically restricting ourselves to study of linear systems in two dimensions. So, we are looking into linear systems in two dimensions.

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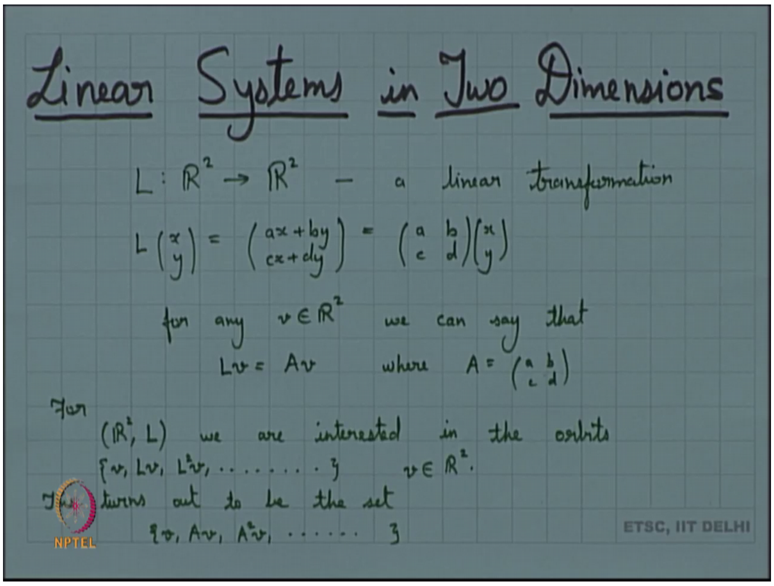

And we can say that you think of L to be; say from R square to R square, a linear transformation. So, typically we can write L as L of x y, since this is a linear transformation this would be something of the form ax plus by and cx plus dy.

So, this basically can be written as the matrix a b c d; multiplied by the vector x y. So, for any vector v in R square we can say that; L of v is some matrix A of v; where A is some two by 2 matrix say a b c d. Now we are interested in the dynamical system; so, supposing we are looking into this dynamical system; R square and L, then we are interested in computing the orbits here. So, basically for the system R square L; we are interested in the orbits, so we have v; L v; L square v and so on.

So, we are interested in this particular orbit; for each v in R square, but if I try to look into what is v, L v because; your L v can always be defined as A times v. So, basically this turns out to be the orbit, this turns out to be the set and what is that set? So, we have v; A v, A square v and so on.

So, we are only interested in computing this set, so ideally in order to study linear systems in two dimensions or maybe in any dimension; whenever we want to study linear systems, we are basically interested in computing; the power of the matrices and when we want to compute the power of matrices, everything your entire problem of finding the orbit first of all reduces to finding the power of the matrix.

So, what is important over here is to find out how do we compute A to the power n. So, let us look into this aspect; now this is something which is trivially done in a say advance course on linear algebra. But since I know many of you will not have that background, so we are going to do some basics today. So, that is what is the intention of this lecture; we will be looking into basics of how to compute a to the power n for various n and we are restricting ourselves again to 2 cross 2 matrices; though in general the theory holds for any dimension.

So, let us look into this concept; this aspect of computing A to the power n.

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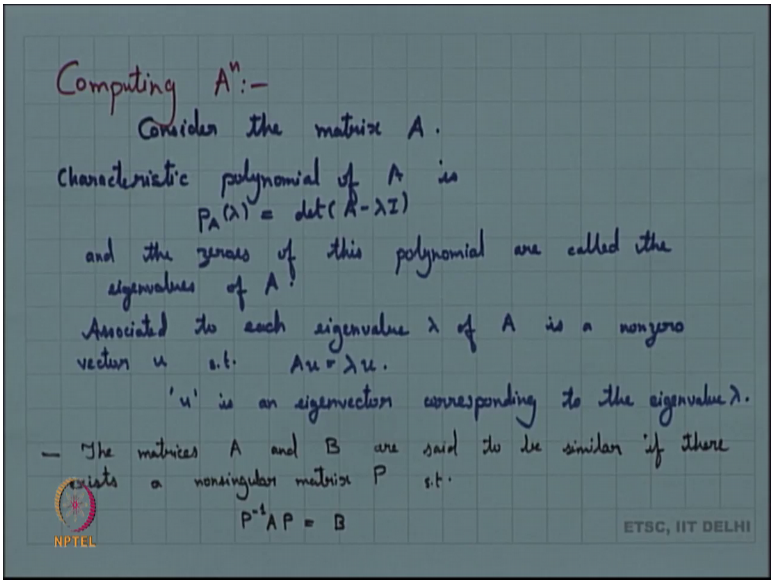

So, we are interested in computing A to the power n for that; we realize that suppose I have a matrix A. So, consider this matrix A now; now I am not looking into the dimension because this general theory is true for any dimension. So of course, we are interested in square matrices, so we are looking into a matrix A and the characteristic polynomial of this matrix A.

So, the characteristic polynomial is basically I can think of this as P A lambda; which is given as the determinant of A minus lambda times I; where again you are looking into the same dimension as A and this is a characteristic polynomial and the 0's of this polynomial are called characteristic values or maybe the Eigen values.

Now, this is which something everybody knows and we also know that associated to each Eigen value lambda of A is a non-zero vector say U; such that A times U is just lambda times U. In fact, many times we compute the Eigen values with respect to this term because U is a vector for which this equation is satisfied. So, looking into any such combination if you get a vector U; such that au equal to lambda times U, then you say that this happens to be lambda is your Eigen value and then such A U is called an eigenvector.

So, U is an eigenvector corresponding to the Eigen value lambda. So, U is your eigenvector corresponding to Eigen value lambda and this is basically your structure what else can we say about matrices? So, we also have this concept of similar matrices.

So, we say that the matrices A and B are said to be similar, if there exists a nonsingular matrix P such that your P inverse A P is B.

So, two matrices are similar; if there is a way of transforming one matrix to the other metrics via nonsingular matrix.

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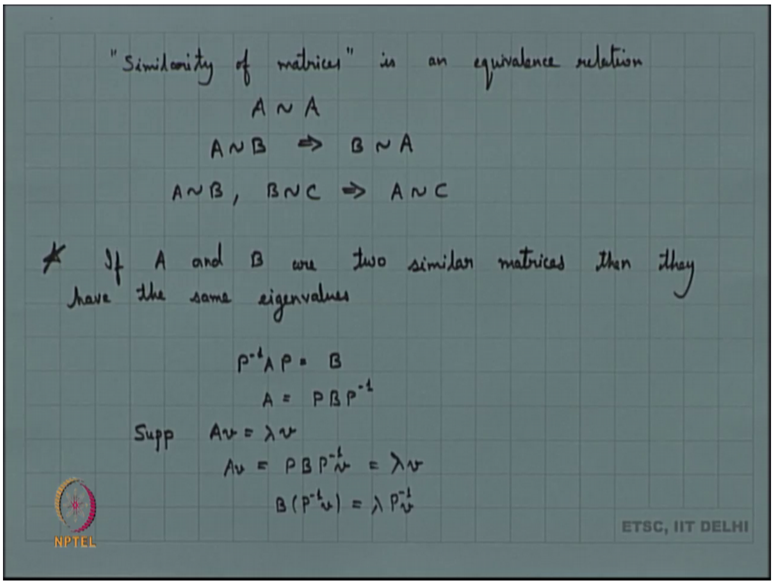

And this is something which I think everybody knows quite elementary that this similarity of matrices is an equivalence relation. What do we mean by that? Well you know that A is always similar to itself and then we know that if A is similar to B; then implies B is similar to A; can you B easily seen, A can be written as P A; P inverse and again this is again some extrapolation. Here we know that; if A is similar to B, B similar to C that would imply that A is similar to C, it just substituting B in that particular situation.

So, we know that this is already we know that this happens to be in equivalence relation. So that means, if you want to just look into some property of matrices; it is enough to consider the equivalence classes and just one element from each equivalence class. Then there is another aspect to the switch, perhaps we all know that is that if A and B are 2 and B; they are two similar matrices, then they have the same Eigen values.

Of course, eigenvectors need not be the same, but they definitely have the same Eigen values and its very simply to see this one. Because if I have this P inverse A P to be equal to B; then I can write my A as P; B P inverse.

Now, suppose your A of v equal to lambda v; then what we know here is that A of v will be equal to P; B P inverse of v and this is going to be your lambda v. So, if we forget this first part; we just look into the second part that would give us that P of B of P inverse v is same as lambda of P inverse v.

So, basically you have a vector a nonzero vector satisfying this equation P of U equal to lambda U and hence lambda happens to be the Eigen value. So, when we are trying to look into computing because we are more interested in computing A to the power n. We want to look into study of similar matrices, we try to look into something more on similar matrices.

Now this is something which I am going to prove; this is an very elementary theorem because we are doing it for two by two, but this is true for any general n dimensional matrix. So, basically any n cross n matrix and you can see that in various books, but since this is not been covered for you at your level, I will be looking into this particular theorem.

So, I look into this theorem and we will basically we throw in this theorem.

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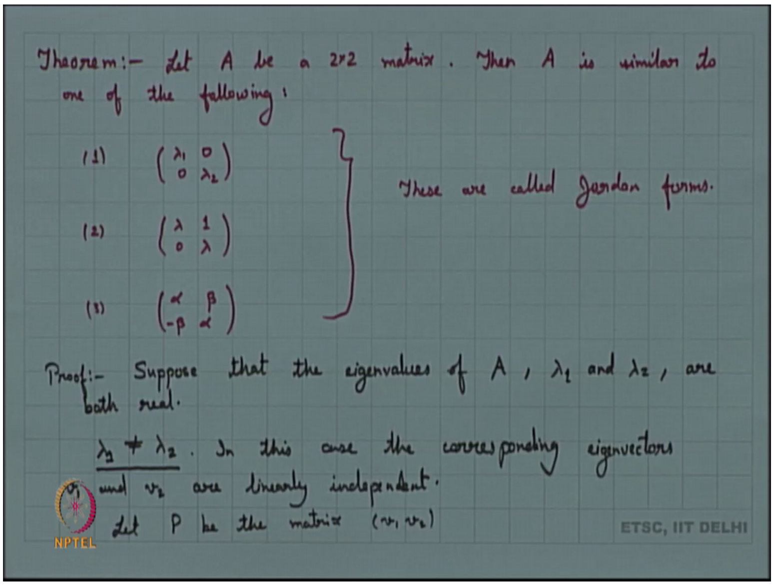

Because that will help you in trying to see how to work out with these ideas, so let A B; a 2 cross 2 matrix; then A is similar to one of the following. Now what are these ones? So, the first case here happens to be a matrix of the form lambda 1 is a diagonal matrix, with diagonal entries lambda 1 and lambda 2. So, this could be one case; the second case could be a matrix of the form lambda 1, 0 lambda. The second case and the third case; it could be a matrix of the form say alpha beta minus beta and alpha this could be the third form.

So, we want share that any 2 cross 2 matrix is similar to one of these and actually this is; what is the theory of Jordan. So, these are basically called the Jordan forms, so these are the Jordan forms and as I said that; this can be considered for any n cross n matrix.

So, let us try to look into the proof of this theorem. So, look into the proof of this theorem; we start with our matrix A. Now, since we know that all similar matrices will have the same Eigen values; we are looking into the Eigen values. Now A is A; 2 cross 2 matrix, so if you look into its characteristic; polynomial characteristic, polynomial will have degree 2 and we know that very well that is polynomial of degree 2 can have at most two roots. And since we are working with a real polynomial now, it is possible that our two roots are complex root.

So, we are looking into the first case; where we have two roots and both the roots are real. So, suppose that the Eigen values; let me name this Eigen values. So, this Eigen values are lambda 1 and lambda 2; so, these are the 2 Eigen values of A and these are both real. Now, first case is the characteristic polynomial; it has both real roots; so I have real case.

Now, when lambda 1 and lambda 2; there are two real roots, there are two possibilities either they are equal or they are not equal. So, let us assume first of all that lambda 1 is not equal to lambda 2. So, we have this lambda 1 not equal to lambda 2; think of that part now, this is something which is very elementary and which everybody knows that; if the Eigen values are not equal the corresponding eigenvectors, will be linearly independent.

So, in this case; so let me write this case; so in this case the corresponding eigenvectors. So, let me name the corresponding eigenvectors the corresponding eigenvectors is v 1 for lambda 1 and v 2 for lambda 2 are linearly independent. So, we know that these are now linearly independent; what we try to do is; we try to form a matrix out of these vectors.

So, we make take a column of linearly independent vectors and try to form a matrix and we very well know that such a matrix will turn out to be nonsingular. So, let P B; the matrix may looking into this particular matrix v 1, v 2. So, this is the matrix with column says eigenvector and then we very well know that P is non singular.

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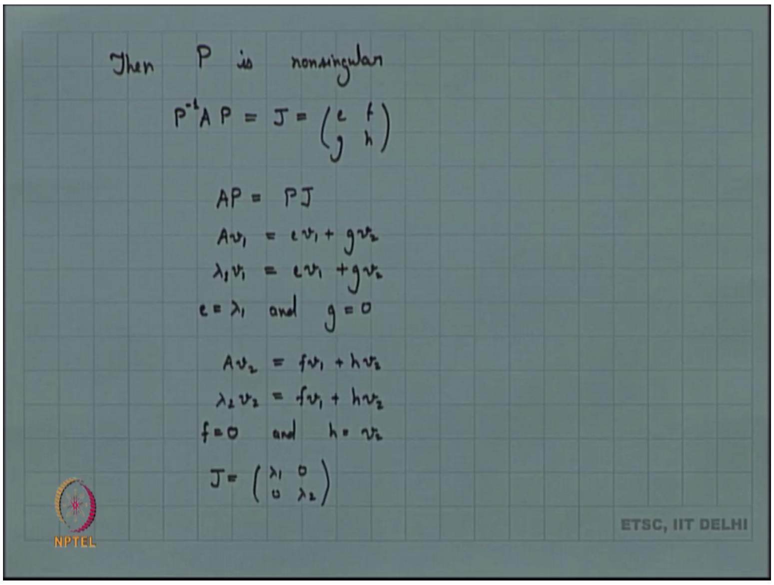

Suppose, now I am interested in looking into this case, what is P inverse A P? What is this going to be equal to? So, we know that this is all I am taking multiplication of 2 cross 2 matrices. So, the resultant will also be a 2 cross 2 matrix, so let me take this to be equal to; I am calling this to be J and that this J be equal to say e, f, g, h; already our A B; we are taken up to be a, b, c, d; so does not matter, we can take this to be e, f, g, h.

So, suppose P inverse A P happens to be e, f, g, h, so what does that mean? Now, so if I look into A times P that happens to be equal to P times J. Now let us look into P A; my P happens to be v 1 v 2, so I am looking into the spot. So, what is my A v 1; take this multiplication A v 1, I can take the similar multiplication here because my P is v 1, v 2 and we have already taken our J to be e, f, g, h. So, in that case your A v 1; it turns out to be equal to e times v 1 plus g times v 2.

Now, A v 1; we very well know v 1 happens to be an eigenvector corresponding to the Eigen value lambda 1. So, your A v 1 will actually turn out to be equal to lambda times v 1. So, we have lambda 1 times; v 1 is equal to e times v 1 plus g times v 2, what do we know about our vectors v 1 and v 2? These are linearly independent. So, if I try to look into these equation; this equation will tell me that e should be equal to lambda 1 and g should be equal to 0.

So, your e happens to be equal to lambda 1 and your g is basically equal to 0. So, we found out the values of e and g; we can similarly find out the values of f and h. So, all we need to look into is what is your; A times v 2? So, your; A times would turn out to be f times v 1 plus h times v 2; f times v 1 plus h times v 2. We already know that v 2; again is an eigenvector corresponding to the Eigen value lambda 2. So, A times v 2 happens to be equal to lambda 2; v 2. So, lambda 2 v 2 is basically f times v 1 plus h times v 2.

Now, again the same condition linearly independency of v 1 and v 2 that gives us that f should be equal to 0 and your h should be equal to v 2. So, what we get here is we get that our J which was a similar matrix to A happens to be equal to lambda 1, 0, 0 and lambda 2. So, if lambda 1 is not equal to lambda 2; we find that this matrix A is basically similar to a matrix of the form 1; what happens if lambda 1 is equal to lambda 2?

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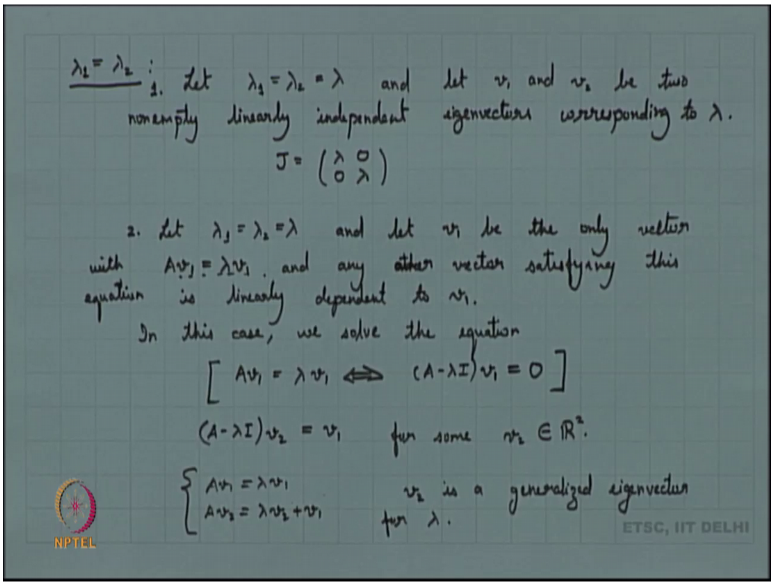

So, I am looking into lambda 1 equal to lambda 2; it is quite possible that we may have two Eigen values to be the same because we are looking into the multiplicity of lambda 1 the characteristic equation.

But then it is also possible that, but in this particular case; we may have the Eigen space corresponding to this lambda; what have dimension to; so, we can get two linearly independent vectors v 1 and v 2 corresponding to the same Eigen value lambda. So, let the first case we look into is; let me write it here 1, the first case we look into is let lambda 1 is equal to lambda 2; let that v equal to lambda and let v 1 and v 2 be 2, non empty linearly independent eigenvectors corresponding to this Eigen value lambda.

Now, we know that what is an Eigen space? Because we have only one Eigen value; this Eigen value has an Eigen space and the Eigen space here, we are assuming to be equal to 2; it is only 2 because we are working with R square, so it is only 2. So, this Eigen space is 2 and we have 2 linearly independent vectors lambda v 1 and v 2. So, again what we do is this reduces to the previous case, the previous case we had v 1 and v 2.

So, we can take P to be equal to v 1 and v 2 that is a nonsingular matrix and then what is your P inverse AP? So, that would be a matrix J and this matrix J would not turn out to be equal to; you can use the same method here. It gives you the matrix 0 lambda; so it is again a diagonal matrix, now with the diagonal entries 0 lambda because here your lambda 1 turns out to be equal to lambda 2; which is same as lambda.

So, this reduces to again the first case; the previous case what happens when lambda 1 is equal to lambda, but the dimension of Eigen space is 1? What happens in that particular case? So, we look into the second case here. So, we have let lambda 1; the same as lambda 2 the same as lambda and let v 1 be the only vector with A times v 1; equal to lambda times v 1 and any other vector satisfying this equation. So, when I call this equation basically I am in this equation.

So, any other vector satisfying this equation is linearly dependent to v 1 so; that means, now I have an Eigen space with dimension 1; what happens in that particular case? So, when we try to look into that particular case, we solve this equation. So, in this case what is the equation that we solve? So, solve this equation that; so we have this equation A times.

So, we consider this part; I am just looking into this part, we have this equation A times v 1 is lambda times v 1; which is same as saying that I am looking into A times; A minus lambda I. So, this is again my matrix when I apply this to v 1, this basically turns out to be equal to 0.

So, what I have here is that A times lambda I in the direction of v 1 is not giving me anything, it is equal to 0. So, we are now interested in looking into something else, so you want to solve this equation. So, we solve this equation which I am saying that A times lambda I; v 2 is equal to v 1 for some v 2 belonging to R square. So, what we try to do here is 1; A minus lambda I; v 1 was bringing A minus lambda I to 0; what we want to do is we want to take up v 2 such that it brings A minus lambda I onto v 1.

So, we try with this particular matrix; we try with this particular solution v 2. So, what do we have here is; so the equations we have here is A times v 1 is lambda times v 1 and A times v 2 happens to be equal to lambda times v 2 plus v 1, we have basically this set of equations.

Now, if we try to look into this particular v 2; this v 2 is not an Eigen value, not an eigenvector for A, but there is something interesting to v 2 is that; it is linearly independent v 1; v 2 and v 1 are linearly independent. And then, if you look into v 2; what you get here is it specifically satisfying some kind of a generalized condition for Eigen vector. So, it is not an eigenvector, but it is a generalized eigenvector for lambda. So, this v 2 is a generalized eigenvector for your lambda.

So, now you have two things here; for lambda you have one; you have a eigenvector v 1 and second what you have is a vector, which is linearly independent to v 1 and this is a generalized eigenvector. Now think of that; now you have taken up the generalized eigenvector, you start looking out for what is your P. So, you start looking into P of course, there is again I would like to say that; if you look into the general case, there is a theory behind why you should take v 2 in this particular form, but we are not getting into that particular theory.

So, we are not getting into Jordan theory at A; I am just trying to explain that this holds to U; v 2 here that you get here will be linearly independent to v 1. So, we try to look into what is your P?

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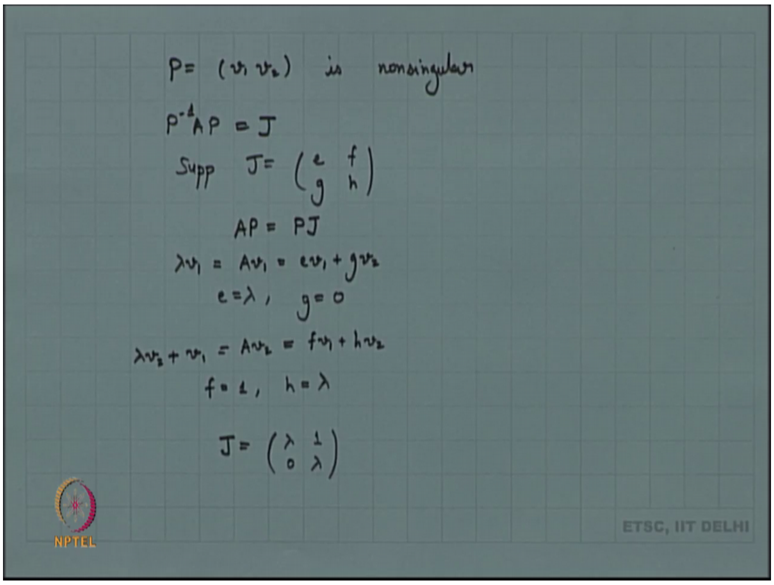

So, your P again is a matrix with your columns as v 1 and v 2; so, this is nonsingular. So, P is this and this is nonsingular; so definitely if I look into P inverse A P; this is basically my J and in that case my J would turn out to be a similar matrix to A. So, basically J is a similar matrix to J and what is your J. In this particular case, we try to compute J; in this particular case let us go back to looking into your J equal to e, f, g, h. And then what you have here is you have these 2 equations, your A v 1 happens to be equal to lambda times v 1, which is basically going to give your g to be equal to 0 and your e to be equal to lambda.

So, this gives your e as lambda and g as 0; on the other hand what the second equation that we have here is A v 2 happens to be equal to lambda times v 2 plus v 1. So, that gives your f to be equal to 1 and your h to be equal to lambda. So, again let us try to write this down again. So, suppose your J was e, f, g, h; then since your A P happens to be equal to P J, so you have A v 1 which was your e v 1 plus g v 2.

And now this A v 1 happens to be equal to lambda times v 1 and hence your e is equal to lambda and your g happens to be equal to 0. On the other hand, you have your A v 2 to be equal to; this is f v 1 plus h v 2, but your A v 2 here happens to be equal to lambda times v 2 plus v 1. And now, if you try to compare what you get here is f is equal to 1 and your h is equal to lambda; your J in this particular case turns out to be equal to the from lambda 1; 0 lambda.

So, this is your J and this is our similar matrix to A; in the case when A has just one Eigen value and we very well know that for similar matrices, the Eigen values will be the same. And the dimension of the Eigen space will also be the same of course, Eigen vectors could be different, but the dimension of Eigen space would be the same.

So, we have that in this particular case; that means, we are looking into the second case when we had real we are looking into the case; when your Eigen values are real and when your Eigen values are equal such that the dimension of the Eigen space is 1; your matrix is similar to matrix of the form lambda 1, 0 lambda. So, this is basically again Jordan form.

We now look into the third case; the third case comes out to be why should we assume lambda 1 and lambda 2 to be real? There could be something else also. So, we look into the case when lambda 1.

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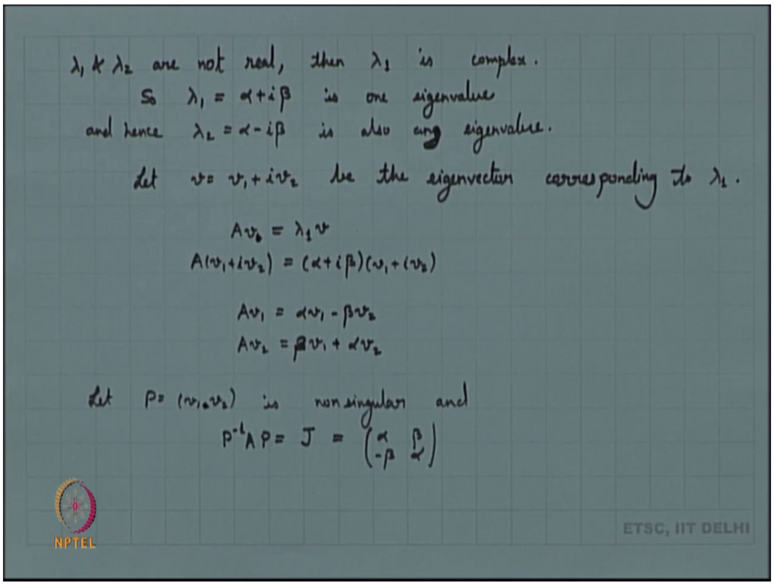

And lambda 2 are not real definitely; if they are not real they will be complex. So, we have lambda 1 equal to; so, your lambda 1 is equal to say alpha plus I times better this is a complex Eigen value. Since lambda 1 is one of the Eigen values, so this is one Eigen value and we very well know that; being a looking into Eigen values and especially when we are looking into complex Eigen values, they always occur in pairs. So, the complex conjugate will also be an Eigen value here.

So, this is also an Eigen value now we have two Eigen values here. So, let us take a corresponding eigenvector; now think of that our matrix is real, our Eigen values complex. So, when we are talking of a complex Eigen value we know that the complex Eigen value will have two components; it will have a real component and imaginary component. So, if you are trying to look into the eigenvector here; the eigenvector also will have a real component and an imaginary component.

So, here let v v 1 plus this be the eigenvector corresponding to lambda. So, now, we have an eigenvector corresponding to lambda 1. So, let us try to see the equation here A v is equal to lambda 1 times v; which gives me that A of v 1 plus I; v 2 is lambda 1, but my lambda 1 happens to be alpha plus I beta and this is my v 1 plus I v 2.

So, that gives me what is your A v 1; your A v 1 turns out to be equal to alpha times v 1 minus beta times v 2 and your A v 2 turns out to be equal to alpha times v 1; sorry you are looking into A v 2. So, that turns out to be beta times v 1 plus alpha times v 2. So, trying to take this part; your A v 1 and A v 2 and we know that A v 1 and A v 2 gives you the column of the similar matrix to A. So, that gives you the column of the similar matrix to A and hence in this case, so again we let our P equal to v 1; v 2 sorry this is a matrix.

So, P is a vector; P is basically a matrix of columns v 1 and v 2, you know that; in that case v 1, this P is nonsingular and what we know here is that P inverse A times P will be equal to J. And what is your J? In this particular case; you just comparing that here; your J will turn out to be alpha minus beta; beta and alpha we have the Jordan form, we have a complex.

If you have at least one because once you have one Eigen value as complex, basically the second one will also be a complex; it will be a complex conjugate. So, if your Eigen values are complex in case of 2 cross 2 matrix; your Jordan form turns out to be of the form alpha beta minus beta alpha and this is one of the forms here which we can look into.

Now, why did we do all these things? We wanted to look into computing A n; so we are interested in finding the orbit. So, we did all this; so this is basically the proof of the theorem, we did all this to compute A n.

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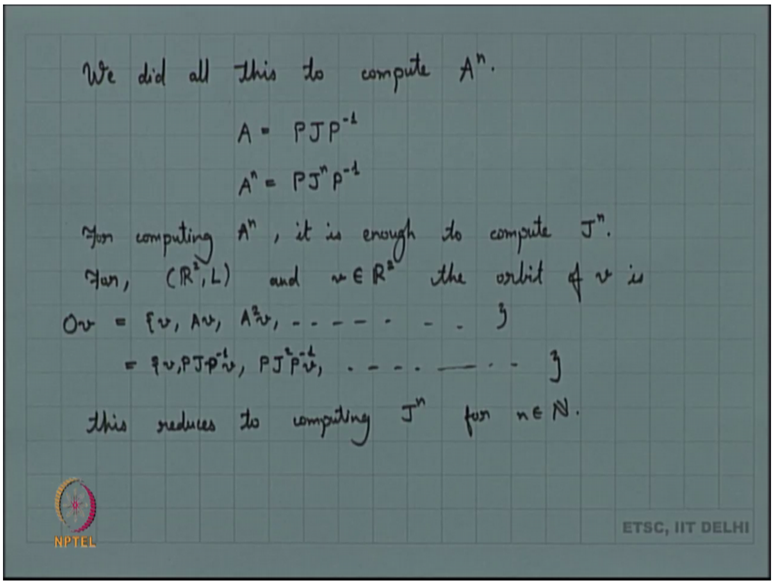

So, what is my A? Here if I look into the fact here is that; A can be written as P; J P in once, where J we can think of to be our Jordan form. So, if it look into a is P I P; inverse, then your A to the power n; what is that equal to? So, this is P J to the power n and P in once so; that means, now in order to compute A to the power n, it is enough to compute J to the power n. And it is easier to compute J to the power n because J is in a very simple form, if you are writing J in a very simple form. So, it is easier to compute what is your J to the power n?

So, what we have here is A to the power n; S P J to the power and P inverse and hence we need to compute A to the power. So, for computing A to the power n, the reason here is that; if I look into the orbit. So, basically I am looking into the system R square n L;. so for the system R square L and any v belonging to R square, the orbit of v looking into the orbit of v. So, the orbit of v is nothing, but v; A v; A square v and so on which turns out to be nothing, but v P; J P inverse v P, J P inverse v then I have P J square P inverse v and so on.

So, all we know is now about orbits; so, we need not look into our matrix A at all, once we know what are the Eigen values; eigenvectors of; we just need to find out what are Eigen values and eigenvectors or generalized eigenvectors of A. It is enough to compute our J, compute our P, compute our P inverse and all we can do is; now we can find out the orbit of v by looking into this structure.

Ideally this reduces the; still does not solve our problem, this now reduces to computing J to the power n. So, this reduces to computing and now when we try to compute this for J to the power n; at now similar what happens when J; so, let us look into this the first page.

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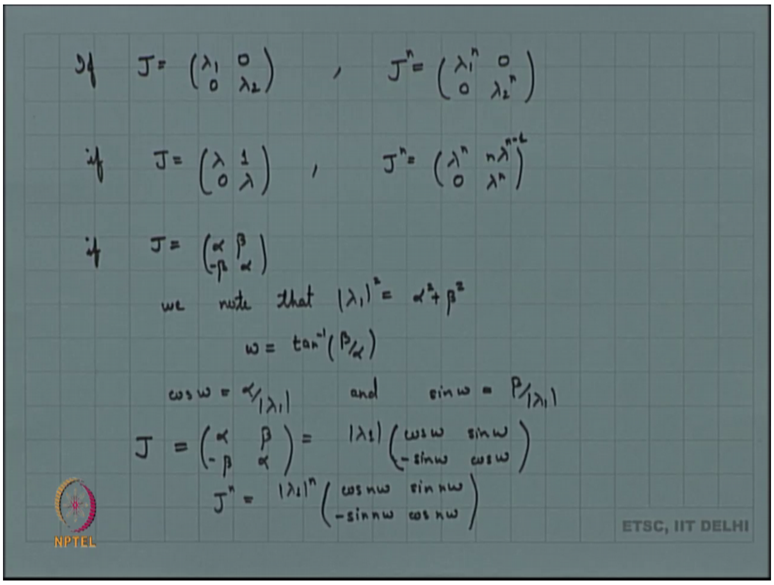

So, what happens if your J happens to be equal to lambda 1, 0, 0; lambda 2, what happens in that case? What is your J to the power n? This is a diagonal matrix.

So, again if I look into this part; your J to the power n would just turn out to be lambda 1 to the power n 0; 0 lambda 2 to the power n. So, computing this is very very easy in case J; happens to be lambda 1, 0, 0 lambda 2; what happens now if your J is of the form lambda 1, 0 lambda? What happens in that case? What is your J to the power n? Here try to seel what is your J square exactly. So, your J to the power n turns out to be lambda to the power n; n lambda 0 and lambda to the power n, sorry this would be lambda the power n minus 1 here.

So, J to the power n turns out to be lambda to the power n; n times lambda of the power n minus 1,, 0 lambda to the power n. Now this is quite interesting fact here what happens when your J happens to be of the form alpha beta minus beta alpha? What happens in that case? This is slightly tricky, but we can think of this looks quite similar here.

So, what you try here is that we note that your lambda 1; I am looking into mod lambda 1 square, what is mod lambda 1 square? That would turn out to be alpha square plus beta square lambda 1 was here Eigen value; it is a complex number. So, looking into the magnitude of the complex number; so that was your alpha square plus beta square.

Now, what we do here is we put some trick here; we take our omega to be equal to tan inverse beta by alpha. And in that case; what we have here is we have our cos omega to be equal to alpha by mod of lambda 1 and sine of omega to be equal to beta by mod of lambda 1; definitely know that cos square omega plus sin square omega should be equal to 1.

So, what we get here is that cos omega is alpha by mod lambda 1 and sine omega happens to be beta by mod lambda 1; that cases the value of alpha and beta. So, in that case; now I am looking into this matrix alpha beta minus beta alpha. This turns out to be; now I am looking at listen to the form of cos and sine of omega. So, this turns out to be first of all; I have here is mod of lambda 1 times cos omega sine omega; minus sine omega and cos omega.

Now, it is very clear what would be our n th power of this? So, this is our matrix J; I am writing my matrix J in this particular form. It is easier to compute, what is J to the power n? So, J to the power n lambda 1 to the power n; I can understand. So, this turns out to be cos n omega sine n omega; minus sign n omega right cos n omega.

So, your J is put up in a very simple form and that makes it easier to compute; what is your J to the power n? You already know how to compute your P n; P inverse. So, what happens here is now you can compute your orbit, for any given v; you can compute your orbit. So, I am today not going to leave you as it is; I am at least going to give you some homework today.

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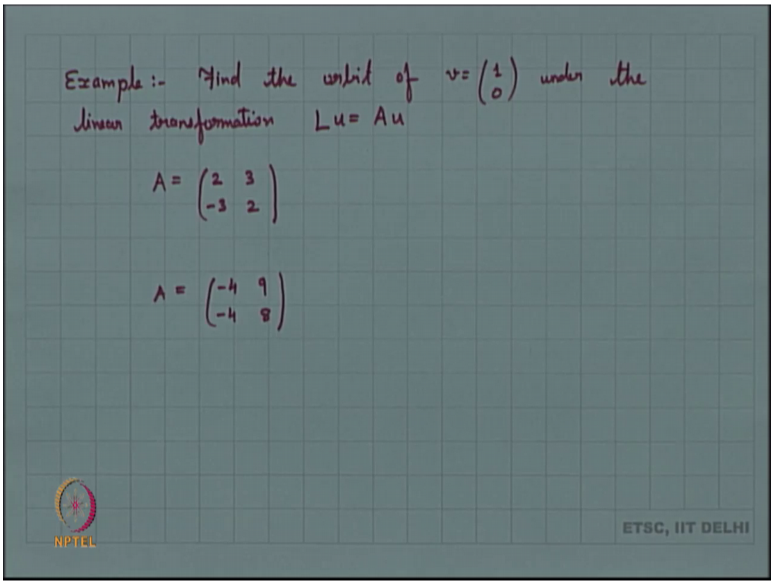

So, I want you to solve this particular example. So, find the orbit of 1; so now I am taking this to be my unit vector v equal to say 1, 0 under the linear transformation. So, my linear transformation is L times u; equal to A times u. So, this is my linear transformation and we are looking into the case 1; A can be given in form of 2, 3; minus 3, 2; A is 2, 3; minus 3, 2; does it ring a bell? How do you compute that?

Now can try something with tan inverse 3 by 2, now I am looking into the second case here; here A happens to be equal to minus 4, 9 minus 4, 8; how do you compute this particular kind of a; what is the Eigen value here? So, the dimension of Eigen space turns out to be equal to 1. So, now you know what is this similar to? All you need to find out is a generalized Eigen vector and compute this part.

So, computing orbits of linear system; it is basically looking into the Jordan form and trying to compute the orbits corresponding to that. I hope this is clear to you, so today; we stop here, we will look into more things details of the dynamics in the next part.