Chaotic Dynamical Systems
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Lecture – 30
Equivalance of the two definitions of Topological Entropy

Welcome to students, so we have seen two definitions of topological entropy. We have seen Adler's definition of topological entropy and we have also seen Bowen's definition of topological entropy. Now in the case of a compact metric space both these because one uses the compactness of the space and one uses the metric property.

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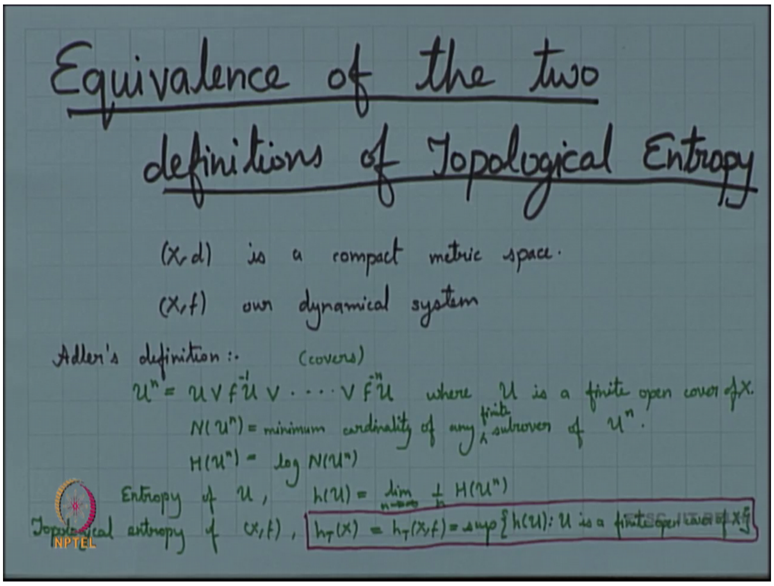

So, when we are in the realm of the compact metric space; both these definitions coincide and today we will be looking into how this two definitions coincide and some properties that we can speak out of that.

So, we start with our basic assumption; so our basic assumption here is that X d is a compact metric space and we have X f; our dynamical system. We recall Adler's definition of topological entropy and Adler's definition of topological entropy is basically using covers. So, we have U n; which is basically the join of U f inverse U f minus n u; take this join and this, where U happens to be a finite open cover of X and you take N; U n to be the minimum cardinality of any finite sub cover of U n, anyway our cover is finite.

So, you take N; U n to be the minimum cardinality of any finite sub cover of U n and then you define the growth rate. So, your growth rate H; U n happens to be or you can think of that to be log of N; U n and then the entropy of u; that is the entropy of a open cover u, happens to be h u, which is the limit as n tends to infinity 1 by n H; U n. And then you define the entropy of u; so the topological entropy of u; sorry of the system X f; is defined as h T X; which is same as h T X f; which happens to be the supremum of h U such; that U is a finite open cover of X.

So, we define this to be our Adler's definition of topological entropy. So, you can say that this is what is Adler's definition of topological entropy that; h T X happens to be the supremum of h u, that is the entropy of a open cover where U can be taken up to be any finite open cover of X. Now we move on to defining Bowen's definition, so what is Bowen's definition?

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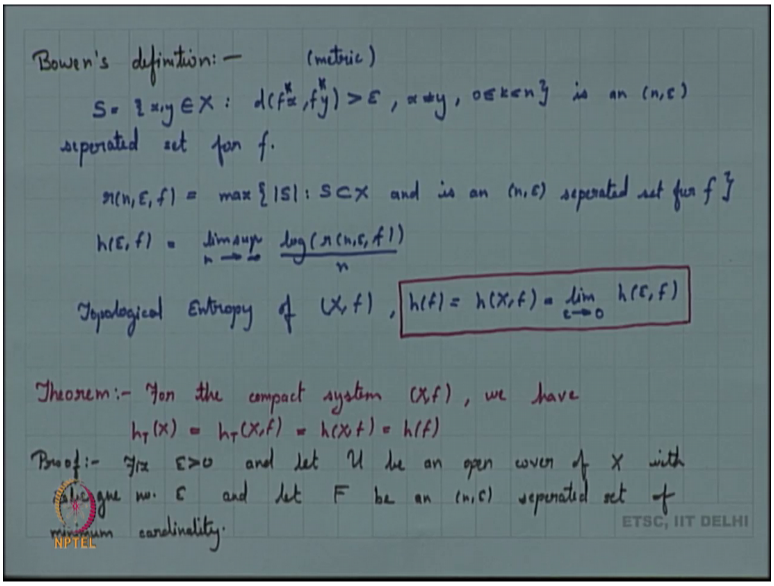

Now, if you very well recall that Bowen's entropy uses the concept of a metric. So, here we define S to be basically the set of all x y in X all such pairs you are collecting such that d of f n x, f n y is or I should say f k x; f k y is greater than epsilon. Whenever x is not equal to Y and 0 is less than or equal to k is less than n. So, this S basically comprising of all such points x and y happens to be an n epsilon. So, this is an n epsilon separated set; for f and then we define r n epsilon f; would be basically the maximum of the cardinality of an n epsilon separated set.

So, your S is a subset of X and is an n epsilon separated set and then we have the growth rate h epsilon f; defined as limsup as n tends to infinity of log of r n epsilon f divided by n. And then the topological entropy of X f is defined as h f, which you can also write it as h; x f just limit as epsilon tends to 0 of h epsilon.

So, this is basically our definition of Bowen's entropy and to see to it both these definition seems to be very different because these both of them are having different approaches. But we will see to it that this is indeed one and the same thing, so for a compact metric space both this definitions coincide. So, let us look into the theorem here; for the compact system X f, we have that is our h T X; Adler's definition of entropy is same as h T; X f we can also push that part and writing in this form which is same as h X f, so this is Bowen's definition.

So, these coincide and we now look into the proof of this part. The proof involves a little bit design here and we will see that; that approach also helps us in saying that somethings are equivalent. So, it gives us more information than what we would ordinarily be seeing in a proof. So, fix an epsilon positive and let U be an open cover of X; now actually we want to relate epsilon with U and since we are working with any U. So, we are looking into an open cover of X with lebesgue number epsilon.

And we let F be an; n epsilon separating set of minimum cardinality. Now, here I want to actually draw your attention to something; if we look back to our definition of Adler, we are looking into an Bowen's definition itself, we are looking into the r n epsilon f happens to be; we are looking into the maximum cardinality here. Whereas, here we are trying to look into minimum cardinality, so we have an n separate in n epsilon separated set, but we are trying to look into the minimum cardinality.

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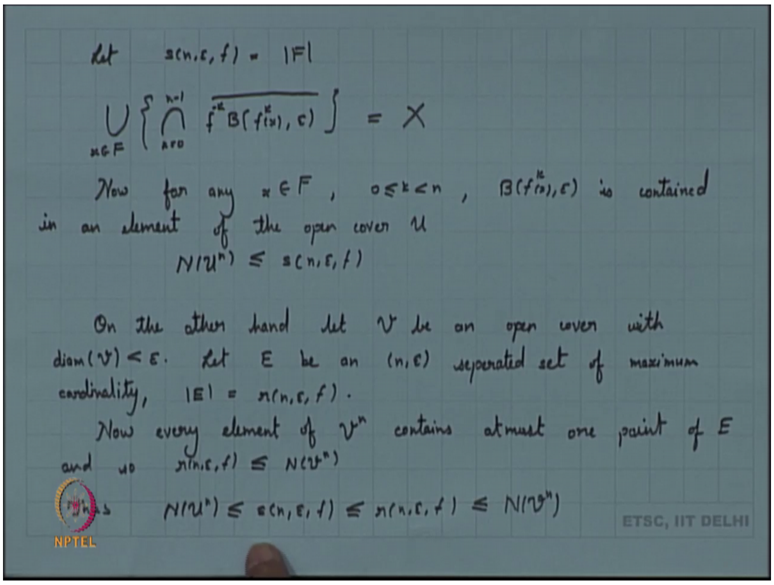

Now, think of that; so we are now defining here; let S n epsilon f be basically the cardinality of f. Now, think of that this is basically the minimum cardinality; that means only these many orbits are there, which are separated and epsilon separated. So, if you take rest of the orbits of X; they will be close enough to one of the orbits, at some particular point.

So, these are the only points which are n epsilon separated and hence what we try to do is; we look into; a ball centered at f k x; radius epsilon, take f minus k of that; take its closure, take the intersection of all these close sets for k going from 0 to n minus 1 and take the union of all these sets for all X belong to f.

This basically would turn out to be the whole of X because all orbits are epsilon close to some of the orbit in this n epsilon set. So, this turns out to be X; now for X in f, any X in f and 0 less than or equal to k; less than n, take this ball B, f k x; epsilon. Now, this is a ball of radius; I think my lebesgue number should be 2 epsilon here; does not matter. So, if you try to take this particular ball; so this ball will be contained in one of the open covers, one of the elements of the open cover because the open cover has lebesgue number of 2 epsilon.

So, this is covered in contains; so this is contained in an element of the open cover U. You find that f k x; one of them is indefinitely contained in an element of the open cover U. So, what is the minimum cardinality of U n that would be covering X? Given minimum cardinality of U and covering X; so for each of this k you have one here and so what happens here is your N; U n is less than or equal to S n epsilon f; on the other hand, let v be an open cover with diameter of v; less than epsilon.

Let E be an n epsilon separated set of maximum cardinality; what happens now? So, basically your cardinality of E is your r n epsilon f; now every element of v n contains at most one point of E. So, you looking into v n; so, you looking into the cover generated by joining v; with f inverse v and so on. Now, if we think of that part and since this is an n epsilon separated set; every element of E, so every element of v n will contain at most one point of E and so your r n epsilon f will be less than or equal to n of v n.

What is the resultant here is; that your n of U n is less than or equal to S n epsilon f; this is less than or equal to r n epsilon f and this is less than or equal to n of v n.

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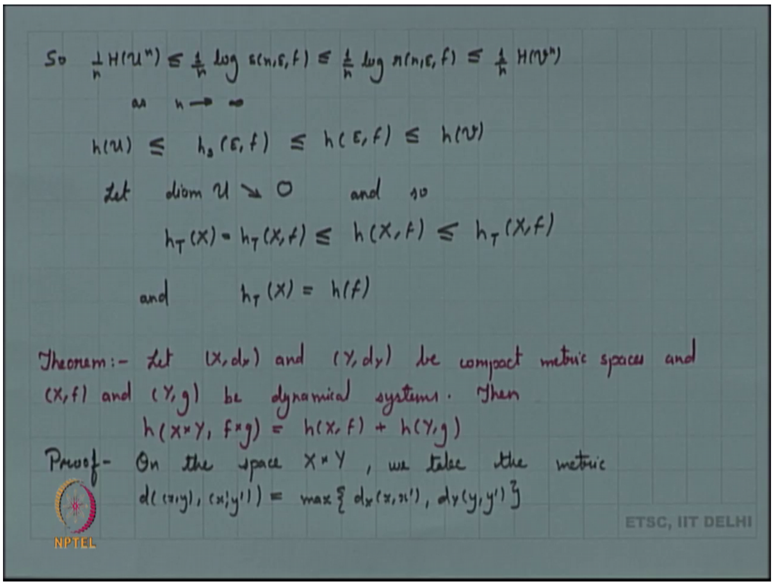

And so, we can say that if I take 1 upon n; H of U n; this will be less than or equal to 1 upon n; log of S, n epsilon f will be less than or equal to 1 upon n, log of r n epsilon f and this will be less than or equal to 1 upon n; H of v n. Now, we can think of taking n tending to infinity, so as n tends to infinity; what we get here is this is basically giving us the growth rate now, so at all places we get growth rate.

So, we have this h of U to be less than or equal to; now this is something which is coming out of the minimum cardinality of an n epsilon set. So, we given an another name to it; we say that this is h s of epsilon f, this is less than or equal to h epsilon f and this is less than or equal to h of v.

Now think of that our U has lebesgue number epsilon, our v has diameter epsilon and all for our separated sets, these are epsilon separated sets. So, as we take epsilon tending to 0; we can think of taking epsilon tending to 0, what we get is we will get a supremum for h u, we will get a supremum for h v and this definitely is going to give us the Bowen's entropy for f.

Let diameter; so, because this is having lebesgue number epsilon. So, we are taking let diameter of U tend to 0 and so, since diameter of U is intending to 0; that means, that all of these diameter is also tending to 0 and which basically means that epsilon is tending to 0; so because epsilon happens to be the lebesgue number there. So, as my diameter of U is increasing tending to 0, so what we have here is that topological entropy of X basically I should write it as X f less than or equal to the Bowen's entropy of X f, this is less than or equal to again the topological entropy of X f because this gives the topological entropy.

And these basically give the Bowen's entropy. So, we can say that this is same as Bowen's entropy and Adler's entropy or one and the same thing. Now this gives another information to us that if we start with an n epsilon set of minimum cardinality; we still get the topological entropy out of it. Though we started with maximum cardinality and to start with minimum cardinality also, we get the topological entropy; we start with maximum cardinality we also get the topological entropy. And hence we can start with any cardinality of n epsilon set, we get the topological entropy.

So, basically this tells us something more that this we can start off with anything. And then if we try to look into Bowen's definition once again. So, if you look into Bowen's definition, it definitely depends on the metric that we have considered. Since the metric that we have considered, gives us what is the epsilon separateness. Now on the real line; suppose I look into the real line, the real line we have been usually read in metric and in case of the usually the Euclidean metric, one could think of that fine the distances are tending to infinity.

So, the n epsilon sets; the growth rate would turn out to be infinity. But if I am looking into something like log X, some entropy like log X, some definition of metric using some kind of contracting functions. Then we would find out that there is an equivalent metric, but under that equivalent metric; the entropy turns out to be something different, the growth rate turns out to be something different. So, if you look into Bowen's entropy; it is highly dependent on metric spaces, but then there is an advantage here that we are able to talk of entropy even for non compact spaces.

So, in that sense Bowen's entropy happens to be more important because we are able to look into it; look beyond what to say compactness. Because for Adler's definition of topological entropy, we need compactness because we are working with open covers and we are working with finite sub covers of an open cover. So, that is one advantage which Bowen's definition gives over Adler's definition. The second advantage that we have here is that; what happens if you want to look into products.

Now, when we try to look into products; we are looking into products of spaces. We are looking into products of open covers, now that actually turns out to be very messy and we may not always have the product to be giving us some nice results on entropy. But here in case of Bowen's example because of this equivalence that we have; we will start with the n epsilon set of minimum cardinality or you start with the maximum cardinality; what you get is the same part.

So, you can have; you can exactly say that the product of you can exactly define the entropy of a product system. So, something which failed for Adler's definition something which Adler's definition could not address to is been address by Bowen's definition. But then again, we know that all systems are not met risible; you have lot of spaces for which you can say that ok fine. So, say for example you have some kind of infinite product spaces, you can talk about compactness there; but then they need not be met risible.

So, Bowen's definition fails for say a large class of spaces for which Adler's definition comes to the rescue. So, if you want to define entropy especially for compact spaces; we definitely have Adler's definition, we have Adler's definition as an edge there, but then the metric structure also gives us the edge of Bowen's definition; which is a little bit easier to see and easier to visualize. It is easier to visualize; how the orbits diverge from each other. So, in that sense Bowen's entropy definitely scores better over Adler's definition.

So, we try to look into whenever we talk of entropy it is always better and it is always easier to work with Bowen's definition. So, look into another theorem here which looks into the product spaces. Now entropy of products spaces, but when we talk of entropy of product spaces; as I said that Adler's definition does not help us here much. So, we need Bowen's definition here and again we are looking into our compact spaces.

So, we look into this theorem; so, we have two compact metric spaces and we have basically dynamical systems arising out of this two spaces. Then the topological entropy now, I am looking into the Bowen's definition here. And since we know that Adler's definition is also equivalent; can say that Adler's definition; Adler's this thing follows from the spot.

But to start with Adler's and trying to prove this part would have been difficult, but what we have here is that the entropy of the product system is basically the addition of the two entropies. We have already seen that; if I have entropy of h and f to the power k and it is basically k times the entropy of X. So, entropy of f; so what we have here is we have something more; which you can think of that part that you look into the product spaces; here we are working with products, not with iterations.

So, with products we are basically adding the entropy here; then you can proof utilizes some ideas. So, on this space X cross Y; we take up the metric, so using the maximum metric on X cross Y; one thing more here is that when you are talking of products, we are looking into finite products. And on any finite product of metric spaces, you can always define this maximum metric; so we start with the maximum metric here.

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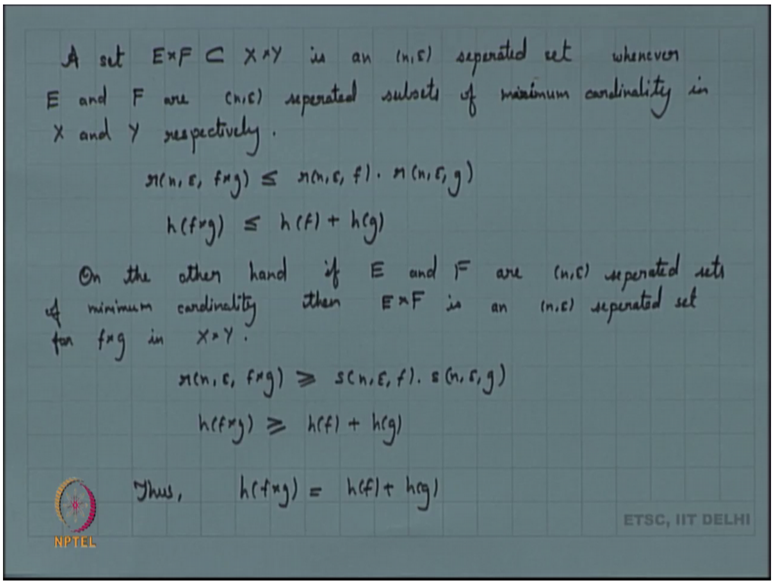

Now, think of this a set E cross f; which is a subset of X cross Y is an n epsilon separated set. Whenever E and f; these are n epsilon separated subsets of maximum cardinality in X in Y respectively.

You start with an n epsilon set of maximum cardinality on X; n epsilon separated set of maximum cardinality on Y take the product. Then definitely it is an n epsilon separated set in X cross Y. So, what we have here is that; if I look into r n epsilon f cross g, that will be less than or equal to r n epsilon; f into r n epsilon g. And hence, if I look into; take the growth rate here and then let epsilon tend to 0, here what we get here; is the resultant will be that the entropy of f cross g will be less than or equal to the entropy of f plus entropy of g.

On the other hand, if E and F are n epsilon separated of minimum cardinality; then E product F is also an n epsilon separated set for f cross g in X cross Y. And so what we have here is that r of n epsilon f cross g, will be greater than or equal to S of n epsilon f into S of n epsilon g. Now what happens in this particular case? Here again I can think of taking the growth rate and taking epsilon tending to 0, what we get here is entropy of f cross g is greater than or equal to entropy of f plus entropy of g.

And so, basically entropy satisfies this property that you stopped with entropy; you start with the system. We try to look into the iterate of this, entropy increases; you try to take a product of two systems. Again the entropy increases because you are adding up the entropies here. So, entropy as such has nice properties; there are other properties of entropy also, but presently we are not looking into that. As far as metric spaces are concerned; we know that the definition of Bowen's definition goes independent of the metric because in the compact metric space; all metrics are equivalent.

So, for compact metric spaces we really have a very good collection of entropy, but then the question here is about chaos.

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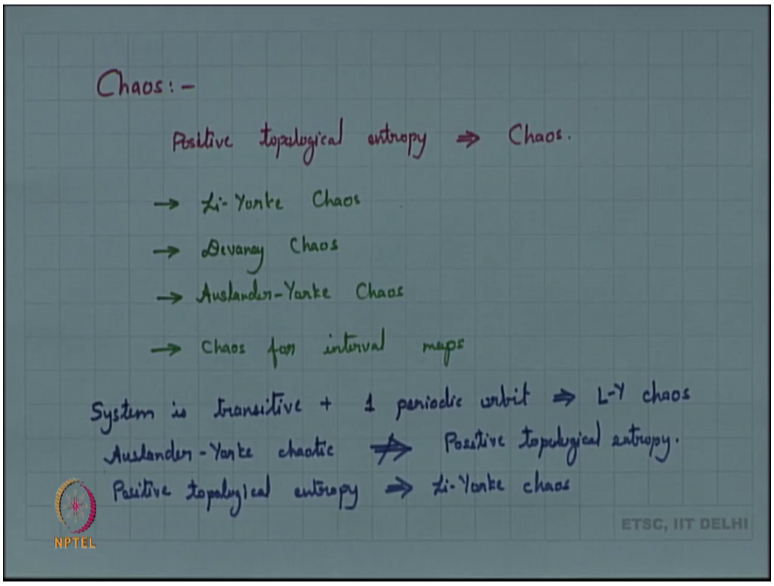

So, we again look back into chaos and we have seen various definitions of chaos and the definition involving entropy is that positive entropy. And in the sense; we are taking as one of the definitions that positive topological entropy is also definition of chaos. But if we try to summarize all the other definition of chaos; that we have taken up, so we have talked about Li Yorke chaos, we have talked about Devaney chaos, we have talked about Auslander-Yorke chaos and I think we have talked about chaos on the interval maps.

So, we have talked about chaos for interval maps, so you would be really interesting to see; where does this definition of topological entropy fit in among this part? So, what is basically the relation among all of these? So, if we try to look into the relation among all of this; Li Yorke chaos says that, you should have an uncountable scramble set. Uncountable scramble set that means that you definitely have orbits diverging off. So, definitely that could be some say sort of evidence of topological entropy; look into Devaney chaos.

Now, what is the relation of Devaney chaos with Li Yorke chaos? So, I think that is known that; if you have say a system which is transitive plus, it has one periodic orbit then that implies Li Yorke chaos. So, a transitive system with one periodic orbit is good enough to imply Li Yorke chaos. So, definitely Devaney chaos for that matter implies Li Yorke chaos, it is a stronger definition. Now the difference between Devaney chaos and Auslander Yorke chaos is basically the concept of periodic points. Devaney chaos is just transitivity sensitivity and dense periodic points; Auslander Yorke chaos simply says it is transitivity as well as sensitivity.

Now, when it comes up to the concept of minimal systems; we have already seen this dichotomy that for a minimal system it is either equicontinuous or sensitive. A minimal system is always transitive, so a minimal system if it is sensitive; it is Auslander Yorke chaotic but it need not be Devaney chaotic. So as such the two definitions are different but where does that tend to; in terms of positive topological entropy?

We already know that we have an example of a minimal system; which has 0 topological entropy. We also have an example of a system, which is again coming up from some kind of minimal system and I am not getting into details over here. So, we have an example which is Auslander Yorke chaotic and which is not having positive topological entropy.

If you have positive topological entropy; that implies Li Yorke chaos and then it comes to what can we say about interval maps. So, in general Auslander Yorke chaos does not imply positive topological entropy. Auslander Yorke chaos and Devaney chaos are again two different systems, it is also possible to get; say a Devaney chaotic system which is not having positive topological entropy.

In fact, you think of positive topological entropy; it need not imply transitivity as well. So, you take; do transitive systems two nice Devaney chaotic systems, union of them. They both will have positive topological entropy, some result which we will state now. And if we take the union; the union do it turns out to be having positive topological entropy; it is not transitive. So, the other way round is not possible; however, in intervals in the case of intervals we have nice results.

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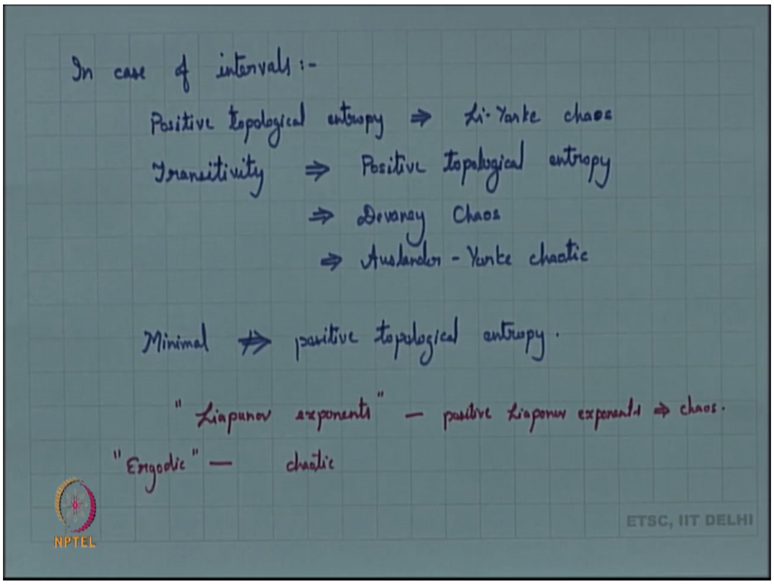

So, in case of intervals say Li and York had proved that period three implies chaos. So, that is what Li York chaotic is; the existence of a uncountable scramble set. But we do have positive topological entropy; implies Li York chaos. Then if you look into transitivity; then transitivity if you look into this aspect, we know that on an interval transitivity implies the existence of a periodic point of period 6; it also implies that there is a dense set of periodic points, as well as you have periodic points of period 6.

So, if we try to look into what are the non wandering points? The whole set happens to be non-wandering. Now entropy of a system is enough to calculate entropy on the set of non wandering points. Here our entire space is non wandering; so, there is a large; what to say inclination for having positive topological entropy. And indeed transitivity on intervals does imply positive topological entropy.

We have also proved that this implies Li York chaos and definitely this is Auslander Yorke chaotic. If we try to look; if we try to summarize various kinds of examples, what you get here is that, there is some kind of inter relation between them. Although there are some things which are not true you have counter examples for that part.

We have already seen an example of a minimal system; which does not imply positive topological entropy. So, we need not have positive topological entropy coming up from all places, but if we try to look into say; again if look into interconnections between them, there are other definitions of chaos; which we have not defined; there is something called distributional chaos. There is something called densely chaos and there are implications among them.

So, if we try to look into say various definitions of chaos and try to look into what would be the interrelation between them? That itself would be a big project, but right now as far as this lecture is concerned or maybe this course is concerned. It becomes too much to deal with looking into our definitions and then looking into what are all the interrelations between them. So, what we apply? What we will do is; we will basically just say that ok fine these things to hold, there is a nice theory behind all of them and we are not getting into that aspect.

Now so this is basically all the concepts; what we have now dealt with this almost all the definitions of what is it chaos or what I would term as a topological chaos. There is another concept of Liapunov exponents; which we have not touched up, so this is the concept of Liapunov exponents. Now these are basically defined for differentiable mappings, so if we are trying to look into a dynamical systems on a differentiable manifold. So, if we are trying to look into; basically a few morphism on a differentiable manifold. We can come across this concept of Liapunov exponents and again if you have positive Liapunov exponents that is an indication of chaos.

So positive Liapunov exponents; so, this is an indication of chaos. The other concept what we have not studied is ergodicity, now this comes in the realm of measure theoretic or measurable dynamics. So, we are not looking into continuous transformations basically we are looking into measure preserving transformation; on a measurable space or an probability space. And then we are trying to look into whether the time average and the space average. So, you trying to look into an orbit; you want to see whether the time average and the space average turns out to be the same.

The time average and the space average turns out to be same, we say that the system is ergodic. And this is also term to be chaotic; which we have not looked into. So, there are a few aspects which we have not looked into, but they also imply chaos. Since in one course, it is difficult to deal with all different aspects, to work with all different concepts. We have not touched this aspect, but this aspect is also an indication of chaos and this also comes up under the realm of dynamical systems. So, today and we will end with this part and more so, we can take up in the next class; I hope there is no difficulty.