Chaotic Dynamical Systems
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Lecture – 03
Elementary Concepts cont'd

Welcome to the students. So, last time we had seen elementary concepts. And today we are going to continue with these elementary concepts. So, what had we seen last time? We had seen; what are periodic points, what are eventually periodic points. We had also looked into the stable sets of periodic points.

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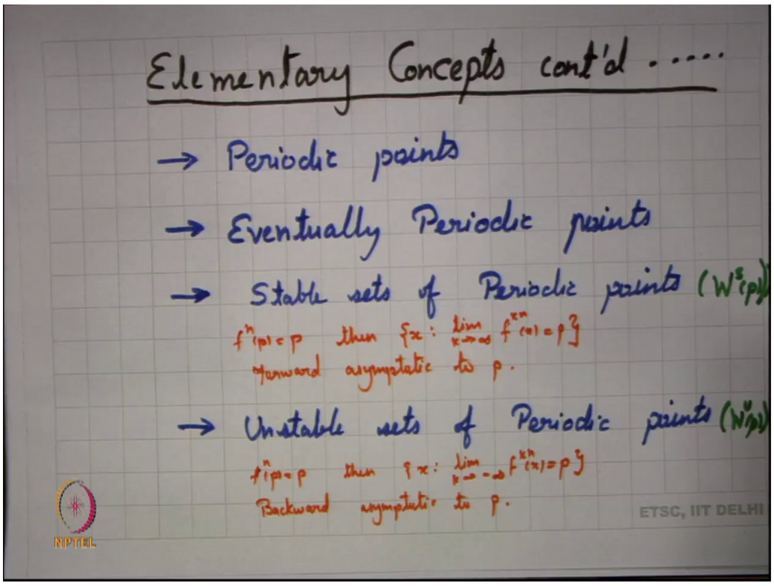

And we had also looked into the unstable sets of periodic points. Now what are the stable sets of periodic points we can recall that once again.

So, these are basically if p is a periodic point of period n, then basically I am looking into all x for which limit as k tends to infinity fkn of x is p. So, these are the points which are forward asymptotic to p, and this is; what is our stable set. We recall again what do we mean by an unstable set of periodic points. So, if you look into the unstable set of periodic points. These are basically those points, say again if my fn p is p, then I am looking into all those x for which limit as k tends to minus infinity, right fkn of x is p.

So, these are basically the points which are backward asymptotic to p; now with this recalling of the definitions. Let us now look into the properties. So, today we will be doing the first proposition that we have in this course. And that is regarding the stable set.

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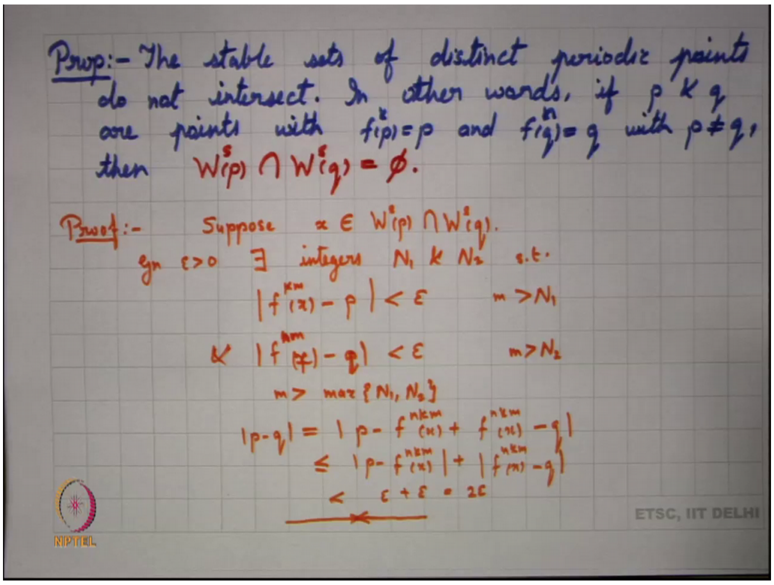

So, what ideally do we mean by a stable set? The proposition says that the stable set of distinct periodic points are disjoint; that means, if I have p and q to be 2 periodic points, and if k is the period of p, and m is the period of q, then we have the stable set of p disjoint from the stable set of q.

Now, let us look into the proof of this proposition. So, ideally for a proof, all we need to do is we start with contradiction. So, we start with the fact that there exists suppose there exist an x which belongs to the stable set of p, as well as the stable set of q. Now since x belongs to both of this. I can say that given epsilon greater than 0, there exist integers N 1 and N 2 such that on one hand I have f km of x minus p, this modulus is less than epsilon, for all m greater than N 1. And on the other hand, I have fkn of f nm of p of x minus p to be less than epsilon for all m greater than N 2. Now this is true since they both belong to the stable set of.

Student: Q.

P and q, sorry minus q.

Student: Q.

Is less than epsilon; now what does this tell me? So, if I take my m to be greater than say maximum of N 1 and N 2, then we have that mod of p minus q, right my triangular inequality I am just using triangular inequality this is p minus f of nkm x plus f of nkm x minus q. My triangular inequality this is less than mod of p minus f of nkm x plus mod of f of nkm x minus q. And we know that both these quantities are less than epsilon.

So, this is less than epsilon plus epsilon which is basically my twice epsilon. I could have started with epsilon by 2 epsilon by 2, this would have been less than epsilon. So, what we find is that given any epsilon, your mod of p minus q becomes less than 2 epsilon; that means, the distance between p and q becomes arbitrary; that is, what leads to contradiction because p and q are fixed points. So, that gives us a contradiction and this contradiction basically comes up from the fact of assuming that there exists a point in the stables which belongs to this stable set of both p and q. And hence the stable set of p and q are disjoint.

So, if we look into the stable set, the stable sets are always disjoint. So, this gives a wonderful kind of characterization of the dynamics of say points on the set, that if we have a periodic point, then if we have some stable sets, then maybe we have disjoint periodic points, then there stable sets will be disjoint. Same thing can be said about the unstable sets also. So, we now go with defining something. And today our basic concentration will be for looking into f a function mapping from an interval I to R.

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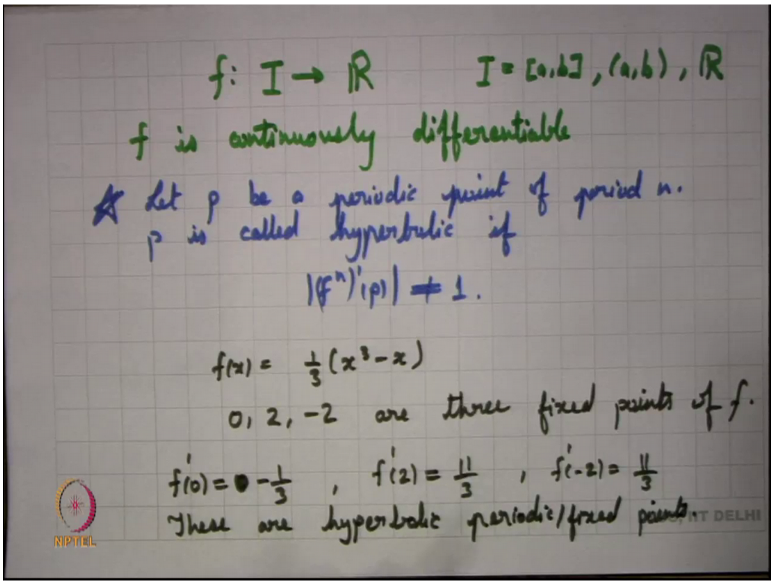

We are only going to look into interval maps here. Where my interval can be something like a closed interval, it could be something like an open interval or it could be R. So, we look into maps with basically today we will be looking into interval maps. And we will be trying to understand the dynamics from some properties of f. Also, I want my f to be continuously differentiable, for looking into for analyzing the dynamics of f.

Now, the first thing we need is some definitions. So, we try to define some points here. So, let me say that let p be a periodic point of period n. Then we say that p is hyperbolic, right, this point p is called hyperbolic, if I take fn I take it is derivative, the value of this at p is not equal to 1. So, we say that we have a hyperbolic periodic point, if the period is n and if I take fn if I take the derivative of fn at p, then that particular modulus of that is not equal to 1.

So, basically this does not lie on the unit. Basically, this distance is not unit. Since we are only looking into this mapping from R to a, R to R then we just want to say that that this modulus is or basically, this value is neither one nor minus 1. So, this is our definition of hyperbolic periodic point. And we can start with looking into some kind of examples. So, let us try to look into some example here. So, we let take the first example here, say let my fx be equal to so, I am looking into 1 by 3 times x cube minus x. All I want to see is that this f will have 3 fixed points, and the 3 fixed points here are 0. So, we have 0 2 and minus 2 are the 3 fixed points, what happens what is the and since these are fixed points;

that means, they are period one points, right. We can try to look into what is the character of these periodic points or these fixed points.

So, if you look into what is f prime of 0, then f prime of 0 happens to be equal to 0. Is it exactly 0, what is f prime of 0? I think it is minus 1 by 3 f prime of 0 is minus 1 by 3. If we look into f prime of 2, right f prime of 2 happens to be equal to.

Student: (Refer Time: 10:40) 10 by 3.

Is it 11 by 3? And if I look into what is f prime of minus 2, again that happens to be 11 by, clearly this values are neither 1 and minus 1. So, these are hyperbolic periodic points. Basically, hyperbolic fixed points will look into the graphical analysis, or maybe we can look into the phase portrait of this periodic point and try to understand how their behavior is.

So, let us draw the phase portrait of this function. So, I am looking into this particular function, I have a fixed point at 0.

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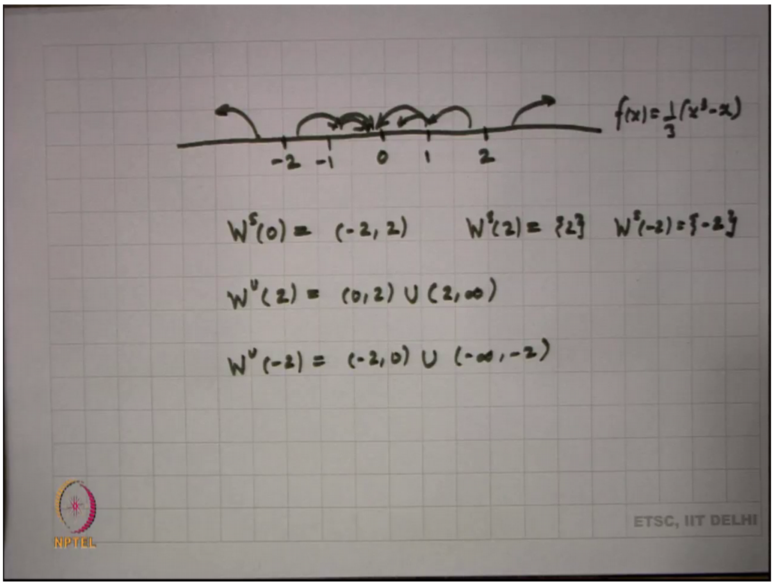

I have a fixed point at 2, and I have a fixed point at minus 2. Now what happens to this particular function? At 0 it remains 0, right? What happens to points between 0 and 2? So, if I try to look into what happens to points between 0 and 2, right? They tend to get closer to what happens to the point one for example, what happens to the point one here? 1 is mapped to 0, if I look into minus 1 right what happens to minus 1? Minus 1 is also mapped to 0. What happens to the rest of the points here? Say, take any point between 0 and 2, we find that it is coming closer and closer to 0, any point here comes closer and closer to 0, what happens to the point which are greater than 2? So, we find that points which are greater than 2, they basically drift away from 2. The points which are less than minus 2 also drift away from minus 2.

So, if I look into my this function my function is fx equal to half of x cube minus x, we find that this is basically the phase portrait of this function. Now it is very easy to see, if I want to look into what are the stable sets here. So, what are the stable sets. So, the stable sets is what is a stable set of 0. So, we find that the stable set of 0 here happens to be the interval minus 2, too interesting. What is the stable set of 2? We find that the stable set of 2 is nothing but singleton 2, right. It is just singleton 2, and the stable set of minus 2 similarly is just the singleton minus 2.

Now, we would be interested in knowing, what are the unstable sets of 2 in minus 2. So, if you try to look into what is the unstable set of 2, we are looking into those points which are drifting away from 2 right. So, those points which are drifting away from 2, or basically they are tending towards 2 in the negative direction right. So, these are basically the points 0 2 union.

Student: 2.

2 infinity. So, the unstable set of 2 happens to be a union of 2 disjoint intervals, right 0 2 and 2 infinity. Similarly, if I want to look into what is the unstable set of minus 2, right? The points drifting away from minus 2, and that turns out to be this interval minus 2 0 right union minus infinity minus 2. So, this is basically the unstable set of minus 2. We try to look into yet another example. I hope this is clear to all of you.

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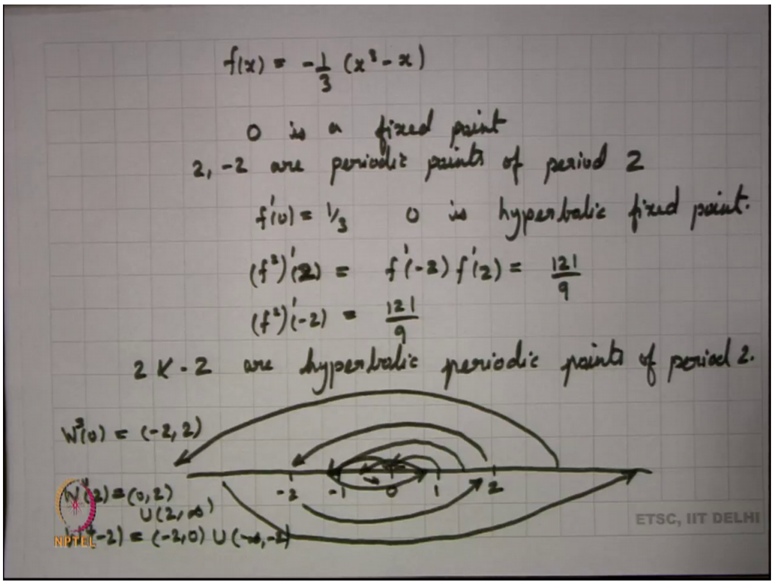

So, let us look into another example here. And this example is just I am just having the same function here, but with a negative sign. So, this is minus 1 by 3 x cube minus x.

Now, what can we say about this particular function. Certainly 0 happens to be a fixed point here. Is there any other fixed point here? 0 is the only fixed point here, and what we find here is that 2 and minus 2 they form a periodic orbit of period 2 right, because 2 is mapping to minus 2.

Student: Minus 2.

Right, and minus 2 is mapping to 2 it is just the previous case in the previous case we did not have the minus sign. So, 2 n minus 2 were fixed points, but here we have a negative sign right. So, 2 is mapping to minus 2 and minus 2 is mapping to 2. So, here we have 0 is a fixed point, and 2 minus 2 are periodic points of period 2. What can we say about? Now interesting fact here is that I should have this 0, right we know that your f prime of 0 here will be 1 by 3, right. So, again 0 happens to be a hyperbolic. So, 0 is hyperbolic. So, it is a hyperbolic fixed point.

Now, we are interested in what happens to 2 and minus 2. So, if we try to look into what is f square prime right of x, what is that turning out to be? And we find that at the point not at the point x, instead we can just look into what happens at the point 2 and minus 2 all we are looking into is we looking into chain rule here right.

So, at the point let me say let what happens at the point 2, right. So, this becomes nothing but this becomes what is f prime at the point minus 2. So, this is f prime at fx. So, this is f prime at minus 2 into f prime at 2. And what is the value over here? So, we know that f prime at minus 2. So, similar calculation right, what will get here is say minus 11 by 3 right minus 11 by 3, the other one is also minus 11 by 3. So, what we get here is 121 by 9.

So, f prime f square prime at 2 happens to be 121 by 9. Which is certainly not equal to 1. And similarly, this also has the same value f prime at minus 2, this also has the same value 121 by 9. So, 2 and minus 2 are hyperbolic periodic points of period 2. We can think of drawing a phase portrait here to understand the dynamics of this particular function. And the phase portrait here is simpler, it is different than the previous case. But again my 0 remains fixed as it is if I look into my 2, right my 2 and minus 2 they form a periodic orbit of period 2. So, what I have here is I have 2 bring map to minus 2, and I have minus 2 bring map to 2. What happens to one and minus 1 here.

So, if you look into 1, and if you look into what happens to minus 1 here, then 1 and minus 1 under this map they both map to 0. So, one is mapped to 0 minus 1 is mapped to 0 we both this both are mapped into 0. What happens to a typical point between 0 and 2 here? So, if you look into any point between 0 and 2, it basically gets mapped to something smaller right. So, if I start gets mapped to something smaller, right on the other hand. If I look into this particular point it gets mapped to something smaller and it goes to the other hand.

So, if I look into typical orbit of a point between 0 and 2, it basically shifts to the negative side to the left-hand side of 0 then again goes back to the right-hand side of 0. So, it oscillates about 0, and then finally, you can say that it basically is trying to converge to 0. What happens to the points which are greater than 2? So, the points which are basically the real numbers greater than 2, will find that they also come up to something larger over here, right. On the other side on the left-hand side and if I say take any point over here, on the left-hand side of minus 2 then that gets mapped to something which is larger and to the right of 2.

So, the phase portrait of this function is very, very clear where points greater than minus 2 and less than greater than 2, and less than minus 2, right. They are also oscillating about 0, but then then magnitude is increasing their absolute value increases. So, typically here again what is my stable set of 0. So, my stable set of 0 here is again minus 2 2, stable set of 2 and stable set of minus 2 are again singletons. So, we are looking into what happens to the unstable set of 2.

Now, I am looking into points which are drifting away from 2, what are those points here? And they are drifting away from 2 at the second iterate right. So, at the first iterate the points come up here, at the second iterate they are again going towards the right-hand side right. So, at the second iterate I am looking into all the points which are drifting away from 2 at the second iterate, and we find the same story here; that this, unstable set happens to be equal to 0 to union 2 infinity. And if I look into what happens to the unstable set of minus 2 again I have the same story. So, this becomes minus 2 0 union minus infinity minus 2.

So, the unstable set of 2 the unstable set of minus 2 both the cases remain the same. So, we have this hyperbolic periodic point where the absolute value of the derivative at the period is not equal to 1. We will now go up to some proposition here. So, let us try to look into some proposition about a hyperbolic periodic point. What is basically the property of a hyperbolic periodic point? Why should we study them that sense?

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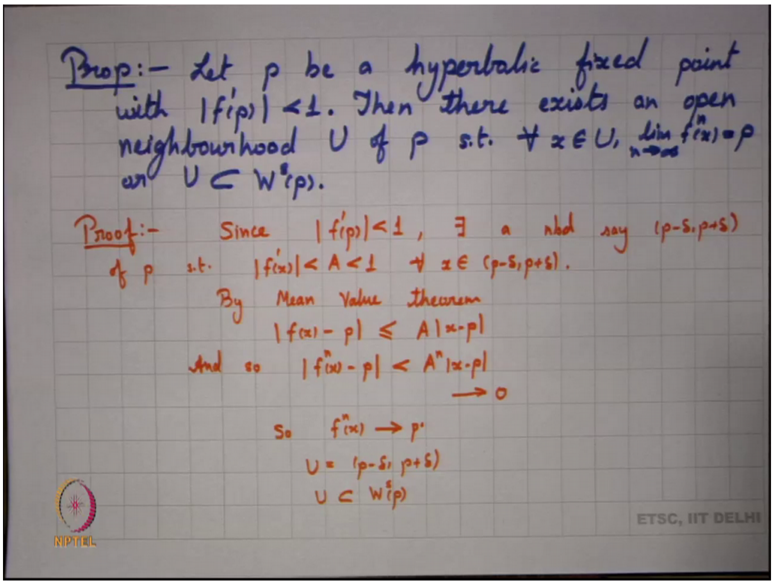

So, if we have p to be a hyperbolic fixed point, right, I am just starting with a fixed point right now. So, if p happens to be a hyperbolic fixed point, such that the modulus of f prime p is less than 1, then there exists an open neighbourhood u of p, such that for all points in this neighbourhood, right. The limit or basically for every x in this neighbourhood fnx the orbit of x is converging to p. In other words, we can say that we have an open neighbourhood of p such that this open set is contained in the stable set of p.

So, the proof here is quite simple. You start with the proof of this proposition. Now all we have been given is that the absolute value of f prime p is less than 1. I can say that here. So, that means, that for some positive delta I have a neighbourhood of p, such that the modulus of f prime of x is less than some quantity which is less than 1, for every x belongs to p minus delta p plus delta.

Now, I can use my mean value theorem. So, by mean value theorem, this is basically less than or equal to right a times mod of x minus p, right, because we will find something between x and p, right. And whatever we find between x and p it is derivative will be less than a right modulus will be less than a. So, we find that this is my mean value theorem we can say that mod of fx minus p is less than a times mod of x minus p. And if I try to look into this repeatedly, right. We can say that since a is some quantity which is less than 1, right. Repeatedly you can say that since a is less than 1, this particular part on the right-hand side would be converging to 0, right. And hence fn of x can be said to be converging to p.

So, fnx converges to p, and hence will whatever proposition says, we can take our neighbourhood u to be equal to p minus delta p plus delta. So, we have this open neighbourhood around p, such that u is contained in the stable set of p. So, all points in this interval are forward asymptotic to p, and hence there is an open set containing the stable set of p. So, this is one of the properties for hyperbolic fixed points that you can see. And that property says that if your modulus if this modulus of f prime p is less than 1, then you have an open set contained in the stable set of p.

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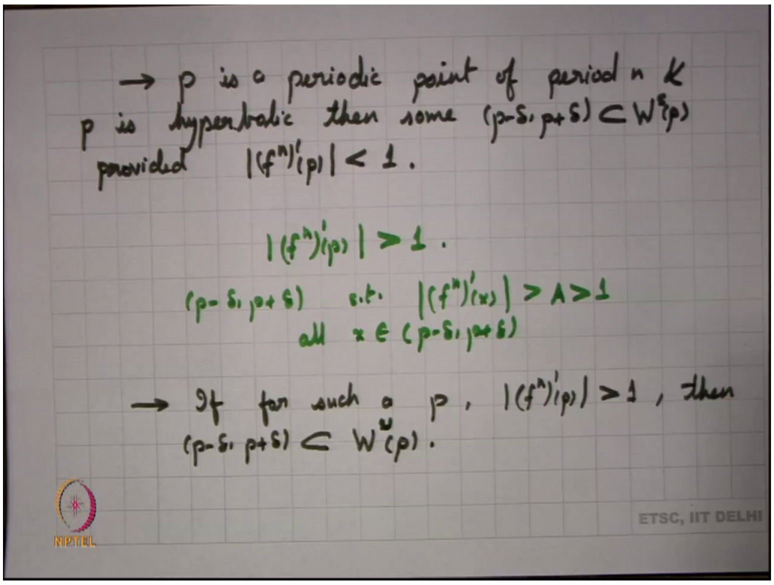

Now, what does basically this give us, right? It is not very difficult to see that in case p is a periodic point of period n. So, if p is a periodic point, and I also want my p is hyperbolic, then some p minus delta p plus delta, right will be a subset of the stable set of p provided n is the period of p right this is less than 1. So, if I have a hyperbolic periodic point, such that the modulus of it is derivative nth at fn is less than 1, then for such periodic points also the stable set happens to be containing an open set. With a little bit difficulty, we can try to look into or we can try to analyze; what happens in case, I have such a periodic point and I have fn prime p is greater than 1. What happens in that particular case?

So, if we again go back to our proof of the previous case, right. So, the previous case we had by mean value theorem we had that this happens to be a quantity a which was less than 1, right. Now if our f prime p is greater than 1 then I can say that I have an open set p minus delta p plus delta, right such that fn prime of x, right this modulus is greater than some quantity a is greater than 1, right for all. Now what does this observation tell you? In that case this particular interval will be contained in the unstable set of p right.

So, in this case if for such a p right modulus of fn prime of p is greater than 1, then you have p minus delta p plus delta to be contained in the unstable set of p. So, your hyperbolic periodic point, right for a hyperbolic periodic point if this modulus is greater than 1, then this may becomes basically this is contained in the unstable set of p. We have a specific name or a specific property of such points. So, what happens now again we are looking into hyperbolic periodic points. So, let p be a hyperbolic periodic point.

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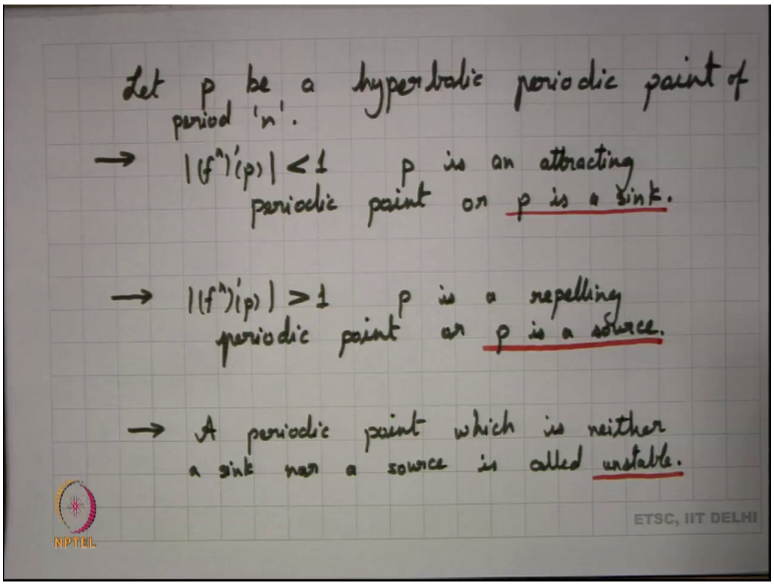

Now, we have seen 2 cases, right. One case is if of course, periodic point of period n. So, let me specify the period here also period n. So, our first case is if mod of fn prime of p is less than 1, then what we find is that the nearby points the points in the neighbourhood of p are basically forward asymptotic to p, and in that case, we say that p is an attracting periodic point. Attracting periodic point means, all the nearby orbits are basically attracted towards p right. So, all the nearby orbits they converge to p, and in that case, we can say that p is a sink, or we say that p is a sink. Everything is all the orbits are sinking towards p. So, we say that p is a sink.

Supposing we have this case that fn prime p is greater than 1, then what happens in that case? It is greater than 1. So, what we find is that the neighbourhood of p is basically in the unstable set; that means, this all the points in the neighbourhood are backward asymptotic to p; that means, they are basically being moving away from p they are repelled from p, right. So, in that case we say that p is a repelling periodic point or we say that p is a source. Why do we say it is a source? Because everything moves away from there, right it kind of works like a source. So, we say that p is a source.

Now, we may come up to cases where p is neither a sink nor a source; that means, ideally when we have a periodic point which is neither a sink nor a source. In that case we say that the periodic point is unstable. So, we say that a periodic point which is neither a sink nor a source is called unstable. We never know what it is what properties it may have. Maybe it is attracting something from one side it is repelling something from the other side. So, for such periodic points we can not say anything. So, we say that they are unstable. So, these are unstable periodic points.

Now, all we try to look into us we try to look into the case, when you have hyperbolic periodic points. So, you are looking into the case when you have hyperbolic periodic points, and now we are looking into what happens to the case of non-hyperbolic periodic points. So, before we get into non-hyperbolic periodic points, we just get into what we do we mean by a non-a hyperbolic fixed point.

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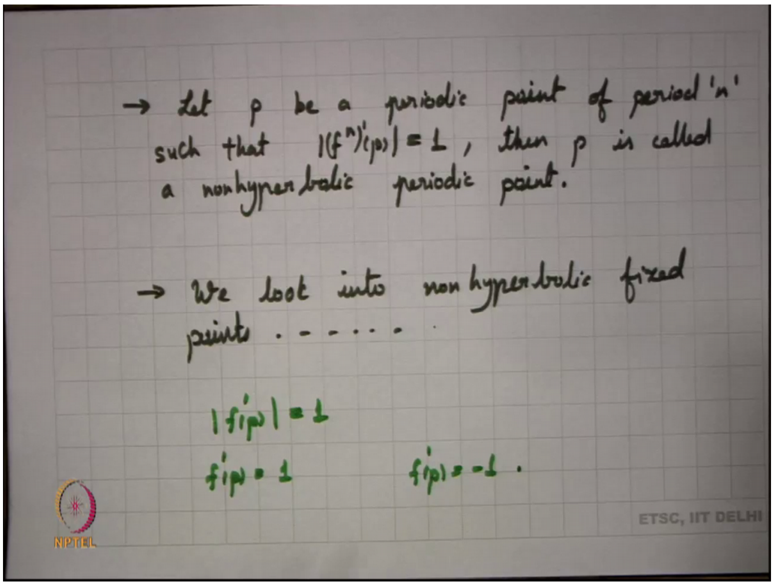

So, we start with this definition. So, let p be a periodic point of period n, such that I have mod of fn prime p, right this is equal to 1 then p is called a non-hyperbolic periodic point. So, basically what we find here is that the derivative or basically the tangent at p right for fn the tangent of fn at p, right. Exactly happens to be parallel right. So, this is 1. And hence we say that p is a non-hyperbolic periodic points. Cases of non-hyperbolic periodic points are much different from the cases of hyperbolic periodic points.

So, let us try to look into let us try to analyze this case. So, we first look into non-hyperbolic fixed points. And the rest of the things will be just similar. Now if I look into non-hyperbolic fixed points. So, non-hyperbolic fixed points will have 2 varieties. One is when mod of f prime of p. So, this is basically when mod of f prime of p is equal to 1. And this has 2 cases, I have that f prime of p is exactly 1. And the other case is that f prime of p happens to be equal to minus 1. So, we have this 2 cases.

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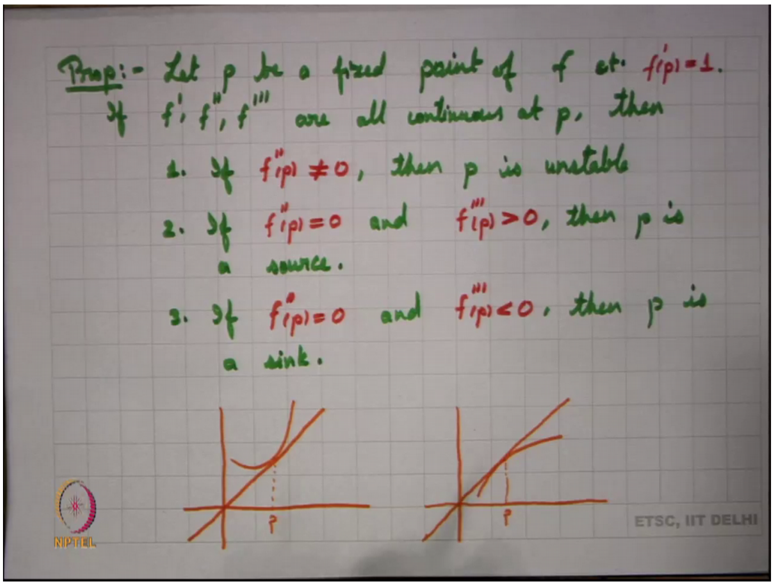

Now, we try to look into the case when f prime of p happens to be equal to 1, and for that we look into a proposition. So, we start with a proposition here. So, let p be a fixed point, such that I just want to emphasize here, that mod of f prime p is 1. Now we know that f we have already assumed that f is continuously differentiable. Here I want to assume something more. So, if f prime f double prime f triple prime. These are all continuous at p, then we have essentially 3 cases.

So, the first case says that if now I already know my f prime of p is equal to 1. So, I am now looking into what happens to f double prime at p. So, if f double prime of p is not equal to 0, then my p is unstable. So, then p is unstable. Now the second case comes up what happens if f double prime p is equal to 0. So, if my f double prime p is equal to 0, and I have that f triple prime p is greater than 0. So, if I have these 2 cases. So, f double prime p equal to 0, and f triple prime p is greater than 0 then p is a source.

Now, we have a third case here which says, what happens if your f double prime p is equal to 0, and your f triple prime p is less than 0. Supposing we have these 2 cases, then p is a sink. So, we have these 3 conditions for the fixed point the non-hyperbolic fixed-point p. For a non-hyperbolic fixed point, we have 3 cases, we are looking into the double derivative we are looking into the triple derivative right. So, the value of the double derivative and the value of the triple derivative gives us what is basically the nature of the fixed point.

Now, I want to you to look into this particular aspect. So, let me just draw a simple figure here. What do I mean by saying that f prime p is equal to 1. F prime p is equal to 1 means, basically the tangent of the line or the slope of the line is one take point. And on the other hand, we have that f double prime p is not equal to 0. Supposing I am looking into the case where f double prime p is not equal to 0; what happens in such a case? So, if I look into this case the first case here, f double prime p is not equal to 0, then I have 2 cases either it is greater than 0 or it is less than 0. So, if it is greater than 0, now if it is greater than 0; that means, it is concave upwards right.

So, your graph happens to be something like this there is a point p right. So, this is basically your point p, right. And the graph is concave upwards. If f double prime p is less than 0, since it is not equal to 0, right supposing this is less than 0, then we find that the graph of p would the graph of f at p would look something like this. This is basically your p and your graph is concave downwards.

Now, we are try to analyze this factor, right try to give a proof of this proposition. So, let us look into the proof here.

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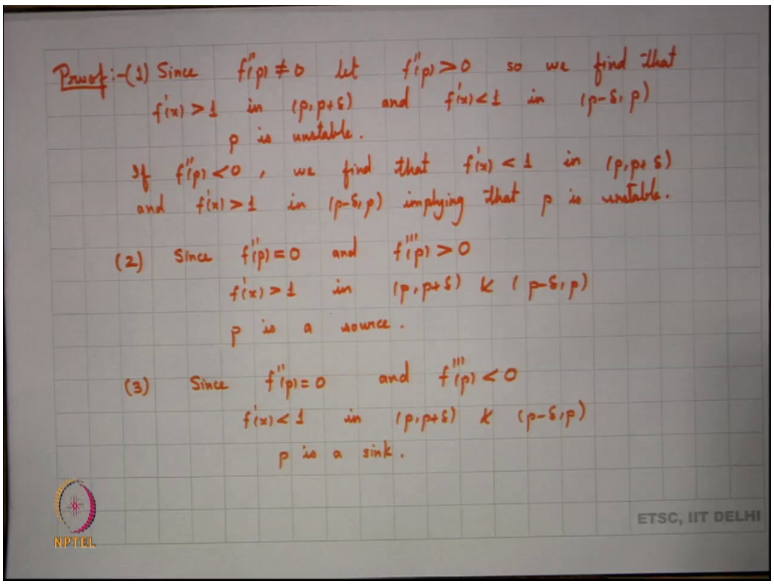

So, we have 2 parts here. Since f double prime of p is not equal to 0, let f double prime of p be greater than 0. Now what happens what is the meaning of saying that f double prime p is greater than 0? That means that, at p if prime p is increasing, right. So, f prime of x is basically increasing here so, we find that. So, for some delta positive, right we find that f prime of x because my f prime of p is 1, right my f prime p my f prime is increasing here in this interval.

So, I find that f prime of x is greater than 1, right in this interval p to p plus delta. And since my f prime is increasing in my f prime p is equal to 1, we find that f prime of x is less than 1, right in this interval p minus delta to p is this clear to all of you. So, what we have here is, we have f prime x is greater than 1, in this context what happens to any x in this interval any x here will be backward asymptotic to p, right. Any x in this interval p to p plus delta will be backward asymptotic to p. And since f prime x is less than 1, any x in this interval p minus delta to p will be forward asymptotic to p.

So, basically p is attracting p is basically attracting. Everything to it is left basically some things to it is left right and p is attracting something. And it is repelling it is basically repelling on it is right. So, this p is can neither be a sink nor a source. So, my point p is unstable. If f double prime p is less than 0. So, we are looking into basically the proof of the first part. So, if my f double prime p is less than 0, then definitely we find that. So, it is a similar observation, we find that f prime of x basically f prime is now decreasing because f double prime is less than 0 f prime is decreasing. So, we find that f prime of x is less than 1, right in pp plus delta, and f prime x is greater than 1, in p minus delta p which again implies again it is the same story. So, this implies that p is unstable.

So, non-hyperbolic fixed point, such that it is derivative at p takes the value one assumes the value 1. We find that f prime p not equal to 0, implies that p happens to be an unstable periodic point. What happens in the next case? So, we try to look into what happens in the next case. So, in the next case my f prime p f double prime p is also equal to 0. Now f double prime p is also equal to 0, what does essentially that mean? So that means, that we are now looking into the third derivative. And my second case says that this is 0, and the third derivative is greater than 0. What happens in this case?

So, we find that in this particular case, your f triple prime p is greater than 0. Since this is greater than 0; that means, your f double prime p is increasing, right your f double prime p is increasing what does that imply? It is greater than 0 in p plus delta p, p plus delta it is greater than 0, and it is less than 0 in p minus delta p plus delta. So, what happens in that case? Your f double prime p is less than 0, right now I am again coming back to the previous case it is less than 0 in this interval f double prime p is greater than 0 in this interval right less than 0 in this interval. What does it imply, right? It implies that your f prime p happens to be greater than 1 in both these intervals right.

So, in both these intervals p plus delta p minus delta p here f prime of x is greater than 1 right. So, this implies that f prime of x is greater than 1 in both say p p plus delta and p minus delta p. Since this is greater than 1, right we can easily use our previous theory or basically our previous observation to say that in that particular case p is a source. If I look into my third case, what happens into my third case? So, in the third case again I have f double prime p is 0, and f triple prime p is less than 0.

Now, f triple prime p is less than 0 will give you that f double prime p is increase is decreasing. Basically, right is it is decreasing in the neighbourhood of p, since it is decreasing in the neighbourhood of p; which means that your f double prime p happens to be less than 0 in p minus delta is less than 0 in p p plus delta and this greater than 0 in p minus delta p. In both cases that would mean that my f prime x will be less than 1, right in both these intervals. So, my p here is a sink.

So, for a non-hyperbolic fixed point for which your derivative the value of the derivative is 1, right we find that this is either a it is either unstable or it is a source or it is a sink. What happens for a periodic point? Again, the calculations would be a little bit, I mean our computations would be a little bit tardy, but we can say that here that for a periodic point a non-hyperbolic periodic point also look at similar cases right. So, they will be either unstable or they will be a source or they will be a sink.

Now, my basic interest is looking into, what happens for a non-hyperbolic fixed point where f prime of p happens to be equal to minus 1. So, if I look into that case, then we have here for that we need to look into some other concept here. And that concept basically is the concept of Schwartzman derivative.

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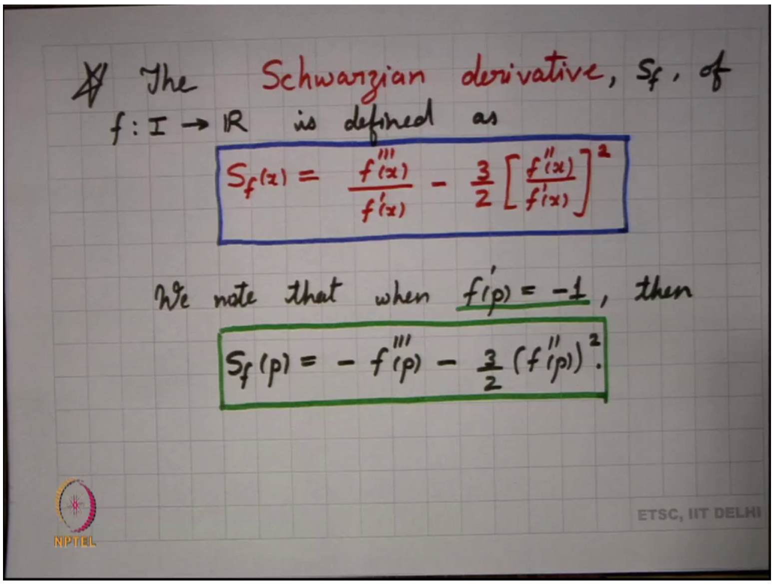

So, what do we mean by a Schwartzman derivative? So, we defined as Schwartzman derivative of a function f to be equal to f triple prime of x divided by f prime x minus 3 by 2 f double prime of x divided by f prime x whole square.

Now, this is basically this Schwartzman derivative for any function defined on the interval. Such that of course, f triple prime is defined and f prime is not equal to 0 and any of the points. Now if I try to look into what happens in the case of a non-hyperbolic point, where my f prime p happens to be equal to minus 1. So, you have a non-hyperbolic fixed point such that f prime p is minus 1 in that case, I simply can reduce that my Schwartzman derivative at p is minus of f triple prime p minus 3 by 2 f double prime p whole square. And we can now try to analyze this because this analysis comes up in terms of Schwartzman derivative.

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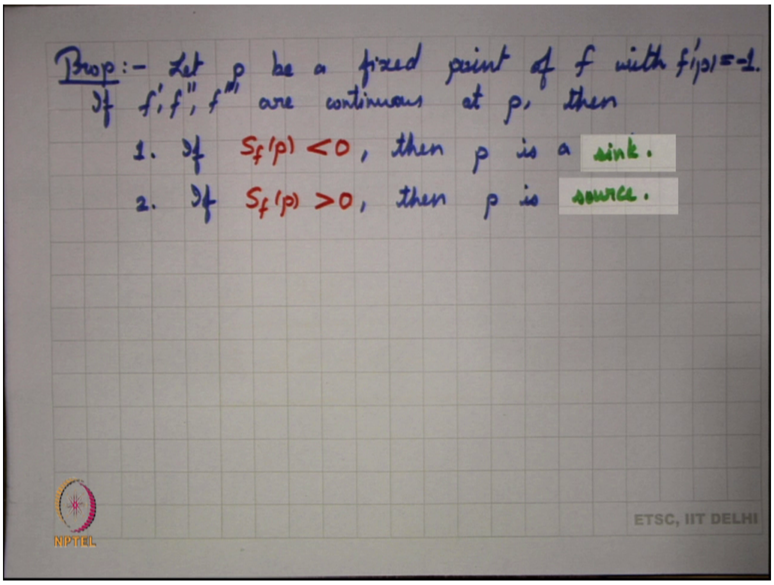

So, we have a proposition here, if I have p to be a fixed point of p with f prime p have been taking the value minus 1, then and again if my f prime f double prime f triple prime are continuous at p, then this Schwartzman derivative is negative, then p is a sink. And if the Schwartzman derivative is positive then p happens to be source.

And this we will not look into the proof of this; maybe I leave the proof of this to you. This is easily proved taking this function j equal to f composite f. When I take g to be equal to f composite f whatever is the nature of p for f, right the nature of p for g remain is same as the nature of p for f. So, they both have the same p has the same nature, under both these functions. But the advantage here is that j prime of p will take the value 1. And hence we can use the previous proposition, but I leave this as a homework right.

So, one can see that we have Schwartzman derivative to analyze what happens to a point p, when it is unstable. Maybe we stop today. We stop here.