Chaotic Dynamical Systems
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Lecture – 28
Topological Entropy - Adler's Version

Welcome to students. So, today we will be formally looking into definition of topological entropy. And especially today we will be looking into Adler's version.

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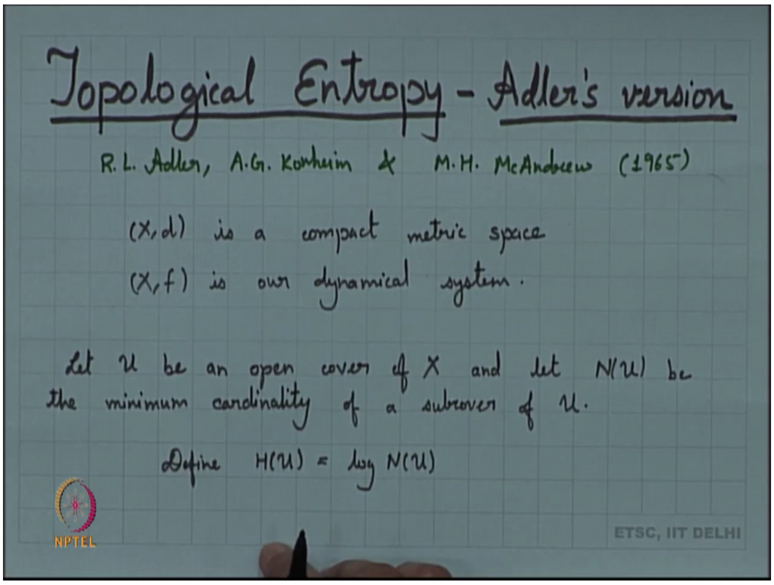

So, this is basically this definition is due to R.L. Adler, A. G. Konheim and M. H. McAndrew. And this was defined the year 1965. So, what does this definition all about. So, again let us go back to what are our assumptions. So, our x d is a compact metric space, and our x f is a dynamical system.

Now, the feature for this definition of entropy is that it basically depends on the compactness of the phase space. So, though metric is not required here, metric does help us in, sorting out some properties of this entropy. But this is basically used whenever you have a compact space, and that is not metrizable this becomes very handy in trying to compute the topological entropy.

Though, if you look into per say the definition, it appears to be a little bit cumbersome to try to compute the topological entropy from this definition. But it very well place it very well shows us what is the structure and what is the reason what behind the topological entropy, how the entropy evolves. So, that is very nicely displayed by this particular definition.

So, since we have already assumed compactness. Let us look into this definition. So, you all are very familiar with properties of compactness, and that is what we are going to use here. Select u be an open cover, now we very well know that an open cover will always have a finite sub cover because of compactness. And hence we let N u be the minimum cardinality of a sub cover of u. So, try to take the minimum possible sub cover of u, and the minimum cardinality something which we denote as N u.

And then we define H of u capital H of u to be log of N u. Let will be any other cover.

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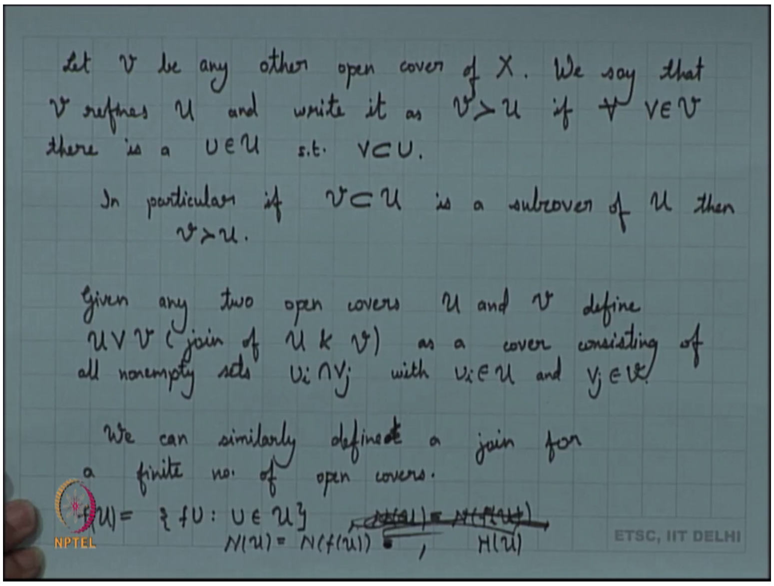

So, let will be any other open cover, then we say that v refines u if and we write it as so, v refines u if for every v belonging to v, there is a u in u such that v is contained in U. In particular if your v is actually a proper subset of u, right. And it is a sub cover of u, then we can say that, right v refines u. So, proper sub cover will always refine the open cover. So, for any to open covers we can see whether we are looking into. So, we can have a class of open covers of x, and then we are looking into basically the chain which comes up from refinement.

There is also another way of defining refinements. So, say given any 2 open covers u and v. We define the join so, we define u join v basically this is the join of u and v as a cover consisting of all non-empty sets Ui intersection V j, with Ui belonging to u, and V j belonging to v. So, you take all us from u vs from v intersect take all the non-empty intersection, take that collection, that collection will again be in open cover. And that open cover will be a refinement of both u and v, right. Because it for everything there will be something containing that, right. Some set-in u containing a set here some set in v containing a set here. So, you will get a refinement. So, this is one of the ways of creating a refinement

Now if we can define a join for any to open covers, we can similarly define a join for any finite number of open covers, right. So, we can similarly define. So, we have this process of joining and this joining gives us refinement. Now here we want to note something if you look into u, right. U is already an open cover. Now we define f u to be basically the set of all f u such that u belongs to u. Nothing of that because our f need not be open, right need not be an open map, right. This f you need not be open, but still we can define f u, right. The f u will cover x fine.

Now, since f u covers x we can think of that we can think of the minimum cardinality of this cover f u of a finite sub cover of this cover f u. And what we find here is that n of u, right would be same as n of f u or may be take this f u and basically that implies I think I haven't define n here. So, maybe I can just go back to this sometime later I think we have defined a N u here. So, start with a N u right. So, I am just looking going back to this factor that n of u will be same as n of f of u. And that happens to be since this is same, right. We can say that H of u, right going back to this definition again, H of u is log of n of u. So, H of u happens to be same as H of f of u. For any open cover you, right. All we know is that H of u will be same as H of f u. What happens here next is that look for a cover.

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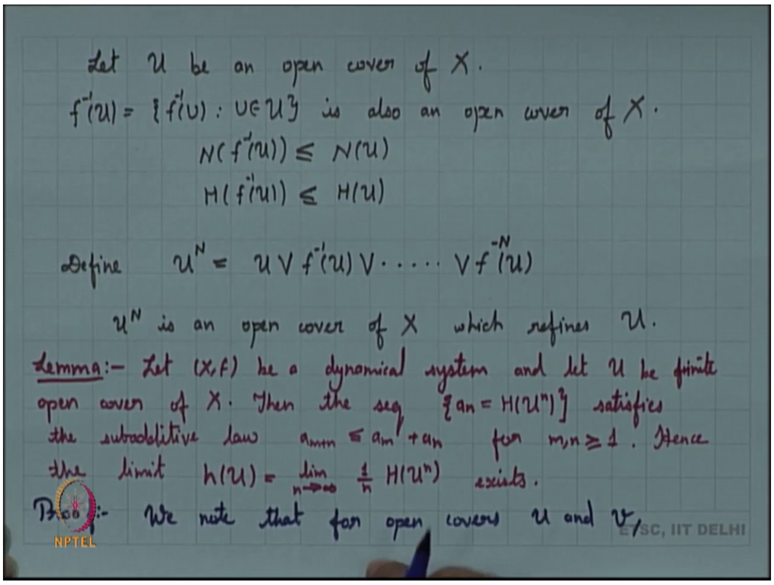

So, let u be a cover of an open cover of x. Now for an open cover of x, right. We know that f inverse u, right. Which we can define it as f inverse u for all u e u, this is also an open cover of x. Now it is quite possible that the sets of inverse u, right. They are larger than u. So, in that sense if I am looking into this as an open cover of u, right and if I am looking into a finite sub cover by f inverse u. It is quite possible that the minimal cardinality, right of the finite sub cover of f inverse u will be less than the finite cardinality for u right. So, what we have here is that n of f inverse u, right. Will be basically less than or equal to n of u. And in that case we can simply say that H of f inverse u will be less than or equal to H of u.

Now, that we need is the fact that f inverse u is also an open cover. And you start with u to be a any open cover, f inverse u is also an open cover. So, if f inverse u is also an open cover f minus to u, right F inverse square, right. Minus to u is also an open cover, right. Similarly, f minus k u is also an open cover for any k any positive k right. So, what we find here is that we can define right. So, we define u capital n to be basically the join of u f inverse u f minus N u. So, these are basically n plus 1 join joining of n plus open covers, but we define this to be this joint. So, we know that a join will any may be a open cover.

So, join gives you an open cover not just gives you an open cover. It is basically a refinement of all these open covers. So, it is a refinement of u also it is a refinement of f inverse u and so on. So, it is basically gives an open cover of x which refines u. So, more mostly and interested in refining u, un is an open cover of x which refines u we look into this lemma so far, our dynamical system x f or let me just assume that.

Let x f be a dynamical system and let u be an open cover or since we are interested in taking up any way we are interested in a finite sub cover we are only interested in the finite sub cover. So, we can very well assume that you itself is a finite open cover of x. So, let u be a finite open cover, then the sequence so, we looking into the sequence I am looking into the sequence a n where what is my a n? My a n happens to be H of u n, right where u n is just the joint that we have define.

So, if I look into the sequence this sequence satisfies the sub additive law. So, sub additive law is a m plus n is less than or equal to a m plus a n, for all m n greater than equal to 1. So, H of u and satisfies the sub additive law. And hence has a limit hence this limit some defining small H of u to be equal to limit s n tends to infinity 1 upon n times H of u n this limit exists.

Now, we already know well I will need to prove. So, to prove this lemma basically we all need to see is that your H of u n satisfies the sub additive. Now then you are just applying the property of sub additive sequence, right. To show that this limit exists, we go into the proof here. So, not that for open covers u and v.

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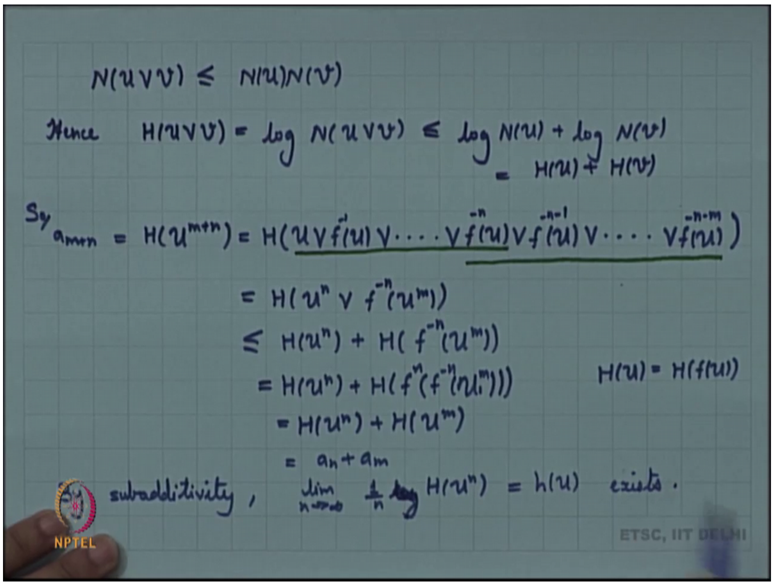

Your n of u join v is less than or equal to n of u times n of v. All we know how do we defined u join right. So, we are just intersecting one element of u and one element of v, and now we are saying that fine when we have a sub cover from this joint, right. Now how many elements can we have now many minimum elements can be there, right. So, that will be always will less than or equal to the minimum the multiplication of the minimal and elements that you get R from the a previous to. So, this is very trivial to observe here and hence what can you say about the join of H m u.

So, if I look in to join of H n v this is basically log off n of u join v. So, n of nu u join v this is less than or equal to again, I can think of log of N u times n v. So, this is log of nu plus log of n v, and this happens to be your H of u plus H of v. So, if I take a join, right. We definitely see that this satisfies this property. So, what happens in case of a finite thing which we are more interested right. So, let us look into what happens to our a m plus n and say that. So, what is a n plus 1 that is basically H of u m plus n. Now we know what is u m plus n, right. It could basically be H of, right I have u of inverse u, right join I can go up if minus N u join f minus n minus 1 u, right.

Fine, I am looking into H of these joints. And again, if I look into that factor, right look into what is this quantity, I am just noting into what is this quantity. This is just u n by definition, and now I am looking in to this quantity, what is this quantity. I can say that this is f minus n of I am looking into this complete join, but what does this complete join, it is basically the join of this part and the join of this part right. So, the first part is just un and the second part happens to me f minus n of so, we can say that this is nothing but this is H of u n join f minus n of, now we just apply what we have previously seen if I have 2 covers, right. H of you join v is less than or equal to H of u plus H of v. So, this is less than or equal to H of u n, right plus H of f minus n u n.

Now think of that we have already seen that for any cover u H of u is same as H of f u, we have already seen that. So, we applied this here 2 f n right. So, this would be same as H of u n plus H of fn of f inverse of u right. So, this happens to be same as H of u n plus H of f n of f minus n of and this is nothing but H of u n plus H of u n. So, we know that this H of u n is nothing whatever a m, right plus a m. So, H of satisfies the sub additive law, right. And hence we can say by sub additivity, limit n tends to infinity H of u n, right. This is equal to H u exists. So, formally defining an entropy now.

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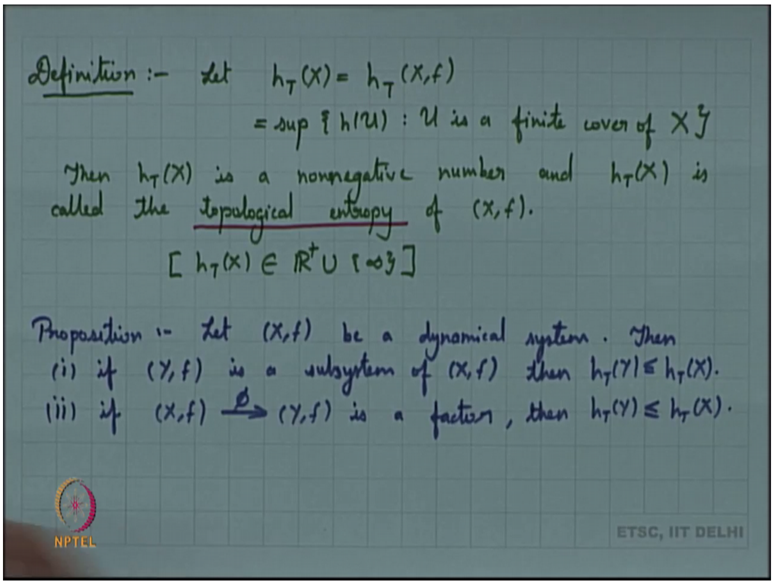

So, I am defining h t of x which I can also write as ht of x f, right, which is basically the supremum of h of u such that u is a finite cover. So, since I know that my h u is defined h u always exist, h u is defined, you can take this numbers h u. And you define the supremum of these numbers h u. So, the supremum of these numbers is denoted as h t x or h t f x. And this h t f x is a non-negative number, right. Since we are taking supremum this is a non-negative number. And h t x is called the topological entropy my h t x is again supremum right. So, it can be in the extended real plane also. So, it can be infinite also, since it is the supremum.

So, my h t x basically you can say that h t x belongs to R plus union infinity. And this is basically my topological entropy. As we have already seen this fact that topological entropy being positive, right is a sign of gas. So, let us look into a proposition here. So, let x f be a dynamical system of course, our x is compact here. Then we have 2 facts if y f. So, we are taking a subset of x y f is a subsystem of x f, then topological entropy of y will be less than or equal to the topological entropy of x. And the second fact here is that if I have a factor here.

So, x f is factoring into y f this is a factor then again, the topological entropy is less than or equal to topological entropy of x. So, by taking a factor we are already always reducing the entropy. And if you try to look into the proof of this the proof of this is simple in the sense that. We know that when you are taking a subsystem, right. If you take any often cover of x, right. That will also be an open cover of y then think of that as also as an open cover of y. And then the minimal cardinality main decrease, right and hence you will get that the entropy will be less, right.

And similarly happens for a factor because if you try to look into a factor take any open cover of y, right take phi inverse that will be an open cover of x. And it is quite possible that the minimum cardinality here will be less than the minimal cardinality the cardinality here. Because we will have other open covers of x also right. So, we are taking supremum over more number of elements right.

So, the entropy may increase here. So, this is a small proposition, but of use is an another position which we shall see right now. So, we take another proposition here. So now, I am getting a chain of refinements. So, I have u 1, right. It is refinement u 2, this is refined by u 3 and so on. So, this is a sequence of open covers with, now here we need a metric space. We need our x to be a metric space here because we are going to define.

The diameter of an open cover, with the diameter of cover some refining diameter of un to be the supremum of diameter u such that u belongs to u n. So, the maximum the supremum of this diameter of all elements of u that forms the diameter of the open cover, and we are taking this sequence of refinements in such a manner, that the diameter of u n is converging to 0. So, of course, this is with respect to supposing I have such a sequence of open covers, then the topological entropy h t x is nothing but it is limit as n tends to infinity of H of u n. So, basically think we can call H of u n to be the entropy of the open cover. So, entropy of the open cover if the open cover is basically decreasing, looking into a decreasing sequence here.

Then you can say that the entropy happens to be limit of this u and the proof here is actually very simple, but it is a nice idea here. So, try to look into the proof here. So, let v be any open cover now we know that our x is compact, we are in a metric space setting. So, every open cover will have a lebesgue number. So, the defined take delta to be the lebesgue number for v. So, this is the lebesgue number for v know what we are taking is we know that this is a decreasing sequence diameter and is tending to 0. So, somewhere the diameter will tend to me less than delta. Somewhere diameter of the open cover there will be some n on words.

Such that the diameter of un will be less than delta. So, delta is a lebesgue number of v. So, we know that un is basically un refines v for every n with diameter of un, right less than delta. So, that means, now you take any open cover, what you find is that you from some and onwards your uns will always all the uns will always be refining this cover right. So, all we need is we just need their what is this cover. So, your H of u n happens to be a decreasing sequence right. So, H of u n so, the entropy of u n, right is a decreasing sequence; obviously, since u n is decreasing the diameter is decreasing right. So, the H of u n will be decreasing, right. And since H of u n is decreasing we know that H of u n refines.

Everything and this is a decreasing sequence. So, since of this is a decreasing sequence we can think of that since this is a decreasing sequence. So, you will always have this will tend to a limit, right. And that will be our topological entropy, because our topological entropy is a supremum of all covers, right. The supremum of their entropy of all covers, but here we have a decreasing sequence of open covers, such that take any open cover it is going to be refined by this open this one our elements of the sequence, and hence your entropy of x.

So, I can say that hence entropy of x is limit as n tends to infinity. H of u n now this is actually a very important idea because this helps us in trying to compute entropy for many things right and though the definition of course, this requires a metric definition, but this definite the benefit of this definition is that; this definition can be used for whole compact spaces which are not metastable. As we shall soon see this is another definition of entropy due to bowman and that definition is practically more useful then this particular definition.

But if you want to use a definition using open cover, right in a metric space setting this proposition is very helpful in determining the entropy. So, let us try to take up some examples for this finding out entropy using this method.

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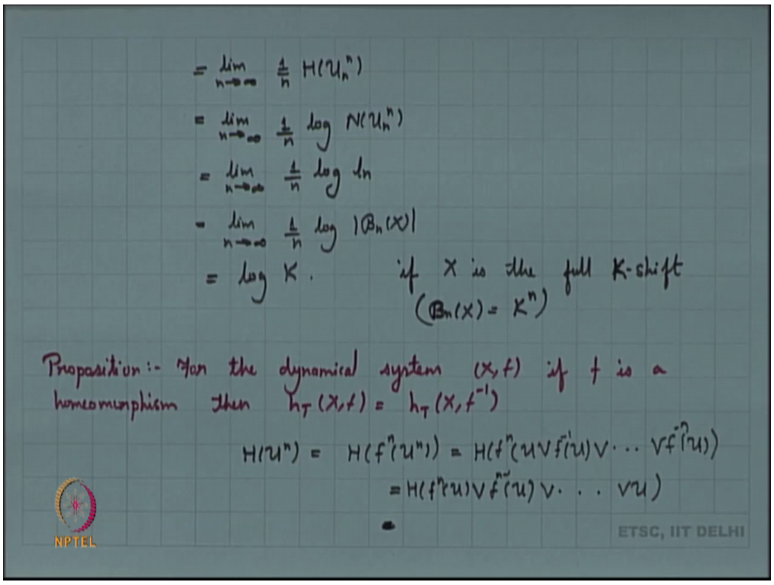

And that will give us a better idea of how to use this definition. So, we look into our first example here, and since we have already discussed the full k shift, right. In the previous lecture we will.

Similarly, discuss the full k shift here and we will try to compute the entropy using this definition. So, for the full k shift that any n any natural number, then what we have is we have p n which I am defining as a disjoint union of all the words in B u x, right of this now what is my element by element x is in X, such that if I take this block x from 0 to n minus 1. So, this happens to be w. We know that there will be different words B u x will have different blocks. So, we are defining the blocks in this particular manner. We know that this forms a partition of x. So, p n gives a partition of x is a partition of x into clopen sets for all words.

So, whenever you take fix any n, right. You know that you will have words of length n and that gives you partition of x into clopen sets. Because these all will be cylinder sets right. So, this gives a partition for partition of x into clopen sets. Now for this partition we define u n m to be the join I am taking I going from 0 to m, right. Sigma minus I of p n. So, start with a partition n p n right. So, this is basically this is again an open cover of x, and think of p n to be an open cover of x. So, this is an open cover of x, and for this open cover, right. We take this we define u n m. Now we know that this is also going to be and open cover. What is H of u n m?

We know that this is going to be sorry, I shouldn't say H I am looking into n now what is n of u n m. So, basically what is the minimum number of elements in u n m which can cover your x right. So, look looking into the minimum cardinality here for this open cover, I find that this is going to be ln, right. Where your ln happens to be cardinality of B u x think of that take u n n, right. Then we know that u n n will be refined by u n plus 1 n plus 1, I am taking words of length n plus 1, right. And I am looking into a bigger refinement there. So, that will definitely refine u n n, for all n greater than n, greater than equal to 1.

And so, this forms are decreasing refinement. So now, I have a decreasing refinement, and what can we say about the diameter? The diameter of u k k, right as your k increases this is tending to 0. Because as you increase the number of words, right. What happens when you increase the number of words? What will be the diameter, because 2 any 2 elements in that basically any in any partition any 2 words will be very close to each other right.

So, the diameter is basically going to get smaller and smaller. So, this diameter converges to 0. So, diameter of u n, I should say u k u k, right. That is tending to 0. So, what we find here is is a decreasing refinement with let me mark this part also with diameter of u k k, right. Tending to 0 as k tends to be infinity. This is tending to 0 and that gives us that the topological entropy of x right can be very well given as a limit of the entropy of u n n.

So, we think of this part that the entropy of x, right. It is nothing but limit as n tends to infinity I have H of sorry h of u n n. Now if you look in to H of u n n, right. What is that equal to so, this is nothing but this is limit as n tends to infinity 1 by n times H of u n n, but if you look into that factor this is nothing but limit as n tends to infinity 1 by n log off N u n n. And what is N u n n that is ls, right? So, this is nothing but this is limit as n tends to infinity 1 upon n, right. Log of ln, which we know what is our ln this is limit as n tends to infinity, 1 upon n log of mod of B u x.

And this is what we had discussed in the previous case, right. This is basically this gives the number of distinct orbits, right. The definition that we had taken up, is same as this particular definition. So, we are looking into this part 1 by n log of mod B u x, and now if our x is a full k shift right. So, if x is the full k shift, we know that this will tend to be equal to log of k.

Because then your b and k will be k to the power n right. So, B u k happens to be so, B u x happens to be k to the power n, right. In that case so, this entropy will turn out to be log of k. So, for the full shift for the full k shift, the entropy topological entropy turns out to be log of k. So, this is basic this is the basic idea of entropy that we had used in the previous class. And now I just want to look into one proposition here for the dynamical system x f.

If f is a homeomorphism then a topological entropy of x f is same as the topological entropy of x f inverse. Now this is very helpful in computing entropy. And one can easily see this holds true because if you try to look into. So, the hint here is simply that the fact that if you try to look into H of u n, right. It will be same as H of f of N u n, right. And H of f of N u n can be written as now you are trying to compute that part right. So, it is written as H of you now have f n of this is u join f inverse u, right. Join f minus N u, right you have this join here. And if you try to look into that part this is nothing but this will give you H of f nu, right. Join f n minus 1 u, right. Join u.

Which is nothing but this basically gives you the entropy of so, this is since this is same, right. You compute the entropy here 1 upon n times this factor, right. What you get is the entropy of f inverse right. So, this is basically the hint here. And we observe that this result is only true for a compact x, it is not true otherwise. Because otherwise you can take say take the positive real line, take the mapping to x then the entropy of 2 x, on the positive real line happens to be lock to, but if you take the inverse there the inverse is x by 2 which is a contraction. And for a contraction we know that everything all orbits are tending to 0, right. So, the topological entropy there happens to be equal to 0. So, this is not true there. We will now look into another example of a system which we know very well. And we will see that topological entropy can also be equal to 0.

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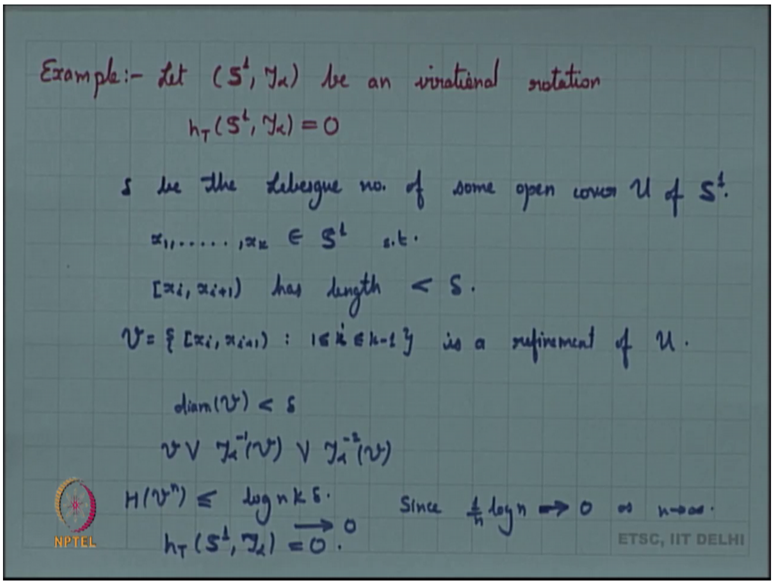

There are cases where topological entropy can be infinite, right. The examples there are highly complicated. So, that is those examples we will not be doing those examples in these lectures, but you have many, many systems for which the topological entropies infinite. So, our example is again an irrational rotation. Now we want to say that if I look into their topological entropy of this irrational rotation, this is equal to 0. And we try to prove this fact again using our lebesgue number.

So, let delta be the lebesgue number. So, start with the lebesgue number, we start with an open cover start with a lebesgue number. What you are not doing is you are taking points x 1 up to x k, right. Take this points in s 1 such that your this arcs xi xi plus 1 if we will try to look into this particular arc, right. This particular arc has length, right less than delta. So, you are taking any k points and s 1 such that this length as this arc has length less than delta.

Now you take v to be a refinement. So, we take v to be the union of all such arcs then, v happens to be a refinement of u. So, if I take v to be basically the union of all this arcs is a refinement of u. Now let us try to compute the entropy of this open cover v. Now if I try to look into v, right. I am just looking into what is the diameter of v. So, the diameter of v right is going to be less than delta. Now what happens here is; you can take the diameter of so, you take v join.

So, t alpha inverse v, take v join t alpha inverse v, then what do that here is we know that this is an isometric rotation. T alpha is just an isometric rotation. So, you are only going to rotate these arcs, right. In some particular angle, the number of points, right where it is going to so, the number of points fine where this intersects. So, you will have at least one point in between xi and xi plus 1 where the one of the image will be intersecting. So, what you have is you have like you had k points now; in the next time you have 2 k points right. So, you will find 2 k points and now you are taking the partition of s 1 with 2 k points, right.

And again, the diameter is anyway going to be less. So, it is going to be definitely less than delta diameter is decreasing there. So now, you have 2 k points. Now when you take another you take the join with t alpha minus 2 v, right. You find that the join happens to be of. So, you will find 3 k points there right. So, you find 3 k you find that you will find that these are 3 k points. So, if you take up any say v n, right. Then in v n how many points do you have here? So, you have basically a partition of s 1, right with n k points at the most n k points. Now with at the most n k points what are want is the diameter turning out to be you have at the most n k points, right. And the diameter there this is less than delta. So, the diameter is again decreasing, right. Delta and then we have decrease delta by something.

So, what do you find here is that if I want to take H of v n, right. This is basically going to be less than or equal to log of n k times of course, I can think of it delta right. So, log of n k delta, this is less than or equal to this particular fact, and if I try to look into 1 upon n times this factor, right. Then 1 upon n times this factor is going to tend to 0 because, 1 by n log n. So, 1 upon n since 1 upon n log n is 0, right. It basically tends to 0 as n tends to infinity. So, what happens here is that if I take H of v n, right. This will be less than or equal to log of n k delta, right. This is log of less than log of n k delta. So, this is basically going to tend to 0 as your n tends to infinity. So, again if I take this limit of H of u n, right. That is tending to infinity that is going to be 0.

So, you have lot of 0s there, right. And this is a refinement of almost any open cover that you take up you will you can find such a refinement, right. You will find some u n because this is the diameter and decreasing you find that this is going to be tending to 0. So, watch what you find here is that your entropy of s 1, right under t alpha is basically 0 c. So, you can take this open cover definition and you can show that basically if you take n is in fact, this is true for any isometric stuff. So, if you take any homomorphism on a compact space x, you have a homomorphism. You can show that this homeomorphism is isometric right.

Then the entropy tends out to be equal to 0. So, then copy in this particular case for an irrational rotation the entropy is 0. In fact, if you start with a finite set, right. And we can take any permutation on the finite set. If that is our dynamical system, then again, we know that the entropy is going to be equal to 0. Though we will see this that interpret does define does imply some aspect some other definitions of chaos.

But it is not true that all chaotic things that we have seen in the previous definition would turn out to be having entropy positive. Positive entropy is also a sign of chaos. Because it shows that there will be multitude of orbits that you can see in the dynamical system. So, in the next class we will basically look into another definition of entropy, which turns out to be because it is more useful it is a metric definition, turns out to be more useful it captures many other features. I hope this is clear as far as this is concerned.