Chaotic Dynamical Systems
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Lecture - 27
Measuring Chaos - Topological Entropy

Welcome to students. So, today we have been looking into measuring chaos and the quantity that measures chaos is topological entropy. So, before we go into what is the motivation for studying topological entropy for understanding any mathematical structure it is very useful if we can attach some kind of numerical value to that structure . So, for example, say we want to study mathematical structures and we are attaching numerical values to it.

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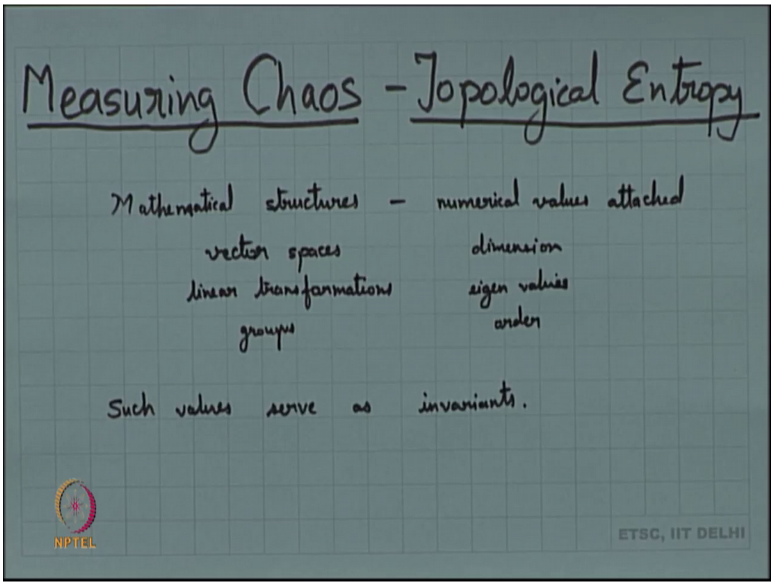

So, for example, we want to study say vector spaces and we know that the numerical value attached to it is the dimension, if you we want to study linear transformations and we find that the numerical value attached to it right we can think of eigenvalues we want to study groups and we can attach something called the order of the group.

So, we have this numerical values right attached to this mathematical structures and then they give us a lot of information about these mathematical structures and then these numerator values attached they not only help us in deciding whether the 2 structures are similar or the 2 structures are isomorphic, but they also help us in comparing the 2 structures. So, such values they serve as invariants and we would like to study one such invariant attached with a dynamical system. So, how does the story evolve? So, basically the story goes back to Shannon.

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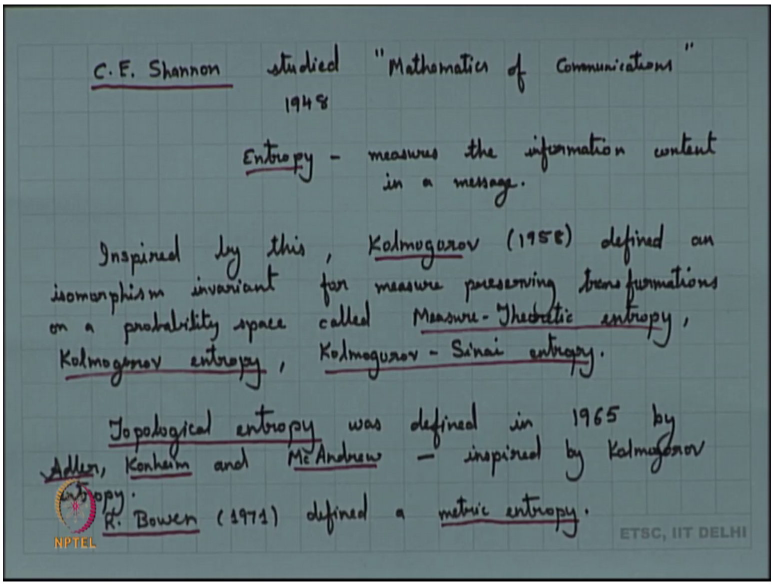

So, Shannon studied when he was studying the mathematics of communications. So, basically this was a mathematical theory of communications way back in nineteen forty eight. So, he wanted to give some kind of a measure to the information that is stored in a message. So, you want to pass message from one place to the other. So, you are communicating messages and he wanted to measure.

What is the amount of information stored in this message now it is very easy to retrieve the complete message if this amount of information in it is very small? So, you can completely retrieve because there was there is no basic loss of information that you can get there, but on the other hand if you have a very complex kind of message which consists of lot of informations, then it becomes very difficult to retrieve the complete or it may not be possible to retrieve completely all the information that was conveyed. So, even when you look into decoding that part it becomes a very complex kind of structure to encode or decode. So, he studied he gave this measure he called this measure as entropy. So, this is; what is the birth of entropy. So, he called this measure as entropy which measures the information content in a message now.

Inspired from this concept of measure of information which actually Shannon had used for looking into the mathematical theory of communication Kalmugarov defined an isomorphism invariant. So, inspired by this Kalmugarov maybe in 1958 defined an isomorphism invariant for measure preserving transformations and the probability space.

So, he defined an isomorphism invariant called this is measure theoretic entropy or it is called Kalmugarov entropy or its also called Kalmugarov Sinai entropy we actually. So, the people important here are Shannon's entropy, Kalmugarov and what he defined was this measure theoretic entropy these are the various names by which it is known now we are not going to study.

Measurable dynamics basically in this particular course, but you can as well study say some kind of measure preserving transformations on a probability space. So, you have the set of all Borel sets you have a Borel algebra and then you are looking into measure preserving transformations; that means, you are looking into that the transformation preserves the measure right after you apply the transformation.

So, when you are looking into that aspect right that is one more way of studying dynamics when again you are looking into only measurable systems or you looking into measure measurable systems where again your transformations need not be continuous you are only measuring them how big the set is how small the set is etcetera that also leads to one very nice theory and there is lot of chaos involved there also, but we are not going to because it is outside the scope.

Of this particular lecture or this particular course that we get into the measurable content also and try to study that. So, we will not be studying the measurable part, but this is here it needs to be mentioned because the entire concept of topological entropy has its motivation from this measure theoretic one now there are many features that you have you see mixing right you see mixing you see transitivity right this all concepts are related interrelated to what you can see for a measurable system. So, you look into measure theoretic dynamics. So, measurable dynamics and if you look in to topological dynamics there are a lot of aspects which are interrelated.

So, one thing aspects have been motivated from the measured theoretic one and then they have been borrowed into the topological case some of the aspects have been taken up from the topological case they have been borrowed to the measure theoretic one, but here definitely this is a very much inspired work from the measure theoretic part now after Kalmugarov defined this entropy it was almost for of long time and say 65 right. So, topologically entropy was defined.

So, topological entropy was defined in 1965 by 3 mathematicians, Adler, Konhean and McAndrew. So, as I said that this was inspired by Kalmugarov entropy, now this topological entropy that was defined by Adler and all. So, this was defined for compact spaces. So, he need compact house of spaces and this was a kind of definition for compact spaces later on R Bowen. So, let me emphasize Adler Konheam, McAndrew also and then R Bowen he defined in nineteen seventy one defined a metric entropy. So, this is not some kind of topological entropy this definition it relies on the matrix on the topology that you define defined a metric entropy and this is now called Bowens entropy and we shall see that these 2 concepts.

These 2 definitions coincide. So, this is the concept of metric entropy basically both this entropy is whether you take it in Adler's definition or whether you take a Bowens definition they are called topological entropy the benefit is that under certain circumstances defining one becomes and working that one becomes easier under some other kind of conditions right defining something some other part and defining Bowens and then using that becomes easier so, but in general this is defined for compact topological spaces Adler's entropy is defined for compact topological spaces Bowens definition goes for in a metric setting.

So, of course, when the 2 things are for compact metric spaces the 2 definitions are the same or they are basically they coincide. So, we will study both these definitions and they form nice this entropy also forms a nice dynamical invariant. So, if your 2 dynamical systems are conjugate, then they would have the same entropy and if you try to look into what is topological entropy. So, topological entropy.

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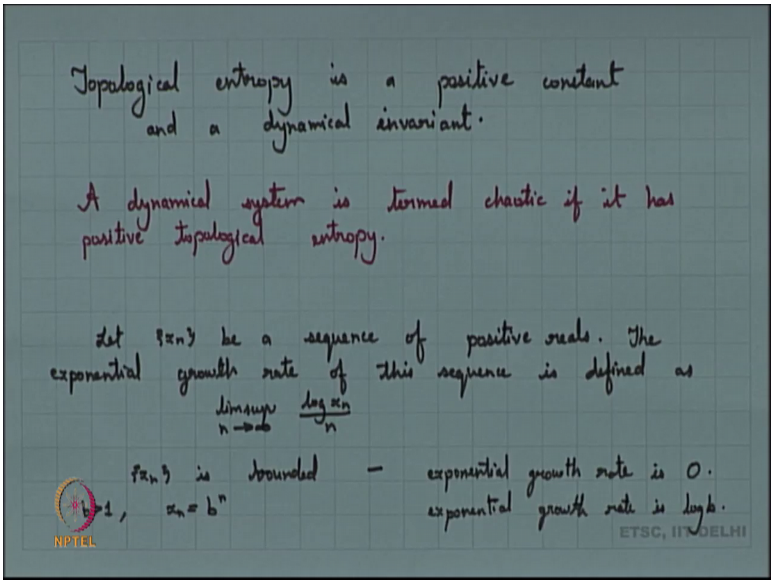

Topological is a positive constant and of course, a dynamical invariant and why do we study topological entropy as I already said that this is something called which can be termed as a measure of chaos. So, dynamical system is called chaotic, if it has positive topological entropy.

Now what we are going to do today is we are going to look into what is the motivation behind this definition how does this definition come into existence and what could one try to visualize it in a very simple terms and then we will go back to Adler's definition and Bowens definition and we will look into some properties of topological entropy now where does this motivation come up from.

So, if you look into that in a various plane words you think of a sequence of positive reals. So, let x n be a sequence of positive reals now you have this sequence of positive reals we are interested in what is the exponential growth rate of this sequence. So, the exponential growth; growth rate of this sequence is defined as you take the limsup of log of x n upon n and we take the limsup as n tends to infinity. So, this gives for any sequence of reals.

This gives the exponential growth rate of the sequence. So, now, it is very simple to see that if my sequence x n is bounded then this exponential growth rate will be 0 right because then log x x n is bounded, but your n is tending to infinity; this exponential growth rate is 0, but if you have your x n to be some kind of an exponential. So, if you are you can express your x n in terms of an exponential form. So, for example, if you have a b greater than one such that your x n is of the form b to the power n then your exponential growth rate right could be is log of b if your x n is of the form b to the power n right this exponential.

Growth rate happens to be log b. So, we are now going to use this idea of exponential growth rate, but then we are working in a dynamical system. So, what kind of sequences do we deal with?

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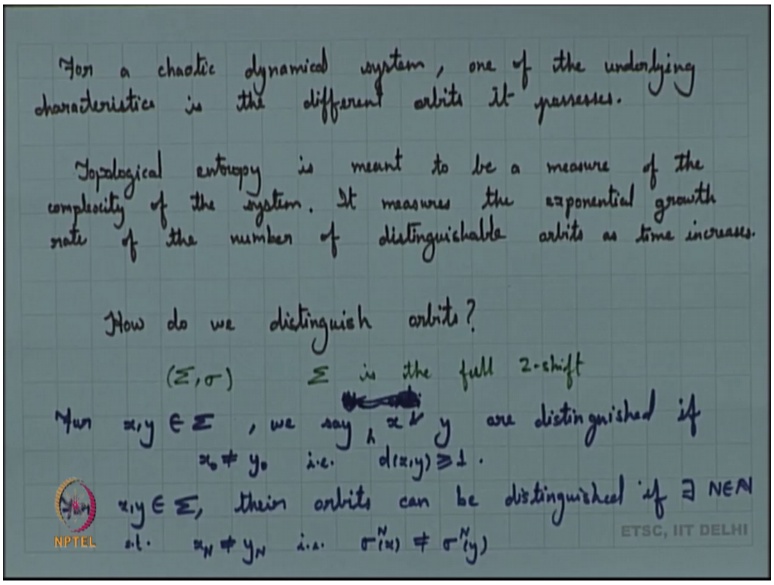

So, for a chaotic system, for a chaotic dynamical system, one of the underlying characteristics is the different orbits now you have already seen that for an interval map right Sakowski's theorem says that if you have an orbit of period 3 you have orbit of all periods right. So, you have different orbits and it is basically this multitude of orbits that you have which gives us a very rich structure for the chaotic system. So, we are interested in this rich structure of chaotic systems and we want to measure this using this different orbits.

So, topological entropy is meant to measure; to be a measure of the complexity of the system the system is more complex if it has more distinct orbits. So, the topological entropy is meant to be a measure of the complexity of the system and what does it measure it measures the exponential growth rate of the number of distinguishable orbits as time increases we are not formally defining topological entropy. So, particularly this class let us assume that the exponential growth rate of the number of distinguishable orbits is our entropy right now we are interested in knowing what do we mean by distinguishable orbits and what do I mean by distinguishable orbit as time increases . So, how do we distinguish the orbits?

Let us look into say the full shift. So, consider the system where your sigma is the full 2 shift. Now what we are interested in we want to see what do we mean by distinguishing. So, we say that for x y in sigma right this we say that x y can be distinguished the orbit sorry the orbits of x and y or maybe I am just saying x y to be distinguished x and y are distinguished if say my x naught; that means, the middle part is not equal to y naught right they are different in the middle part; that means, we know that this is the metric space right. So, the distance between x and y the sequence is x and y the distance would be greater than equal to 1. So, we are looking them as apart. So, x and y can be thought of us apart

Now, what do we mean by saying that the orbits can be distinguished. So, we say that for x y in sigma their orbits can be distinguished if there exist an n in n such that your x n is not equal to y n now what does that mean it is same as saying that your sigma to the power n of x is not equal to sigma to the power n of y; that means, now at the nth stage you can distinguish these 2 orbits if x and y are same right say maybe something greater than n right we cannot distinguish the orbits because if they are they will be having the same orbits up to n steps right. So, we can distinguish their orbits only if they are basically within that bracket of time that we want to see within that bracket we are able to distinguish them.

So, we say that the orbits can be distinguished at time n right if of 2 n we can get some kind of value k for which sigma k x is not equal to sigma k y, then we can say that we can distinguish them now we know we have studied the full shift full 2 shifts very well right.

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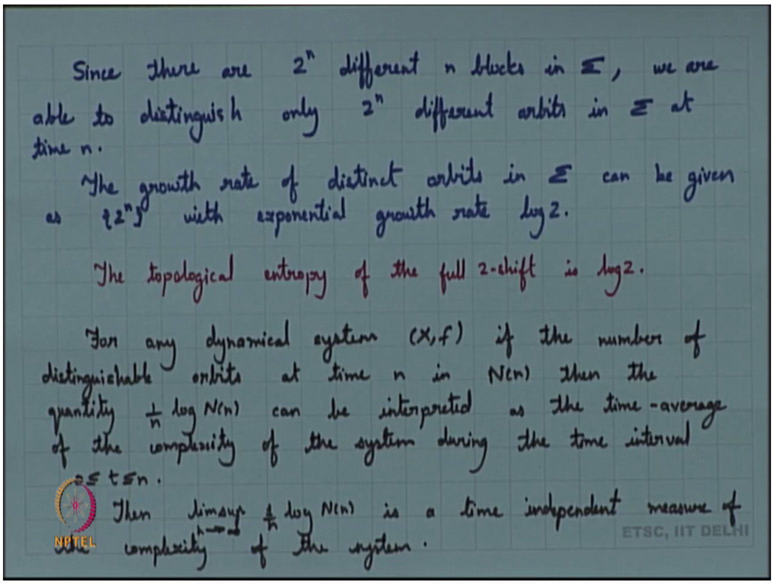

So, we know that we can able since there are 2 to the power n different blocks right. So, if you want to how many distinct n blocks are there we have we know there are 2 to the power n different n blocks Right. So, we are able to distinguish only 2 to the power n different orbits in sigma at time n. So, if we want to look into time n what happens after time n how many orbits can we distinguish.

We can distinguish only 2 to the power n orbits. So, what is the growth rate of distinct orbits and sigma? So, the growth rate can be given as now I can write it as a sequence what happens at the time n; how many orbits can I distinguish right. So, this can be given as 2 to the power n the sequence 2 to the power n right. So, the growth rate of orbits and sigma I can write it as a sequence 2 to the power n and what is the exponential growth rate of this what would be the exponential growth rate here log 2 . So, we can say that for the full 2 shift right the topological entropy is lag 2 right as we said we roughly say that growth rate of this orbits is giving as the topological entropy.

So, we can say that the topological entropy of the full 2 shift is log 2. So, let us let us note it down also. So, the topological entropy let us work with our dynamical system. So, we say that for any dynamical system if the number of distinguishable orbits at time n is say a function n of n then the quantity 1 by n times log n of n right can be interpreted as the time average of the complexity of the system during the time interval . So, now, my time interval is 0 less than t less than n. So, during this time interval, we can say that the time average of the complexity is one upon n log and then the limsup is n tends to infinity of this quantity. So, this is like 1 by n log of n n.

This is a time independent measure time independent measure of the complexity of the system this time and we can take this time independent measure of complexity to be our topological entropy right we will formally define topological entropy in the next lectures, but presently we take this to be our topological entropy and we have seen that for this for the full 2 shift right this time independent measure of complexity happens to be log 2 which is our topological entropy. So, let us take x to be any shift space. So, what we do is let x be any shift space.

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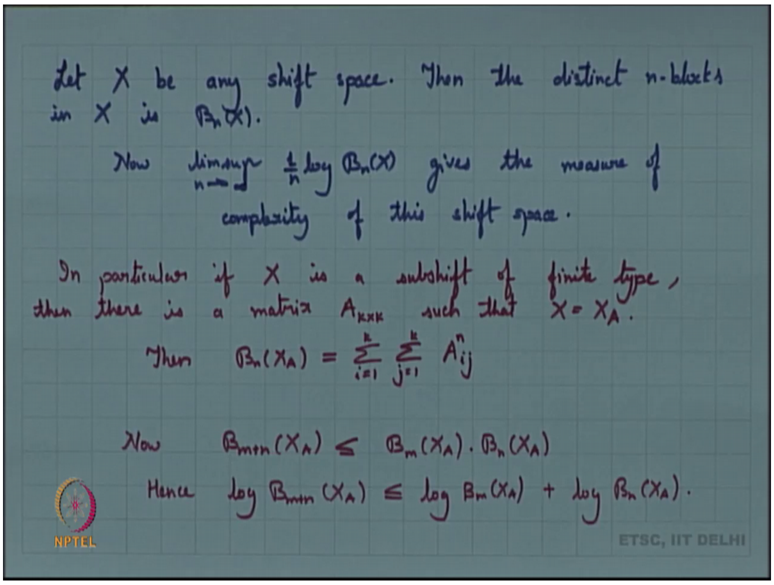

Now, it is easy to see all these things in terms of shift spaces and that is why we are looking into the example of shift space.

So, let x be a any shift space, then that distinct n blocks in x right we know that we have defined that to be B n x. So, now, if you want to measure this complexity right we just need to take this limsup as n tends to infinity of 1 by n log of B n x right this gives the measure of complexity of the shift space. So, for any shift space if you know how B n x is working how B n x functions right it is very easy to find out what is the measure of this complexity very very easy to find out what is the logical entropy. So, what happens in particular if our x happens to be a sub shift of finite type. So, in particular if x is a sub shift of finite type.

Then we know that there is a matrix a now how many symbols are there in X A depends on that. So, let us assume that a is a k cross k matrix. So, there is our k cross k matrix a such that your x is same as X A then we know what is B n x in that case what is that equal to we know that from any given vertex I right we need to go to a vertex j in n steps right from any vertex I to any vertex j if you travel in n steps right that is what is going to give you the number of all such paths right that is what is going to give you your words of length n in x right. So, this we can write it as summation. Now I am taking this vertex vertices I am naming the vertices one to k. So, taking the summation one to k this again the summation j going from one to k right I have a to the power n right i j.

So, I am looking into the i j th entry of a to the power n and if it sum upon of them then we know that what we get is all the different blocks all n distinct blocks right of length n in x i and another think that you can may look into from this very definition is that if I look into what is the m plus n blocks in X A then we know that this the number of m plus m blocks in X A will always be less than or equal to the number of m blocks in X A right multiplied by the number of n blocks in X A because any n block can be extended to an m plus m block any m block can be extended to an m plus m clock right and if you multiplied that 2 right that will always be a some number which is greater than the number total number of m plus m blocks . So, we have this. So, what happens is that the number of blocks they satisfy this sub additive condition. So, what is the sub additive condition?

We can simply say that hence, if we take log of B m plus n of X A this is less than or equal to log of B m X A right plus log of B n X A. So, our blocks m blocks right the sub shift of finite type that satisfy this particular condition now we will look into a small lemma here and this lemma is basically from real analysis, I would encourage you to prove this give a proof of this though we are not going to consider this proof here, but you can try to prove this.

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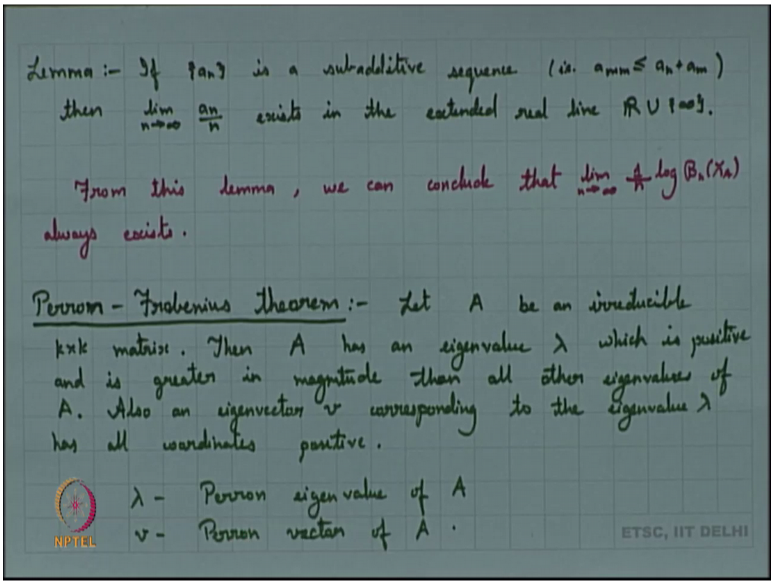

So, if I have a n right to be a sub additive sequence; that means, what I have here is a n plus m is less than or equal to a n plus a m supposing this satisfies the sequence a n satisfies the sub additive condition, then limit as n tends to infinity a n upon n exists in the extended real line; that means, I am looking into r union infinity. So, this limit a n by n. So, if I take the ratio a n by n, this limit exists this limit can also be in finite right, but it will all definitely exist.

So, this limit is either a real number or it is infinite now from this lemma right we can conclude that limit as n tends to infinity right one upon n log of B n x B n X A right will always exists. So, it could be infinite we are not sure about that, but we are not this limit would always exist and since this limit would always exist we can always define a topological entropy for the sub shift of finite type now.

What exactly does this topological entropy sub shift of finite type turn out to be for that we again have to go back to linear algebra, but we may have to go back and study some more result there. So, again there is a result is a very famous result which I am just stating here we are not going to prove that result I am just stating the result here. So, the result here is. So, the result here is Perron Frobenius theorem; the Perron Frobenius theorem what is this Perron Frobenius theorem. So, let A be an irreducible that A be an irreducible say k cross k matrix. So, A is an irreducible k cross k matrix, then A has an

Eigenvalue lambda which is positive and is greater in magnitude, then all other Eigenvalues that a can have also if I look into this eigenvalue lambda.

Which is not only positive, but it is greater than magnitude than all other Eigenvalues then this Eigenvalue lambda has a corresponding Eigenvector v. So, we can say that also an Eigenvector v corresponding to the Eigenvalue lambda has all positive all coordinates positive; that means, v can be written as because a is a k cross k matrix right. So, this vector v would belong to r k and since this vector v is an r k you can think of all this vs right to be strictly positive. So, v is v 1, v 2, v k, where each of this v k is strictly positive now when we think of this part right this is such a lambda right is called a Perron value let us say Perron eigenvalue of A and such A v is called a Perron vector right it is basically Eigen vector, which as say that it is a Perron vector of A.

So, Perron eigenvalue and Perron vector right these 2 are concepts defined for irreducible matrix a now let us try to look back into let us start to implement this basic this this theorem into our discussion. So, let us go back to our discussion now.

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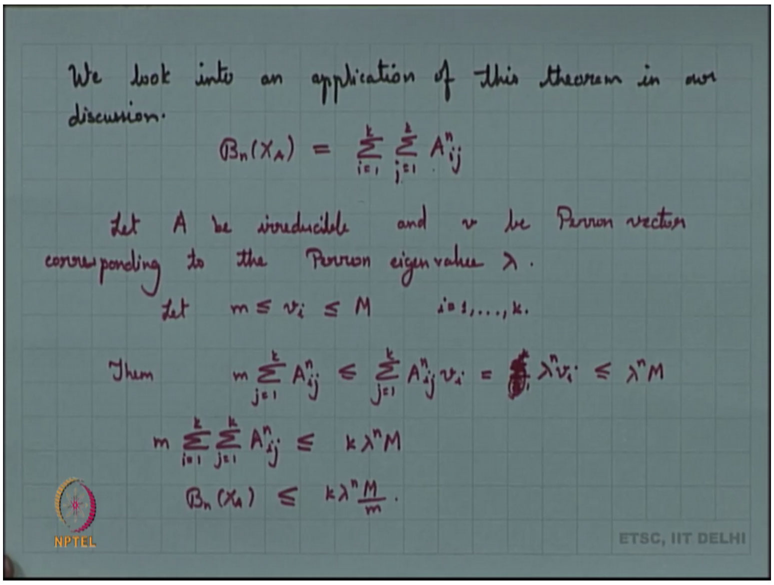

So, look into an application. So, if you again go back to our discussion we started with a sub shift of finite type and now we are looking into and we also looked into that if we look into sub shift of finite type right all the n blocks permissible n blocks right can be given in terms of summation i going from one to k summation j going from one to k right A to the power n i j. So, let a b irreducible, now, let us assume that a is irreducible and let v be its Perron vector corresponding to the Perron eigenvalue say lambda. So, v be the Perron vector corresponding to Perron eigenvalue.

Now since we know that v is a Perron vector, right, we know that each of this v i is positive. So, let us assume 2 constants m and n such that m is less than or equal to v i is less than or equal to capital M right for all I going from one to now let us again go back to this once again and try to see how it this fits in this information helps us in determining this quantity. So, first of all we see that if I take M times summation j going from 1 to k A to the power n i j this would be less than or equal to summation j going from 1 to k A to the power n i j of v i.

But if you try to look in to what is this because a has eigenvalue lambda right and b is the corresponding eigenvector then this would basically turn out to be nothing, but summation j going from one to right this turns out to be nothing, but this is I think this turns out to be my lambda to the power n right times v i by looking into this factor this turns out to be lambda to the power n times v i and this is less than or equal to lambda to the power n times capital m right because we know that v i is less than m . So, we find that this holds true and this holds true for each of the i. So, hence we can sum of for each I we have this relation and hence we can take all this I, sorry, i equal to k. So, we can take all the k equations right this k inequalities here and we can sum them all up.

So, we try to sum them all up and what is that sum turning out to be equal to; so, we say can say that m times this summation i going from 1 to k summation j going from one to k A n i j is less than or equal to now let me not write this term here again right I am simply saying that this would be let me not write this term here just look into this particular term. So, this is less than or equal to k times lambda to the power n m and hence what we can say here is now again let me look into this part what is summation I going from one to k summation j going from one to A n i j this is nothing, but my permissible blocks of length n.

So, if I look into this equation again I can say that the permissible block of length n right in X A is less than or equal to k lambda to the power n m by m. So, this gives an interesting fact here I am again trying to keep this here because we need to define this part. So, or maybe I just write it down this part again what we have derived is.

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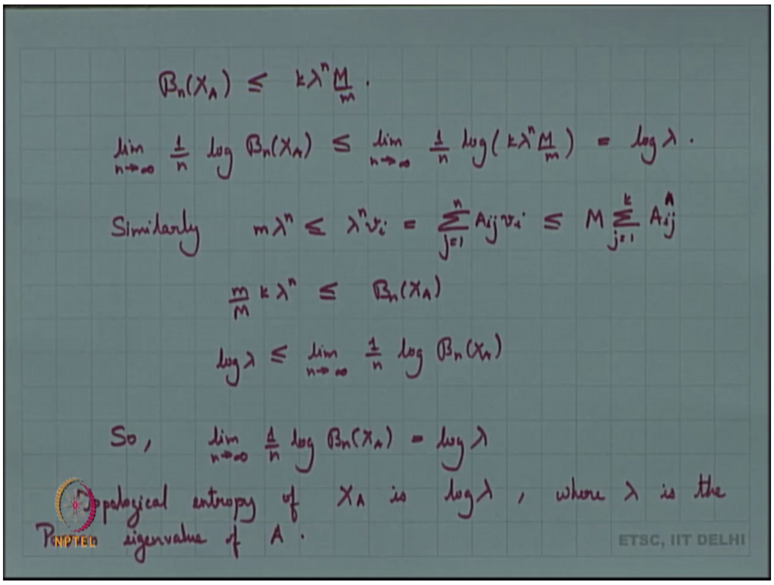

That the permissible blocks of length n in X A is less than or equal to k lambda n m by m and then we can say that limit n tends to infinity now we already know that this limit exists right. So, limit n tends to infinity 1 by n log of B n X a, this will be less than or equal to limit n tends to infinity 1 by n log of k times lambda to the power n m by n and we very well know that this would turn out to be nothing, but log of lambda.

So, our one upon n log of B n X A right if we take the limit as n tends to infinity let us some quantity which is less than or equal to log of lambda, but then we can similarly just as we saw this case we can similarly see that if I take m times lambda to the power n right that is going to be less than or equal to lambda to the power n times v i right which I can say that this would be same as summation j going from one to n A i j v i right and this is going to be less than or equal to capital n times summation j going from one to k A i j to the power n right. So, we know that this.

We can look into this in equality and from this inequality what we would get here is that m I m right k times lambda to the power n is less than or equal to B n; X A just repeating the process of the previous case. So, what we get is we get this fact here and. So, from here we can say again we can apply the same part here we can say that log of lambda right, happens to be equal to less than or equal to limit n tends to infinity 1 by n log of B n X a. Now what does that give that gives that this limit is equal to log of lambda, right. So, I can say that limit n tends to infinity 1 by n log of B n X A is same as log lambda and what was your lambda your lambda happened to be the Perron Eigenvalue of a and. So, we can say that the topological entropy of X A right the sub shift of finite type is log of lambda where lambda is the Eigen is the Perron eigenvalue of A, the topological entropy. Now for a substitute a finite type right can be thought of as log lambda; let us now look into some example here. So, we discuss some example here.

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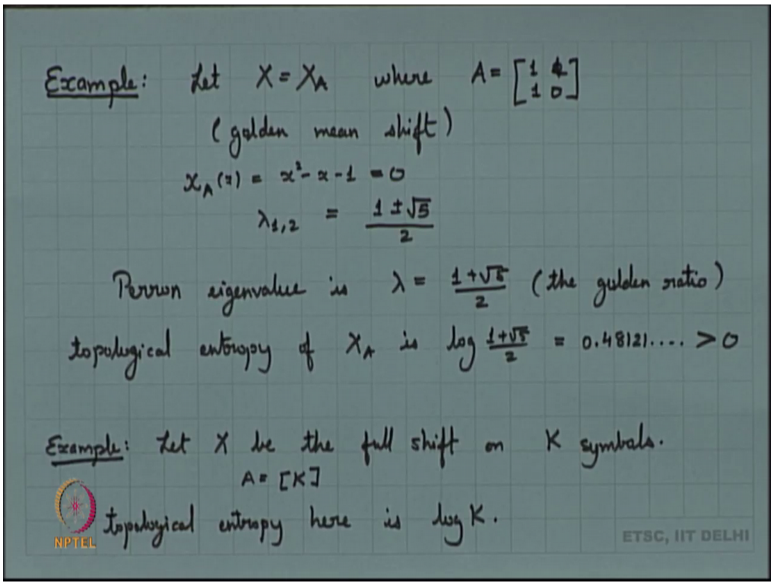

So, let me take x to be equal to X A where a is the matrix simple matrix 1 1 1 1 0, you think of this simple matrix we know that this is going to give us the golden mean shift what is the characteristic equation of A.

So, the characteristic equation of a happens to be equal to x square minus x minus one right and equating it to 0 gives me the eigenvalues here. So, the eigenvalues here I get 2 eigenvalues here right and these Eigenvalues are basically 1 plus or minus root phi by 2. So, here what we get as the Perron value the Perron eigenvalue that think of it as lambda to be equal to 1 plus root phi 2 and we all know 1 plus root phi by 2 right, it is a golden ratio it is because of this reason we call this shift to be the golden mean shift right. So, this is the golden ratio and hence we call it the golden mean shift right and now we know that what will be the entropy of the golden mean shift.

So, the topological entropy of golden mean shift log of the golden mean right and this somehow turns out to be something like 10.48121 something of that is all right and very well we know that this is positive. So, we can turn this sub shift of finite type to be chaotic right this sub shift of finite type is chaotic because of the other definition that we are taking up that anything with a positive topological entropy happens to be chaotic there can be another example that we take up. So, let us take a full shift on k symbols, let x be the full shift on k symbols.

Now, we very well know that if x is a full shift on k symbols, I can always write it as a sub shift of finite type right the underlying matrix being a one cross one matrix right with the single entity k now for this matrix right the Perron Eigenvalue will always be k and hence the topological entropy will be is log of k. So, if you have a full shift on say s symbols right your topological entropy is log s.

In the next class, we will go into the formal definition of topological entropy, we will try to look into some properties of topological entropy and we will try to see that this concept that we just studied and this was just a discussion that we did right it matches with the definition I hope there is no difficulty nothing over here.