Chaotic Dynamical Systems
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Lecture - 25
Subshifts of Finite Type

Welcome to students. So, today we will be looking into a special kind of shift space which is called sub shift of finite type. Now sub shift of finite type is something which can be seen as like many dynamical systems can be seen as sub shift of finite type. So, they can be characterized as A sub shift of finite type. These dynamical systems consist of maybe you have piecewise linear maps on a in interval, or you have some kind of maps some kind of smooth maps; that means, we are looking into differentiable maps on differentiable manifolds.

So, we recall what we did last time.

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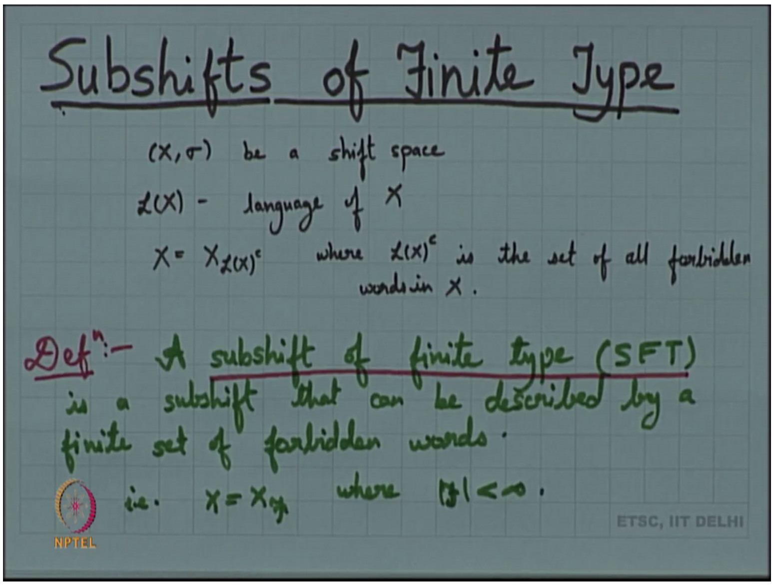

So, last time we had seen is that you have x sigma to be a shift space, and we had studied that l x is something which we denote as the language of x. This is all the permissible blocks, right. Coming up in sequences of x.

And then we had seen that x can be written as x l x complement; that means, the language is good enough to specify what shift space it is when your language x complement is basically the set of all forbidden words, forbidden words in x we can say. So, we take up this definition now. So, A sub shift of finite type which basically we can also denote it as SFT is A sub shift it is a shift space that can be described by a finite set of forbidden words; that means, my sub shift of finite type is that particular shift space, where this is where the forbidden words is finite. Now one can think of this part. So, this is basically our sub shift finite type. See, what happens is if I have forbidden a word w.

Then any word for which w happens to be A sub word is also forbidden. Because w cannot come up right. So, as such if we try to look in to in any shift space if we try to look into what are all the words that are forbidden, right other than the full shift, there can be infinitely many words, but not all words, but what we need is we need a finite description of the forbidden words, and that is what sub shift of finite type is.

\Sub shift of finite type does not mean that actually the words which are forbidden or finite. Because any word which contains that word as A sub word is also forbidden. So, as such they are infinitely many words, but our description of f happens to be finite. So, basically that gives us a very easy description of what are all the forbidden words. So, that is what is sub shift of finite type.

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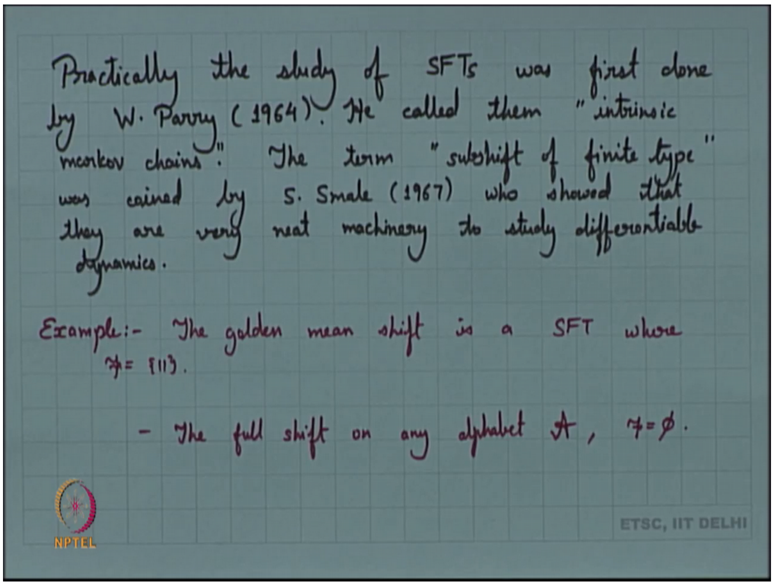

Now, I am going back little bit into history, practically the sub study of sub shift of finite type was first done by parry. And he actually had called them intrinsic markov chains. Now these sub shift of finite type are also sometimes called topological markov chains.

So, he had studied this in terms of markov chains. And he had find he had look the first systematic study was done by parry. Of course, the existence of it was known even before that, and we are not sure about since when it started. But the first systematic study was done by parry, in terms when he studied piecewise linear maps on intervals.

Where he call them intrinsic markov chains and he studied, sub shift of finite type. But the term sub shift of finite type was actually coined by smale. We have already seen the smales horseshoe. So, it was in a study of differentiable dynamics a celebrated paper there where he coined this word, and he showed that they are a very neat machinery to study differentiable dynamics.

As we have seen that differentiable dynamics also happens to be a part of considering dynamical systems. And we are not going to study differentiable dynamics in greater details in this particular course. But what he could what smale could realize is that this is something which is underlying in any nice very nice kind of smooth map that you can think of. And he coined the word sub shift of finite type.

So, let us try to look into some examples here. We look into the first example here. And I think we all know this example very well. So, the golden mean shift is A sub shift of finite type where we know that the forbidden words are just the block 1 1, right. We can describe the forbidden word by just the block 1 1. So, this is A sub shift of finite type. The other example one can think of is the full shift on any alphabet A. Now if you look into that part, right. We know that in this particular case the forbidden words is an empty set.

So, this is a these are also trivially sub shift of finite type. But there is more interesting stuff to this part, and a very nice observation is something which we can see. So, let us start with the sub shift of finite type, right.

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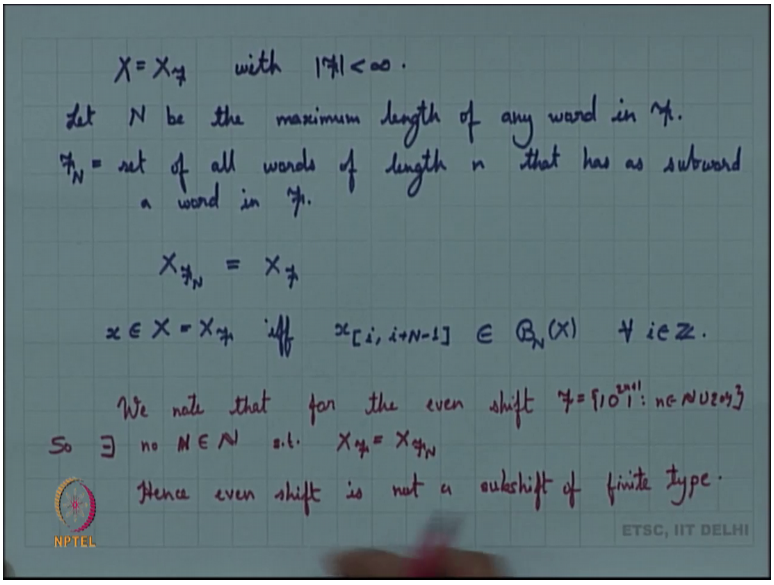

So, we know that x as x f with mod in finite. Now since we have finitely many words in f, we know that all these words will have some length right. So, let N be the maximum length of any word in f. Now think of that. We are looking into one word which has length N and f as finitely many words. So, we can always extend the words of f to a word which has length n. So, let f N that has as A sub word a word in f. So, we are extending every word in f to a word of length n and we call that.

New collection to be f N, now think of that this f N will also be finite, right. So, if I consists consider A sub shift of finite type generated by f n, right. This will be same as the sub shift of find finite type generated by f. And the simple reason for this will be the languages of both will be same. The languages are same we have the same subspace. So, these are the 2 things. So, these 2 are equal.

Now, we can say what happens now in these particular cases, that we can specify what is your x. So, your x belongs to x which is same as x f. If and only if you consider this block x i i plus n minus 1 and we see that this is basically a block of length n in x, right. And this is true for every i in z.

So, you start with that. So, what we are looking into is your perhaps made a window of length n and you are tracing this window. So, you are actually moving this window you are tracing this sequence, right with this window of length n, and you say that fine this sequence is an x if. And only if you can see that all this all elements in this window, right. They are basically blocks of length n in x.

And conversely if we can see that everything is there in the everything like whatever the window specifies, right. The block that the window specifies when you move when you trace the window across the sequence if that in they allowed blocks. Then that sequence would be definitely a point in x. So, you can specify x by looking into just one window.

Now, we are not here something else here. We note that for what happens for the even shift your f happens to be 1 0 to the power 2 n plus 1 1 such that n belongs to n union 0. Now try to look into this f, right. There is no n for which I can write my x f as x of f N. So, there exist no n, and I should say N, capital N in N such that x f is x of f N. And hence the even shift is not A sub shift of finite type. Now let us look into some definition here.

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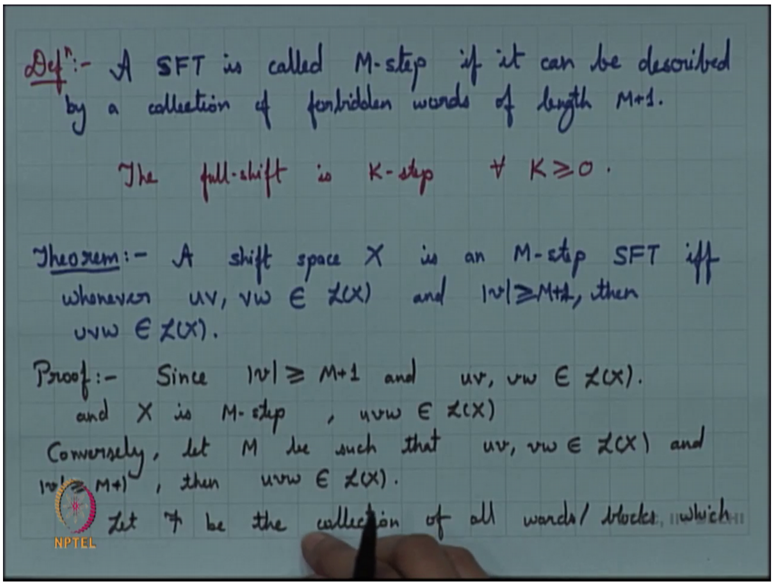

This is again an important communitorial aspect of the subj shift of finite type.

So, shift the finite type is sub shift of finite type is called M step, if it can be described by a collection of forbidden words of length M plus 1. So, supposing your forbidden words, right. Have length n plus 1 then we say that the shift is basically M step now. This n step happens to be very important aspect in trying to look into some more communitorial aspects here, but let us look into the fact that if A sub shift of finite type is M step, then it is also M plus ne step right.

It is also M plus 2 step. And so, it is k step for every k greater than equal to M. So, let us look into the example of the full shift. What can you say about the full shift? What is a full shift? Forbidden words are empty. So, I can describe the forbidden words in in terms of length 1, I can just describe the forbidden words in terms of length 2, I can describe the forbidden words in terms of length 3, right. So, the full shift is k step, right. For all k greater than equal to 0, can be describe that in any possible way. So, full shift is a very special kind of sub shift of finite time. So, let us take down this theorem now.

A shift space so now we are starting with any shift space x is an M step sub shift of finite type if and only if whenever I have this blocks u V and this block V w, whenever these 2 blocks are in lx language of x. And the length of V is greater than M, then u V w this entire block will also be in the language of x. So, we say that A sub shift space is any shift space is an M step sub shift of finite type if whenever you have 2 blocks. So, I have basically block u and block v.

So, I have words u V and w in the language and my V has length greater than M, and when I am combining those 2 words u and V and V and w these 2 are also there in the language. What happens is then we can glue all the 3 words u V w and that word will also be there in the language. So, this is one of the properties of sub shift of finite type, but it is also one of the defining characteristics of sub shift of finite type which is M step. So, let us look into the proof of this.

So, we look into the fact that this is less this is greater than we can think of this to be greater than equal to M, no problem in taking V to be equal to M. So, you can take this to be greater than equal to M, and I think it should be it should be just greater than M, because we are talking of M shift. So, it should be M plus 1. So, greater than equal to M plus 1 you can think of that part.

So, V is greater than equal to M plus 1, and what you have here is that u V and V w these are in the language of x. X is M step. Now what is the meaning of x being M step is that the forbidden words are only given in terms of words of length M plus 1, but here we know that V u V is not forbidden V w is also not forbidden. So, when I take the word V u V w, right. That word also cannot be forbidden, because the forbidden words will only have length, right. Will have lengths something less than M plus 1, right. Here you can specify that by M plus 1. So, if I take this word u V w this word will also. So, you take any M plus 1 block here.

That clock will always be permissible block, right. It will never be a forbidden block. And so, this belongs to your language of x. So, if you have a shift space which is M step M step sub shift of finite type, then this property holds. Conversely, we let M be such that is in language of x, and your mod of V is greater than or equal to M plus 1, then is in the language of x. Supposing this property holds true where I am taking M to be that particular value for which this happens. Now we take f to be the collection of all blocks of length M plus 1 which are not in b M plus 1.

So, let f be the collection of all words or I should say all blocks which are not there in b M plus 1 of x.

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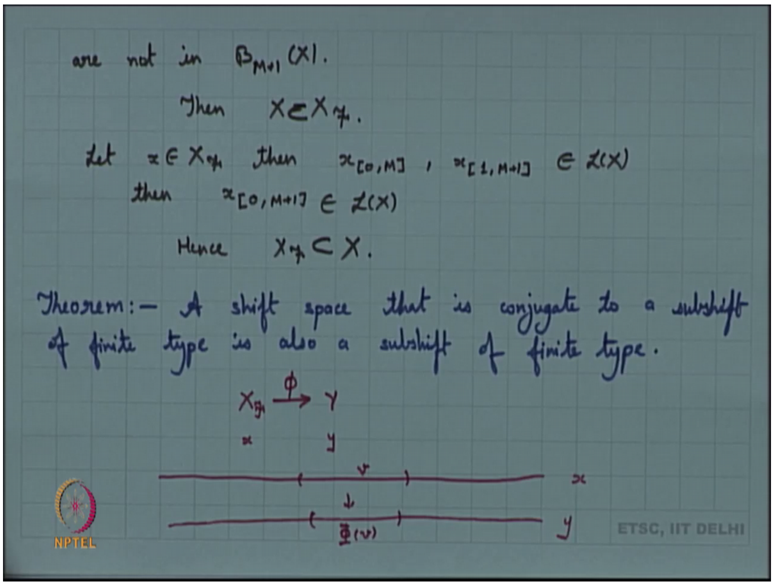

So, we are collecting all those blocks. So, we have specified that M such a number for which this property holds true. We go to all the permissible blocks of length M plus 1, and then we try to see what are all the blocks of M length n plus 1 which are not there in this particular collection. And that collection is something which we call as f. Then one thing is very clear, that x will always be a subset of the shift space given by this forbidden block f.

So, by this set f, right because all the words of length M plus 1, right, which are forbidden come up over here. So, x is definitely a subset of this part. On the other hand, let x belongs to x f, then we know that if I take x from 0 to M. Now this is a word of length n plus 1. So, this word and if I take this x from 1 to n plus 1, right, these 2 words are of words of length M plus 1.

So, these words are definitely in l x is these are permissible M plus 1 blocks there. Now by using that property which means that we can say that if you take x from 0 to M plus 1, right. We are combining these 2 words, we take this word from 0 to M plus 1, then this also belongs to l x. So, that means, we are starting with permissible M plus 1 blocks, and using this permissible M plus blocks we can generate the whole x.

So, this belongs to the language of x. So, you start with any point of x f, you find that the block of length M plus 1 is in the language of x; that means, the language of x f is same as the language of x, or it is contained in the language of x in that sense right. So, we have x f is sorry this is a subset here. So, we have x f is a subset of x. And combining these 2 we can say that x is same as x f. And we note a simple result here, a shift space that is conjugate to A sub shift of finite type is also A sub shift of finite type. Why do we have this part. So, let us look into the fact that; your x and y are 2 shift spaces, and there is a conjugacy here.

So, I have a conjugacy phi here and my x is A sub shift of finite type. We can say that x is A sub shift of phi is an M stuff sub shift of finite type. So, I can say that this is my M stuff sub shift of finite type. Now since we have a conjugacy here we can think of for every x here there is a unique y here, and then when I write this sequence x here, and write this sequence in terms of words here. So, it is supposing this is my sequence x here, and supposing this is a sequence y here. Then we know that this particular phi because this is a conjugacy right. So, phi happens to be mapping which is induced by a block map. So, there will be a block map here right.

Corresponding to phi there will be a block map, such that you start with any word we here. What you get is that this is basically mapped to the word, right, which I can say yes phi of V here. So, this block is always mapped into this block, right. And correspondingly since this is a synchronous action the action of a block map is synchronous, you can find that every word in that can be gives a word here, right of the same length of course defined by the block map f. So, depending on what is a memory and what is the anticipation, if this is M step; that means, I just have to trace this by a window off length M, right to know whether this is a sequence there.

Similarly, because length M would give me some other length n here, all we need to do is just trace this by a window and this is a homeomorphism. So, all I need to do here is just trace this by a window of length n to see that this particular word exists. So, if my x is M step, we can say that there is n such that wise n step. And so, any shift which is conjugate to A sub shift of finite type is itself A sub shift of finite type. We will later look into the fact that this is in general not true for factors, but maybe that would be later, right.

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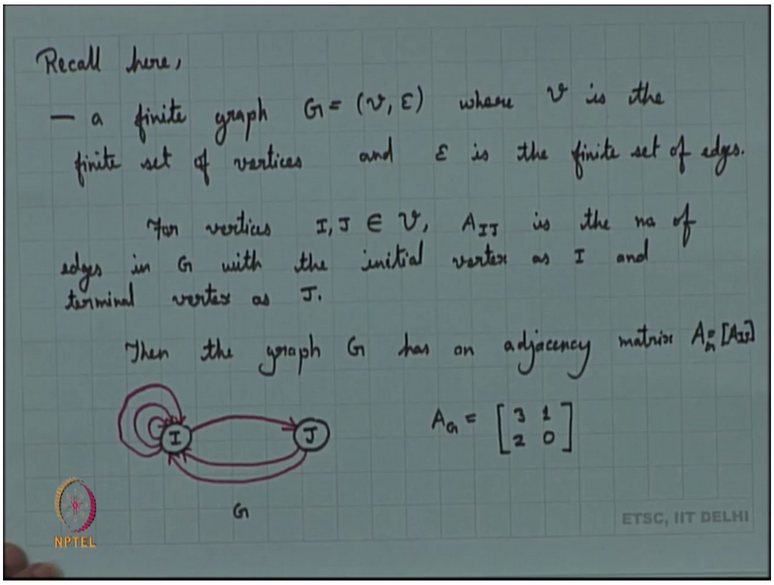

Now, let us recall some other results, which perhaps you would have studied in some different context.

So, we recall here a finite graph G which is basically written as a pair V and E a finite graph G is basically a pair V and E, where V is the finite set of vertices, and E is the finite set of edges. So, we are talking of a graph G. And we will be more interested in a directed graph, because we would like to see what happens to whenever we talk of an edge. The edge should have some kind of an initial vertex, and some kind of a terminal vertex. So, we are always interested in this directed graph G. So, what G here is a directed graph. And what we find here is that for vertices supposing I have this vertices a I and J, right in V. Then we look into this number A I J; this is the number of edges, with the initial vertex terminal vertex as J. Then we say that this graph G has an then the graph G adjacency matrix.

You can call it which is given by A I J, right for all I J. So, let us look into this simple graph here. So, I have this vertex I and I have this vertex J. And for this vertex I and vertex J, I have these edges. So, I have one edge here, I have 2 edges in this manner. And I have 3 edges here. Then corresponding edges adjacency matrix can be given in terms of I write it as A G, right. I can write this as A G, right. Which is with respect to this is my graph G basically. So, this matrix becomes A 2 by 2 matrixes. And I know that from I going to itself how many vertice, how many edges are there?

So, there are 3 edges here from I going to J there is 1. From J going to I there are 2. And from J going to itself there are 0.

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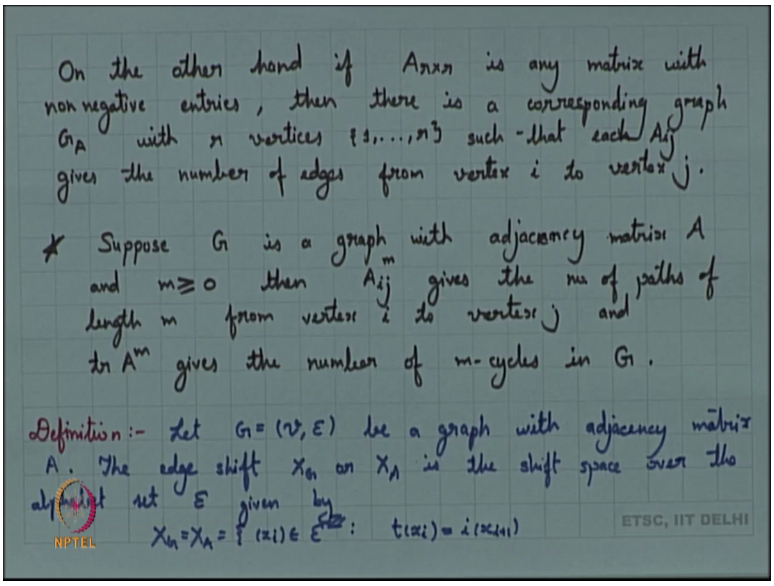

So, this gives my adjacency matrix. On the other hand, so, I am come back to this figure again. If a r cross r is any matrix with non-negative entries, then there is a corresponding graph G which I call as G a with r vertices. So, let me call those r vertices to be say one up to r and such that. So, this vertices are such that each A I J gives the number of edges from the vertex i to the vertex j. So, from a graph we can generate a matrix and from a matrix.

We can generate a graph now another aspect which you can see in terms of this graph; for example, if I have this to be my matrix, then this will definitely generate this graph. So, a graph in matrix and that terms are identical, because one can always be used to define the other right. So, let us observe 2 things here is; what happens in a graph G. So, suppose G is a graph with adjacency matrix A and say M is greater than or equal to 0. Then if I look into A i j to the power M. So, that gives me the number of paths of length M from vertex i to vertex j.

And the trace of a to the power m gives the number of cycles M cycles in G. What do you want to say here is that supposing this is a graph this is the adjacency matrix. Now we pick up 2 vertices. So, here we anyway have 2 let us pick up I equal to I and J equal to I also. Then A i j to the power m gives me how many edges will be there, right. Starting from I of length M coming up to J. So, a I J to the power M. So now, we are looking into supposing I am looking into this square, right. So, this square will be equal to yes what is the first entry here A 1 1 square. What is A 1 1 square here? 11, right. So, that means, if I am looking into how many edges will of length 2, or how many what to say what is a path of length 2, right. Starting from this vertex I and coming back to this vertex I, it should be 11, how?

It can take this path, then it can take this path; that is one option. It can take this path and it can take this option, second it can travels around itself third, right. Again, you can think of that it can take the second path. And it can then take the first path right. So, if you count that that total happens to be 11 which actually can be given in terms of just square the matrix, right. And you get what is the entry there. On the other hand, I want to know how many cycles are there.

So, cycles by cycle I mean I am starting with the same vertex and coming back to the same vertex in M step. So, supposing we want to look into how many cycles are there. All you need to do is look into the trace of a to the power m. Because if you look into the trace of a to the power m, you know what happens here is, this gives me the number of cycles with start at one vertex and come back to itself this gives me the number of M cycles which started vertex 2 and come back to itself. So, if I look into the trace we get the M cycles here. Now let us try to exploit this particular property of the graph here to make another definition.

So, we start with this definition. Let G be a graph with adjacency matrix A. Then the edge shift which we define it as X G or we can say that it is X A, right is the shift space over the alphabet set E given by the terminal vertex of x i should be the initial vertex of x i plus 1, because every edge will have a terminal vertex and an initial vertex. So, we start with the terminal vertex x i. So, wherever x i ends, right, wherever the edge x i ends.

The edge x i plus 1 should start from there. So, that gives us a shift space. So, the graph generates our shift space, right. And this shift space is something which we call as the edge shift. Now note what is the meaning of that shift is; we are looking into all the sequences are basically we are looking into all, by infinite paths or by infinite walk on the graph G. So, just walking on the graph G, and any by infinite walk on the graph G, generates this kind of edge shift.

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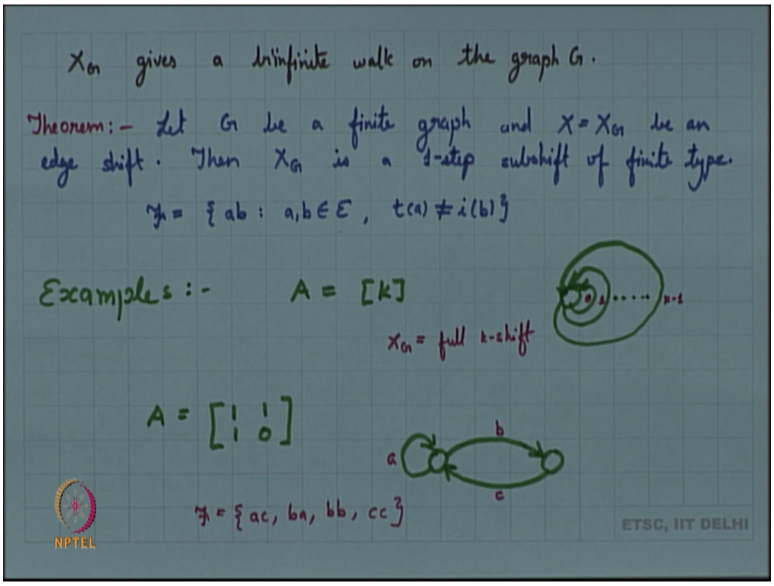

So, what we have here is that x G gives A by infinite walk. Let us start with the theorem here. Let G be a finite graph and this x is a shift space generated by this graph G. So, this is an edge shift, then your x G is a one-step sub shift of finite type.

So, you have an edge shift and the edge shift is basically a one-step sub shift of finite type. Now here we can simply note that if you start with this collection f, right. Which is the set of all words of the form a b, where your a and b are basically edges, right. Such that the terminal vertex of a is not equal to the initial vertex of b.

So, we are collecting all such words a b, right. Such that the terminal vertex of a is not their initial vertex of b. So, all these a b will be forbidden. You cannot have any work, right. Basically, first a and then b it is not possible. So, this will be forbidden, and this forbidden block will be finite right. So, the edge shift that you generate will be a one-step sub shift of finite type. So, any edge shift is a one-step sub shift of finite type.

Let us try to look into some examples here. So, we look into examples here. So, let me take a to be a very trivial kind of matrix, this is the matrix k 1 cross 1 matrix k. What is that mean? How many vortex will it have. If I look into a graph generated by A how many vertex will it have? It will have A 1 vertex. So, I can say that this is one vertex here. And how many edges will be there here k edges, right from the same vertex to itself. I have k edges here and now if I looking to naming each vertex, right because this each of this edges will be different right. So, I should name each, right. There will be finitely many of them. So, I can name this 1, this as 0, this as 1, and this as k minus 1.

So, what is the edge shift that it is going to generate? What is your x G going to be equal to? What will be your x G here? It will be the full k shift. All permissible words are possible, right. Everything is possible there is nothing the basically your forbidden block is empty set, right. Nothing is forbidden here.

So, this is the full k shift. Shift on k letters, because I am taking I could take these names to be anything right. So, this is the full k shift. The other example we can think of is; let a b equal to I am taking this matrix 1 1 1 0. Now I know not here we will have 2 vertices. So, we have 2 vertices here, I have one edge going from the first vertex to itself. I have one edge going from the first vertex to the second vertex. And I have one edge going from the second vertex to the first vertex.

And I do not have any vertex going from sorry, no edge going from the second vertex to itself. So, that means, I can name this right. So, maybe I call it a, I call it b, I call it c right. So, there are 3 edges here. And now I can say that this specifies A sub shift of finite type this specifies an edge shift, right for which, what are your forbidden blocks? So, think of that. After a b comes up right, a b is permissible right, but after a c cannot come up right. So, a c is forbidden here. What next is forbidden? After a a always comes. So, a is not forbidden here, but look into this b part, after b c comes right. So, b a is forbidden here. So, this is forbidden after b c comes, but b does not come.

So, again b b is forbidden, and again look into c after c both a can come and b can come right, but after c c cannot come up. So, c c is forbidden. So, this matrix gives us an edge shift, right. For which this is a forbidden. So, I get a one-step shift here, right. The edge shift such that these are the forbidden blocks. We note here that every edge shift is a subject of finite type. Can we say about, what can we say about the converse part? Then we say that every sub shift of finite type is an edge shift.

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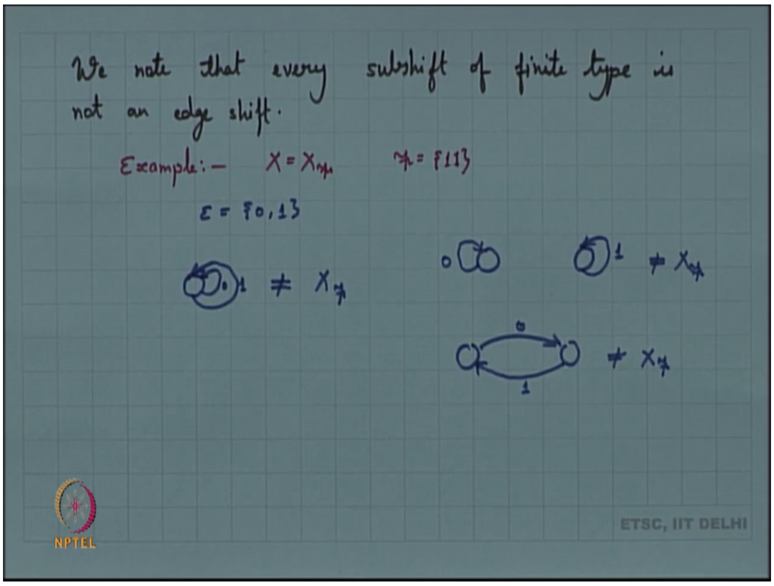

So, we note that every sub shift of finite type is not an edge shift. And the trivial example that we can think of here is; let us look into this example of the golden mean shift. So, I am looking into this golden mean shift, where my forbidden block is 1 1. Now this is the one step shift. So, one can think of fine this could be an edge shift, but think of this factor. Supposing I say that this is an edge shift, then it should there should it should be having some graphical representation; that means, that I should be having some number of vertices and some number of edges.

Now number of edges here are only 2, because my alphabet set is just 0 and 1 right. So, this is the only alphabet set here. So, number of edges we know that only 2, how many vertices are there that is a question here.

So, if you try to look into the number of vertices here. We find that here since my edges set is just 0 and 1, right. We try to look into what are the numbers of vertices here. Supposing the number of vertices is just 1. Then I have in one vertex, right. And on that vertex, I have to shift edges 0 and 1 coming here, but that gives me the full 2 shift. So, this is not equal to the this cannot generate my even golden mean shift here. So, this is not a so, this possibility is any way not possible then we try to look into what happens here is supposing we have 2 vertices.

So, if we have 2 vertices here, right there are 2 possibilities. One possibility is because I have 2 edges right. So, one possibility is 0 goes into takes one vertex to 1 and 1 takes one vertex to the other. But this is a kind of a very shift nothing known there is no interaction between each other. The only points that can be generated will be the fixed point 0. The fix sequence of 0s and sequence of ones right. So, this anyway not my golden mean shift at all right. So, this also is does not give me my golden mean shift here. The other possibility that remains is I have 2 vertices and I have one edge.

So, 0 going here, and I have the second edge say one going here. Now if I try to look in to this aspect also, right. The only possible sequences we get here is 1 0 to the power infinity, right. The block 1 0 appearing itself and we get the block 0 one appearing infinitely often, these are the only 2 possibilities.

So, this also does not give me the golden mean shift. So, basically when we talk of edge shift, right. Every sub shift of finite type is not an edge shift, but there is a way how to do away with this particular problem. And we shall discuss that in the next class.