Chaotic Dynamical Systems
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Lecture - 24

Shift Spaces

Welcome to students. So, in the previous lecture, we had looked into introduction to symbolic dynamics, and the main reason was getting into it was that we now can use some kind of communitarian tools to understand chaos. So, to understand the underlying dynamics right we need some we need whatever tools are possible, we try to study some tools also and that gives us a general idea of looking into dynamical systems in a better way.

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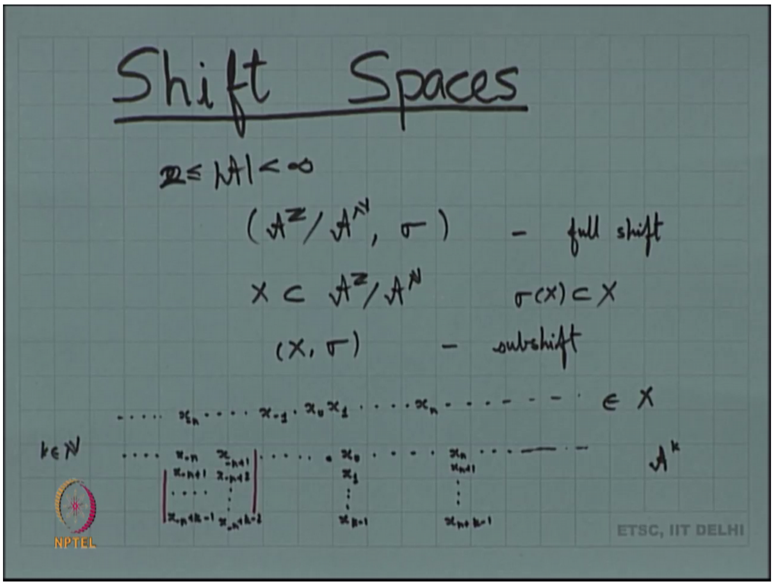

So, what we had looked into was that if you start with say an alphabet set sorry 2 less than or equal to mod A less than infinity. So, you look into a finite alphabet set A such that it has at least 2 elements, then A to the power Z or A to the power N, this set of all by sequences with elements from A, letters from A and the set of infinite sequences.

So, this gives a compact metric space and on this particular compact metric space, you can apply the shift map and can study the dynamical system. Now this is a shift space and this is basically called a full shift. The other idea is to use x as a subset of again it could be A to the power Z or A to the power N 1 sided or 2 sided, and then you look into those x for which are invariant under x. So, you try to look into shift invariant, subspaces of the full shift and that gives rise to another dynamical system, which we call it as a sub shift. Now a typical element of the sub shift right you could thinking terms of say let us take the full shift 2 sided full shift. So, we can think of the typical element to be something of the form say x n, x minus 1, x 0, x 1, x minus x n and so on.

Now, this is a typical element of x one can think of that, but many times we are not interested in looking into this. So, this is basically some kind of a one dimensional shifts, we basically we are interested in some kind of a hire block version of the same shift. So, the same sequence can now be written as.

So, maybe I take fix some k in N, and for this particular k we can think of the sequence we can write the sequences x minus n, x minus n plus 1 and x minus n plus k minus 1. Then again I start here because here my sequence here the element here would have been x minus n plus 1, then we start with this factor x minus n plus 2, then we go up to x minus n plus k minus 2, and we continue with that at the zeroth place we have something called x 0, x 1 right and this goes to x k minus 1 and again we go back to x n x n plus 1, and x n plus k minus 1.

So, basically we have fix the k in n and we are now thinking of this particular. So, we representing the same sub shift in terms of a higher block shift, where we say that this higher block is admissible. So, basically this is an infinite sequence we say that this is admissible if we have this factor. So, you look into this factor, this part right of the block agrees that this part of the block. So, if these 2 blocks agree then we say that fine this is like I am writing the same shift in terms of a higher block, and this is called a higher block version of that, and in this you can think of all of these to be alphabets right and.

So, your alphabets now are in A to the power k. So, these are your alphabets and now you can think of the same thing what is your shift doing is. Your shift is doing thing, but it is shifting 1 side basically it is shifting this point 1 side to the right. So, this is basically a higher block version of the shift and maybe not today, but in future you may use it this is a very nice expression now, the same shifts sub shift can be expressed in this particular form, and this is helpful in many ways.

So, this is how you can describe a shift space, but then the question here arises that we have started with a finite alphabet set. Do we always need to describe a sub shift with a finite alphabet set? So, we see that that is not always the case, although the finite case gives us lot of more in features or it gives us very nice characteristic, which you cannot find in an infinite alphabet set.

But let us try to take an example here. So, we start with an example, now think of this example we first to recall right. So, this is something from number theory, which we all know that very well.

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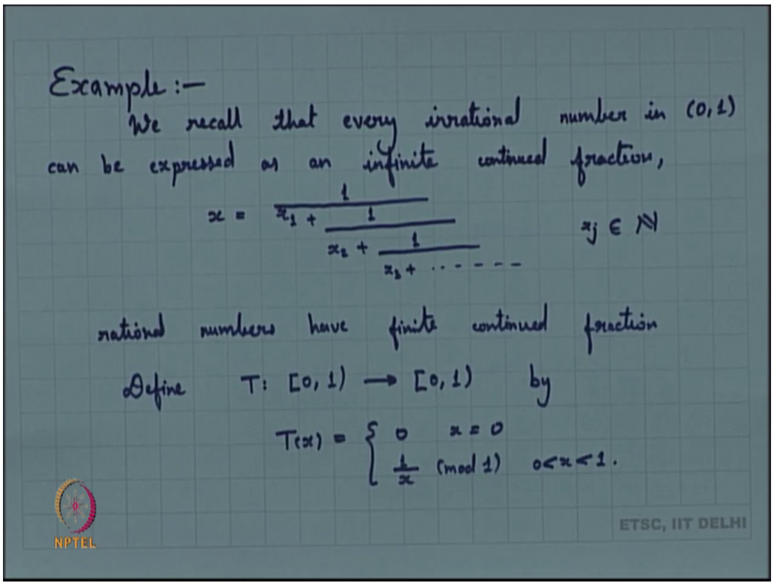

So, we recall that every irrational number in open interval 0 1, can be expressed as an infinite continued fraction. So, I think you will know; what is the continued fraction. So, what we do is we start with an irrational number in 0 1, then we can express x as say this is e 0 1. So, this is 1 by I have x 1 plus then again I have 1 by x 2 plus, then again I have 1 by x 3 plus and so on,

So, we know that every irrational number in 0 1, can be expressed as an infinite continued fraction. So, this is infinite and all my x js right these are elements of the rational number and what happens for a rational number. So, this rational numbers have finite continued fraction. So, their continued fraction is not infinite, it does not go on and on, but they have a finite continued fraction, you can express them in terms of a finite sequence here.

Now let us look into a map here. So, we define T from and I am starting from closed to this closed 0 open 1, and we define T by T of x is 0 if x equal to 0; that means, 0 is a fixed point here and if it is not equal to 0 we define it as 1 by x mod 1. So, this is a mod 1 function here. So, this is holds whenever 0 is less than x is less than 1. So, T of x is defined as 1 upon x and if we want to see what is the action of T, then the action of T can be seen as say you start with this continued fraction.

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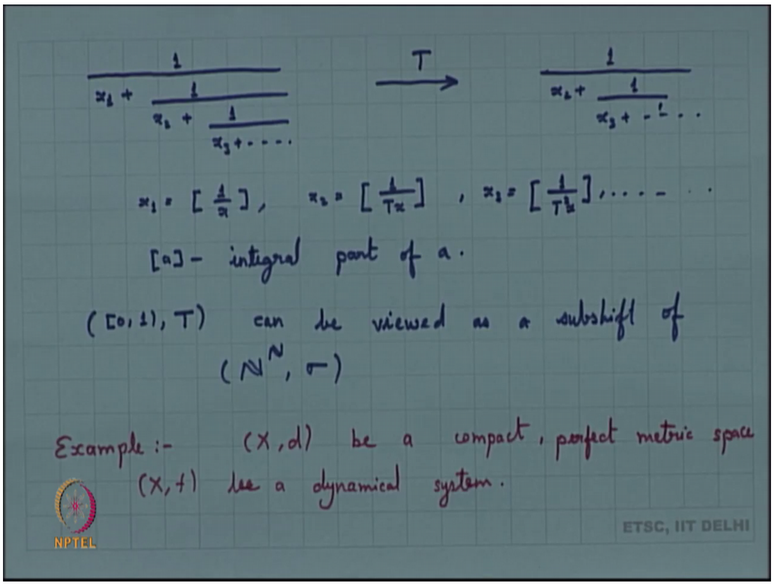

Now, I am writing x in terms of a continued fraction. So, my continued fraction is 1 upon x 1, plus 1 upon x 2 plus 1 upon x 3 plus and so on and what happens under the action of T now this is non zero. So, I am just putting it up as 1 upon x right. So, this numerator becomes this denominator becomes numerator here, and then when you take mod 1 right.

Basically x 1 is a rational number. So, when you take mod 1 this x 1 gets cancelled out. So, what do you get over here is another continued fraction, which you can write it as 1 upon x 2 plus 1 upon x 3 plus and so on.

So, under the action of T this continued fraction is being mapped to another continued fraction and we are representing every x which is nonzero right in terms of continued fractions. So, what we see here is that x 1 happens to be the integral part of 1 by x, your x 2 happens to be the integral part of 1 by T x, your x 3 happens to be the integral part of 1 by T square x and so on So, basically this represents right the integral part .

So, find that this is an integral part and now what we can realize here is that, if I look into this dynamical system open from 0 close from 1 T. If you look into this dynamical system then this can be viewed as a sub shift of.

So, I am taking the system I am looking into all N. So, my alphabet set is N here. So, n to the power N and sigma. So, we can think of writing this entire continued fraction in terms of n infinite sequence x 1, x 2, x 3 and so on and then we can view this dynamical system in terms of a sub shift of this part. So, you can express these dynamical system also in terms of a sub shift, but we note here is that neither of the systems are compact right our alphabet set is infinite n to the n it is non compact. So, this is a non compact. So, both the system are non compact

So, it is very natural to ask whether we can have any compact version of this part. So, then look into that in terms of directly our dynamical system here. So, let us take xd. So, I am looking into another example here now. So, let x d be a compact I want to take it a perfect metric space I do not want isolated points. So, compact perfect metric space and what we are interested here is that, we are interested in a dynamical system on this particular x. So, let x f be a dynamical system now we are interested in this particular space x. So, we think of the space we consider the space X to the power N.

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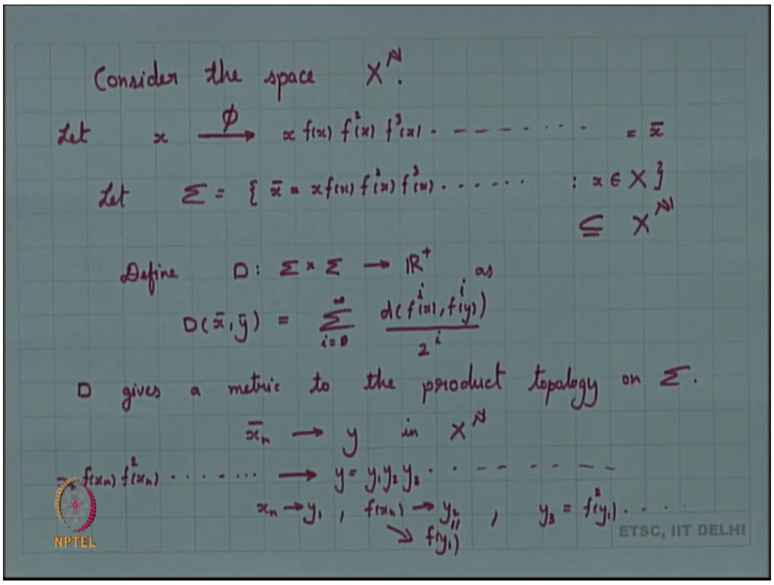

Now, think of that, our x is a compact metric space. So, X to the power N itself is a compact metric space right and its a compact metric space its a perfect space.

So, this is a nice perfect space where we end out this with the product topology. Now we are looking into some correspondence here, some looking onto x corresponding to this particular point. So, my point is x f of x f square x right f cube x and so on. So, let me call this particular point as x bar, and let me call this correspondence as phi. So, we are interested in this correspondence all points of x are corresponding to this particular point. Now if you look into this correspondence this particular point is an element of x to the power n. So, we basically a looking into x to the power n, let sigma I am looking in to sigma to be the set of all such x bar right equal to x, f x, f square x, f cube x and so on such that x belongs to x. Now if I look into the sigma this sigma is basically a subset of x to the power n.

Now, let us define a metric d all sigma. So, we define D from sigma cross sigma to R plus and we can define this as if I am taking any x bar and y bar then d x bar y bar happens to be equal to summation I going from 0 to infinity, I have D of f to the power i x f to the power i y right divided by 2 to the power i. Now this d is definitely gives me a metric because it satisfies all the properties of metric and it will be equal to 0 only n with when x is equal to y, because once x is equal to y all f I x will be same as f i y right f is a continuous function here. So, using this property you say that this happens to be a metric. So, this D, D gives a metric to the product topology on sigma.

Now, there is another fact of sigma that we can think of. What happens if I take the closure of sigma, what is a limit point of sigma in x to the power n? So, supposing I have x n bar converging to some point y right in x to the power n what happens here. Now I can write my y to be equal to say y 1, y 2 right y 3 and so on. Now if I think of that aspect what happens here is my x n bar is converging to y; that means, now I can write my x n bar I can write it as x n. So, this is like x n, f of x n right f square x n and so on. Now this is converging to this particular point, we have this metric here what can you say about the relation between x n and y 1, what is the relation between x n and y 1 here? X n is converging to y 1 right. So, what we have here is that x n is converging to y 1.

Now, since x n is converging to y 1, we can also say that f of x n will be converging to y 2 right. So, f of x n is converging to y 2, but what we know very well is that f is a continuous mapping and since x 1 converging to y 1, f of x n will be converging to f of y 1. So, this is basically converging to f of y 1 and hence my 2 should be same as f of y 1 and in the same way we can say that, your y 3 will be same as f square y 1. So, if we are looking into limit points right we are looking into the limit points of any sequence in sigma, we find that that limit point is also an element of sigma. So, the basic deduction here is that sigma is closed right.

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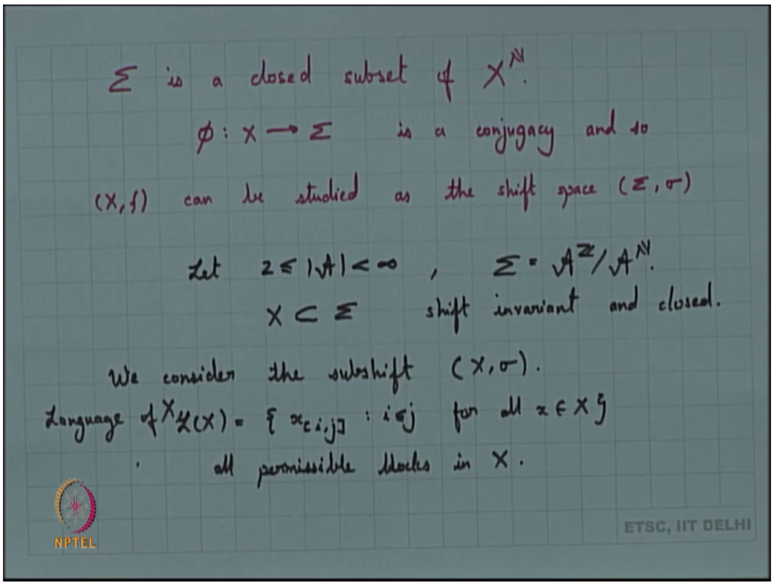

So, sigma is a closed subset.

Now, it is very interesting here, sigma cross subset of X to the power N and we already know that there exist a correspondence between x and sigma right. So, we have our phi right a correspondence between x and sigma, there is there is give a conjugacy right this is 1 right. This is basically a surjection; this is a surjection plus continuity right because our metric is dependent on the metric of x right see that continuity here. So, basically this is a conjugacy and so, what we did use here is that my system x, f now my x, f was any dynamical system right it can be considered as can be studied as the shifts space sigma sigma, it is very clear here right that if I apply sigma on x bar right.

What I get here is f x f x square x f x cube x etcetera which happens to be equal to f x bar right. So, f x corresponds to f x bar right and hence we can find this correspondence here and so, we say that this can be studied as a shift space. So, here we have a compact shift space because this is a close. So, this is also compact. So, now, we know that this 2 compact spaces are equivalent and we can study any dynamical system in terms of shift space, but what happens here is that when you study a dynamical system in terms of shift space in this manner, we are losing lot of the combinatorial properties which we are interested in. And most of this combinatorial properties we give because a happens to be finite.

So, this is like representing a right a finite set because a finite set can be part of a finite subset of the rational numbers, and that particularly gives us lot of commentarial properties. So, still we do not know exactly I think all commentarial properties come pertaining to this particular shift space right is not yet investigated. So, still one can think more on this and try to ascertain certain properties about this, but I am not sure how much 1 can succeed over here, but let us now look into those aspects where there is already been lot of progress and that is what is the main concern for us here. So, we are looking at dating back to getting back to our alphabet set, such that it contains at least 2 elements.

So, this is our alphabet set and now, for us our space sigma happens to be A to the power Z or e to the power n, and we are taking x to be subset of sigma right which as shift in variant and closed. So, closed shifting variant subset and we are interested in the sub shift. So, we consider the sub shift, the sub shift is x sigma. And we recall we have already studied that any sub shift will have a language right. So, the language of x say the language of x right which I can write it as L x happens to be basically the collection of all distinct blocks. So, I am looking into all blocks such that I is less than or equal to j right for all x and x. So, we are looking into this distinct blocks here. So, this is our L x and this is basically all permissible blocks and there is something that we had already noted about our language which we shall again see here and that is.

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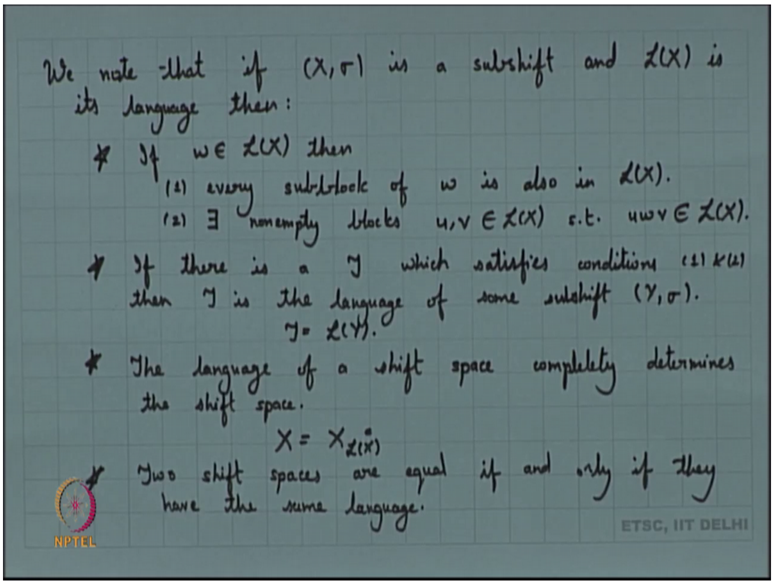

So, we note that if this is a sub shift and L x is its language, then the first thing we can note here is that if w is in L x think of that; w is an element of L x and we have already seen that L x is a set of all permissible blocks right. So, L x set of all permissible blocks. So, this like if w is an Lx; that means, w appears in some x somewhere at some point of time. So, now, what happens of w is in an L x, then there are 2 things that happen here the first thing is that every sub block. If a block itself is appearing in x right its sub block is also going to appear in x. So, every sub block if w is in L x then every sub block was w is also in the language and second thing is that they are exist not empty.

So, they are exists non empty blocks I can say non empty blocks u and v in L x, such that u w v is also an element of L x basically what does that mean is that you take any block in the language you can extend it right to the left as well as to the right in such a manner in that the extended word is also there in the language. So, its always possible to extend it and this is true since my w happens to be say some element its some block in x now x is an infinite sequence or by infinite sequence. So, you can always find words right to the right of w and to the left of w right appearing in x and so, this will be there in the language.

Now, since these are appearing in x these are also in words of L x right. So, for any w in L x we note that these 2 conditions definitely hold, but that gives us an interesting fact here and the interesting facts can be seen in terms of if there exists some T. So, this is a set T right which satisfies conditions 1 and 2 then T is the language of some sub shift I can call it y sigma. So, what I want to say is that T happens to be the language of y. Now why is it true you think of that you take any element of T, and for that element every sub word belongs to T and you can always extend it on 2 sides? So, what you can do is you can always extend that on 2 sides.

So, you basically you are taking 2 empty words you can always extend it on 2 sides, what you get is some kind of an infinite sequence here right in finite or by in finite sequence here and there can be extended we can call the set of all such by infinite sequences to be in the space y right and so, you can say that if you have a property. So, this is basically a property see typical property which tells you the characteristic of any language of a shift space. So, this is basically your language and so, we can say that this to a large extend the language of a shift space right it determines the shift space. So, the observation here is that the language of a shift space completely determines the shift space, and why can we say Ss, because consider because we have all these alphabets.

So, we look into all those words which are not there in the language then these from the forbidden blocks in x and so, your X can always be written as X language of X compliment. So, it can be this forbidden blocks right are specified by those words which are not there in the language of x and so, your language completely it determines your shift space and of course, 1 can deduce from here is that 2 shift spaces is are equal if and only if they have the same language. So, this 2 shift spaces are equal right if the language is same; that means, the shift space is same and now this gives us some kind of a definition which we can give in terms of a language. So, we give some kind of a definition to the language.

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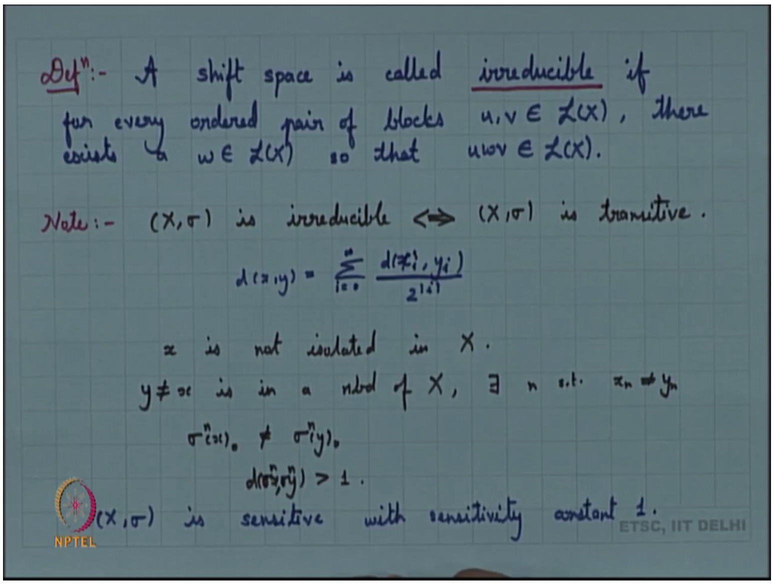

So, let us try to look into this definition now what is this definition. So, we say that a shift space is called irreducible, if for every ordered pair of blocks u, v the language of x right there exists w in the language of x. So, we have another block w in the language of x so that u w v this block is also there in the language of x. So, if we have such a property for the shift space, then we say that this shift space is irreducible. Now think of that what do we really mean by irreducible.

So, we take a small note here and this note says that, if my shift is shift space is irreducible if and only if this is transitive. So, same that a shift space is transitive is equivalent to saying that a shift spaces is irreducible, and we can try to look into the proof of this it is very simple here. Supposing we assume that is irreducible, we want to show that this is transitive.

So, we know that the basic open sets will be the cylinder sets. So, you can start with a word right which gives you the cylinder set, now we have these 2 words. So, there is a lot about coming in between. So, we know that there will be some sequence there will be some n such that now we look into the length of this word. So, that gives us an n such that it takes 1 cylinder set, after that many e traits to the other cylinder set right. So, what you get is irreducibility is, gives you transitivity, on the other hand if you have the system to be transitive right; that means, that there is say 2 cylinder sets right and from the 2 cylinder sets after some iterates go there. So, you will have some kind of a word with that particular lens such that given u, v you have a w such that, u w v is in the language right.

So, these 2 concepts are equivalent concepts, we said under shift spaces irreducible same as saying that the shift spaces transitive. There is an aspect to this, now think of that. We have seen that the metric on x right it can be given in terms of d x y happens to be, something like summation I going from 0 to infinity d of I have x i y i right upon 2 to the power mod i. Now think of this factor here right what happens if my x is not isolated. So, my because I we always with non isolated stuff, but now if x is not isolated what happens in that case. So, I will always find a y in the neighbourhood of x, because x and y are not equal right so; that means, at some point x i and y i are not equal.

It is not possible that x n x i and y i will be equal everywhere, because otherwise x would be same as y. So, if y not equal to x right using a neighbourhood of x, what happens in that case? You get an n such that x n is not equal to y n. So, there exist an n right such that x n is not equal to y n and what happens in that particular case? We know that look into this fact we know that if I am looking into say sigma n x right in the zeroth coordinate, and sigma n y in the zeroth coordinate these are not equal because x n is not equal to y n. Looking into the metric of that we can say that the distance between x and y will be greater than 1.

So, this distance is greater than 1 and what does this tell us. Sorry distance between sigma n x and sigma n y right this is greater than one, what does this tell us? Given any x non isolated, there exists a y in the neighbourhood such that for some iterate n the distance between them their orbits gets greater than 1 and so on this is sensitive. So, any sub shift which is perfect right which does not contain any isolated points will always be sensitive. So, what we deduce from here is that x sigma is sensitive and not only sensitive its sensitivity constant is 1 right. Now we are more interested in looking into when can we say that 2 shift spaces are conjugate.

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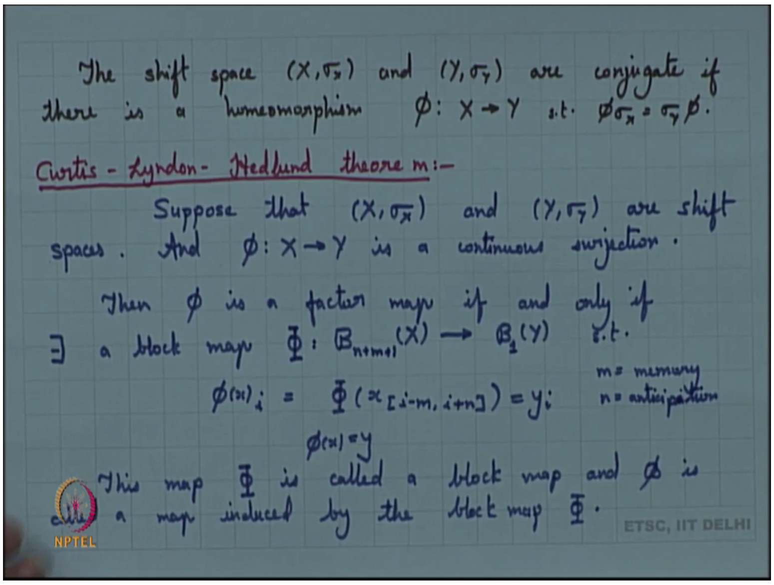

So, now looking into that aspect when our 2 shift space is conjugate; the shift space are conjugate, if there is a homeomorphism say phi from x to y such that phi of sigma x is same as sigma y phi right it the action of sigma x and sigma y.

Then we say that this happens to be topologically conjugate. And we know that under conjugacy the dynamical properties are preserved and if now phi instead of being a homeomorphism if it is just a continuous surjection, and we say that phi happens to be a factor. Now in that case we say that y happens to be a factor of x, but very interesting is what is this factor map? Now we have 2 shift spaces x and y. So, what kind of map is this factor map? So, for this we have a very nice theorem. So, this is basically Curtis Lyndon and Hedland theorem. So, what is this theorem all about, suppose that x sigma x and y sigma y are shift spaces and we have a map phi from x to y this is a continuous surjection, at least the continuous surjection it would be more.

Then this phi is a factor map; that means, it is intertwines the action of sigma x and sigma y. So, this is a factor map if and only if there exists a block map let me call it capital phi. So, this is a block map, and this block map is on the block of all n plus m plus 1 blocks in x right this is a map taking this as your arguments to the 1 block in y such that your phi x at the value i is same as this block map phi right acting on this block x going from i minus m to i plus n look into this block, and this is what is going to give me y i. So, my y i happens to be element of b.

So, phi x equal to y right what we have here is phi x equal to y, and individually coordinates we can think of that that phi x at i is basically determined by this particular block map. So, this map phi I am calling it capital phi here, phi is called a block map and our map that is our factor map right is called a map induced by the block map phi. Now we have this particular term n and m. So, what do these mean. So, basically here my n m stands for memory and my n stands for anticipation. So, how does this map work out let us try to see that aspect, and then we will try to prove this theorem.

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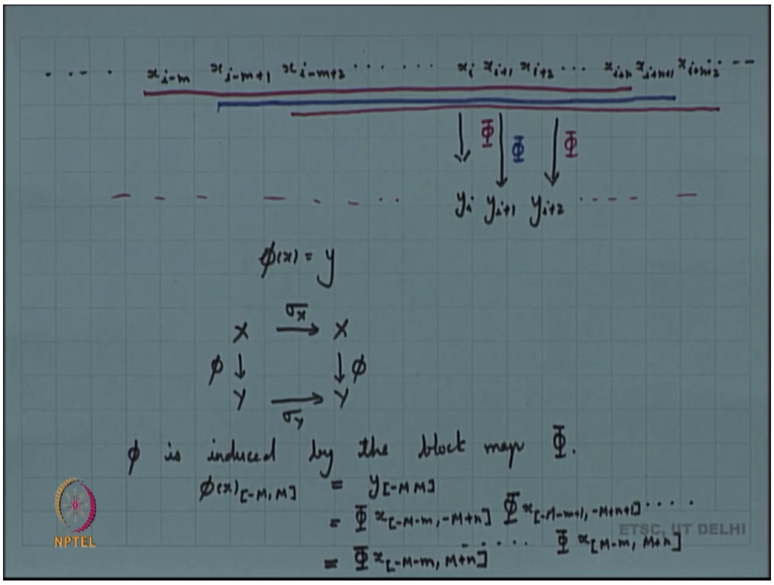

So, let us look into this point right. So, I have this x written as x say I have i minus m, right then I have i minus m plus 1, then i minus m plus 2 right this goes on then I have an x i, i have x i plus 1, I have x i plus 2 and this goes on I have x i plus n, x i plus n plus 1 and I have x i plus n plus 2 and this goes on. So, this is basically my sequence x now what happens here is what is our block map doing is that it assumes this value on x. So, it first of all reads this block x i minus 1 right to x i plus n it reads this particular block and then depending on this particular block right it basically gives me a value right which we can call it as y i . So, after reading this block right it gives this value y i.

So, this is basically what the actions. So, this phi which we can think of this as the block phi the block map phi it reads this particular block, and it generates a value which we call it as y 1 on the other hand it will again read this particular block. So, it reads this block x i plus i minus n plus 1 right then x i plus x i x i plus 1, it goes up to x i plus n plus 1. So, its particularly reading this particular block and by reading this particular block it gives a value. So, from here it gives a value right which we call it as y i plus 1 and then again we can think of this particular block. So, this particular block reads this value. So, this is what this reaction of the block map here and this.

It gives a value here which we call it as y i plus 2. So, this is basically what the block is doing. So, block reads. So, the block map reads, within the whole sequence the block map reads a certain block, and then it depending on how we define it right it generates a single value and that is what that is how we get this particular sequence right which we can say that now we define this as phi x equal to y. So, this is basically a synchronous action of the block map right on the entire sequence and that is what gives our map 5. So, what is the Curtis Lyndon Hedland theorem? It says that if phi is a factor map then it must be induced by a block map.

On the other hand if there is a map which is induced by a block map, it must be a factor map right. So, these 2 concepts are equivalent and that is what we shall try to see here, all I would like to note here is that when we are thinking of a one sided shift then our memory is 0 because in one sided we are not looking into the memory at all, we are just looking into what happens after the parts of memory happens to be equal to 0. So, let us now look into the suspect. So, we have this particular x right and I have sigma x taking x to x, we have y and we have sigma y taking y 2 y and there is a map phi here. Now we want to see whether this map commutes, all we know is that phi is induced by the block map.

Now, let us note something else here, I want to say my phi x right and what is phi x between; minus m to m if we want to look into that factor, then we know that phi x between minus m to m would be same as this y because phi x equal to y, this is same as y going from minus m to m and how it is points of y how is this block y generated because this is generated by this block map right how is this noise generated.

So, this y is generated as I have this block that phi x right this phi acting on minus m minus m to minus m plus n, then I have phi x this block phi x generated from minus m minus m plus 1 to minus m plus n plus 1 right we have this factor, and this is like phi, this is acting on the block m minus m to m plus n and we can say that this is nothing, but this happens to be the block phi right acting on x from minus n minus m to m plus n right. So, this block x on this particular values and now we look into what is the action of sigma on this particular phi. So, let us try to see whether this intertwines are not, we already given a block map we want to see that this is a factor map all we want to see is that this is into a this diagram commutes. To see the diagram commutes let us see that your sigma x of phi right.

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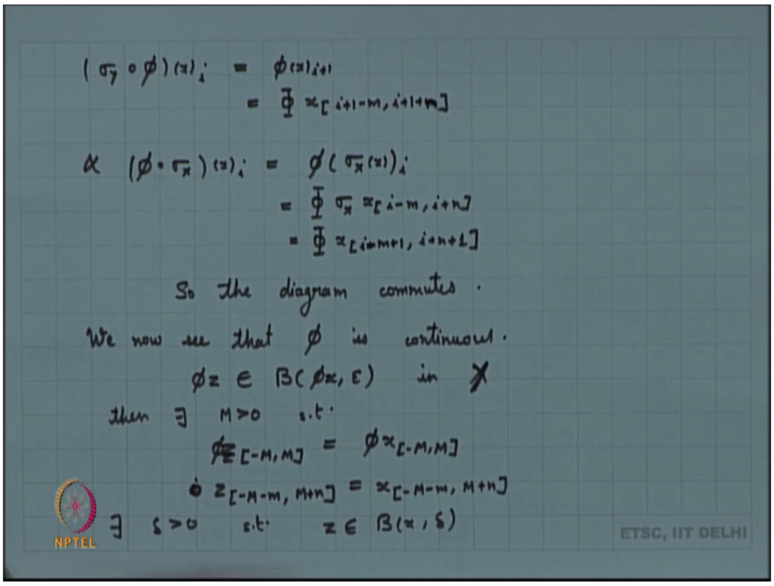

And if I that operate on x at the ith position, then that would be same as because I have taken sigma x on phi x right at i which is nothing, but this is my phi x at i plus 1 and my phi x at i plus 1 is given by this block map phi right x going from i plus 1 minus n, to i plus 1 plus n and my phi of sigma x right x the position i is given in terms of phi if I can think of this as this is sigma x of x at i and this I can think of this is be given in terms of a block map here.

So, this would be block of sigma x right and we have x at again I have i minus m to i plus n here. Now sigma x i am just shifting this block by 1 right. So, that gives me phi of x at i minus m plus 1 and i plus n plus 1. So, you find that these 2 are equal right and so the diagram commutes. So, we find that if i is given by right if this continuous surjection is given in terms of a block map, then definitely it is a factor map because the diagram commutes. And also we want to see that such a phi has to be continuous we also need to see that phi is continuous here. So, we know that the diagram commutes all we want to see is that phi is continues. So, we shall see that phi is continuous we now see that phi is continuous, now that is again simple what we can to take up is we can take phi of z in a ball right at centered at phi of x of radius epsilon, supposing this happens in your x sorry this happens in y right. So, in y you have a ball of radius epsilon centered at phi of x, then we know that because this is an epsilon ball around phi of x, there exists an integer m right.

Such that I can say that my phi of z sorry phi of z here agrees from minus m to m this block agrees with phi of x right minus m to m. So, there exist an m because phi of x is an epsilon ball around phi of x, phi of z is an epsilon ball around phi of x. So, we know that the central block for both of them will agree and what does that mean? Central block agrees which means that I can use this previous observation and say that, this would mean that my phi my there or we can directly write it that z at minus m minus m to m plus n right this would be same as x at minus m minus m to m plus n right, which is making use of this particular.

Fact here this is same. So, that would mean that this block has to be same right this block has to be same x and z have to agree on this particular block right so; that means, that my z n x they agree in this particular block, and what does that mean? Now that there exists there should exist a delta positive such that z should belong to a ball of radius delta right centered at x, and that is enough to give me continuity of phi right. You start with any x and z right and start with any x point x right take an epsilon ball around phi x, you find that there exists a delta ball around x such that all elements have of it under it under phi are mapped into this ball. So, this gives you continuity. So, we seen that phi is continuous.

So, if there is a map given by a block map right, then it is a factor map right. So, that is a factor map and now we want to see the converse part, we want to see that every factor map can be realized as a block.

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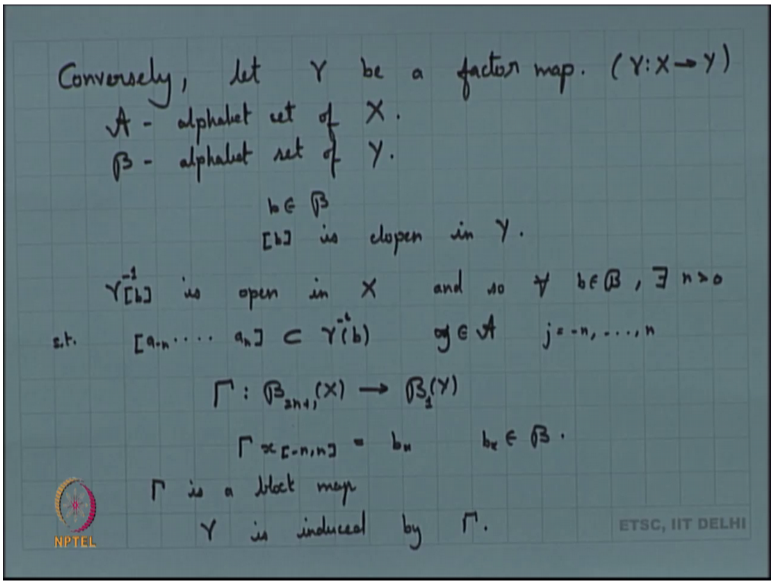

So, what happens conversely? So, conversely let gamma be any factor map. Now let me say that A is the alphabet set of X. So, my gamma is something like gamma goes from X to Y, my X and Y are shift spaces and this is a factor map, if I am taking my A to be an alphabet set of X and B is an alphabet set of. So, we have an alphabet set of X and alphabet set of Y. Now think of that I am looking into this particular thing as a factor map and I know that if I take any b belonging to B right then the cylinder set right is clopen in Y.

Now, this cylinder set is clopen in Y. So, it is compact also it is closed. So, it is compact and then these clopen and sets right for each b in B right these clopen sets will be all disjoint and hence because this is a factor map right what can you say about gamma inverse b. So, gamma inverse b is open in x, now gamma inverse x b is open in x and so, for every say b in B there exits an n positive, such that there is some word right some cylinder set a minus n up to a n right some cylinder set which is contained in gamma inverse b where my sorry a j belongs to A right for all j from minus n to n.

So, since I am looking into finitely many my b is a finite alphabet set right. So, finitely many letters, we can find some n which is common to all of them such that you have such a cylinder set belonging to gamma inverse b. And each gamma inverse b is open . So, also disjoint right. So, we start with. So, all we start with this a map gamma right from B 2 n plus 1 of x to B 1 of y right which is defined as gamma of x from minus n to n right it gives you some b x, where b x belongs to B. Then this gamma is a block map because of this factor right this gamma is a block map, and your factor gamma is induced by this block map gamma. So, if we have, conversely if gamma is a factor map right then it is induced by a map.

So, this is; what is Curtis Hedlund theorem. So, whenever we talk in shift spaces whenever we talk of block maps, whenever we talk of factor maps right they are basically generated by block maps. We end our class here.