Chaotic Dynamical Systems
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Lecture – 22
Strongly Transitive Systems... Cont'd

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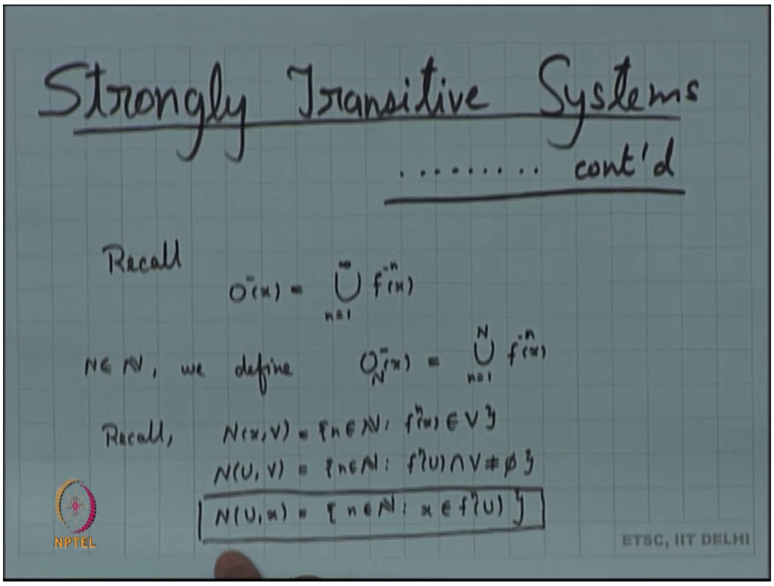

So welcome to students, in this lecture, we are going to look into strongly transitive systems. We are going to continue with what was done in the previous lecture and we recall here. So, the rest of the things like x is a perfect space compact space right and our system right that remains the same, you simply recall the definition of the backward orbit that we had taken up. So, the backward orbit is basically the union of f minus nx right on n going from 1 to infinity.

Now, we may not be interested in the backward orbits right the complete the full backward orbit. So, we would like to see that we want to look into a backward orbit up to certain level N. So, for that for n in N N we define this backward orbit of x for the level N we define this factor to be the union of f inverse of x or f minus n of x right when I am looking into my n going from 1 to capital N.

So, basically we are just interested in fine or one can say that one is interested in a finite backward orbit. So, you are looking into backward orbit up to a certain level n, you are not looking into the complete backward orbit. So, we define this factor and again I would like to recall the hitting time sets. Now we know that we have seen those hitting time sets N xv right which is basically the set of all n in N N such that fnx belongs to V we had looked in to the setting time set when we were discussing recurrence.

And then we have the setting time sets N U V to be the set of all n in N such that fn of U intersection v is non empty and now we define another kind of hitting set which we write it as N U x which is basically the set of all n in N such that x belongs to fn of U. So, we can think of this new kind of hitting set that we have defined here. So, I am looking into this part this is something new that we are considering here. So, we are looking into all those images of U right all those images of U under n fn U for which contain the point x.

Now, we will look into a theorem and we have discussed some properties of that, but we will now push that up in terms of a theorem.

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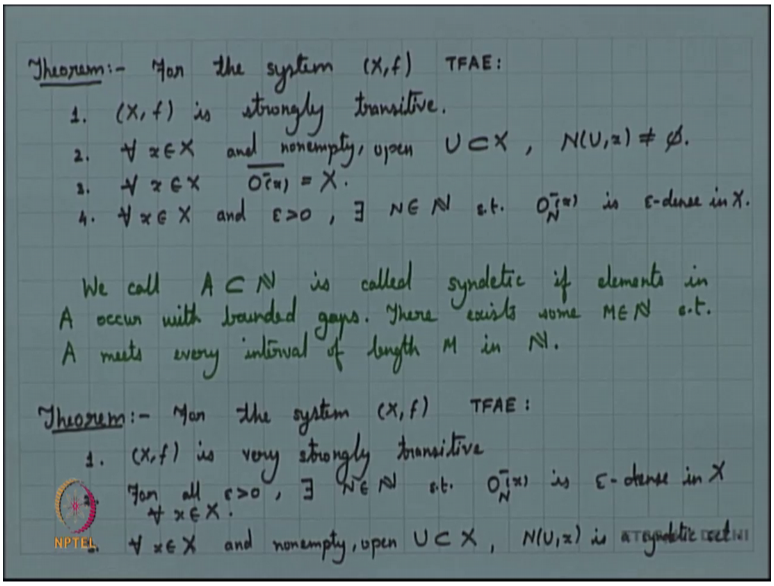

So, we have a theorem here that for the system X f, the following are equivalent. So, the first thing that we want here is that X f is strongly transitive, the second is we want that for every x in X and non-empty open U contained in x, we want this new kind of hitting set that we have defined N U x to be non-empty. So, for every x in x and you can take any non-empty open U in x then we find that this hitting set is non empty.

In fact, the compactness of x will basically say that those hitting set will be infinite then we have for every x in x the backward orbit of x is dense in x, we have already seen that these 2 properties are equivalent this is just a demonstration that yes your x can be written as union of f n U right and. So, definitely your N U x will be non-empty and from compactness we can say that it will be infinite actually then this is also the same thing which we have already seen back in the previous lecture X f is strongly transitive implies that for every x in x the backward orbit is dense.

And the fourth property which I want to write here which is something new is that for every x in X and epsilon positive, there exist an n some integer such that if I am looking into my backward orbit of x of level n then this is epsilon dense in X. Now I just want to recall that; what do we mean by epsilon dense right we say that something is epsilon dense its it basically it forms a net right. So, it is an epsilon net basically in X.

So; that means, you take an epsilon ball centered at each point here right that covers the whole of x, fine. So, this is what we want to say is that for every x in X, you stack any x here, you take a positive epsilon depending on that x in that epsilon you get an N such that this finitely many points here that you get here right this will be epsilon dense in X and that is very simple to see that because you know that your orbit is dense in X.

So, you are trying to take. So, you start with say you start with the inverse image of x right take epsilon neighborhoods there right. In fact, you take epsilon neighborhood for every point in the backward orbit and then since x is compact right by compactness you will get something like a finite sub cover right. So, what you get is something finite right. So, you this finitely many epsilon balls will be covering x and so, you get that the backward orbit of level n happens to be dense.

But before we go to something else I would like to recall a definition we have n to be a set of natural numbers. So, we call a subset of n to be syndetic if the elements of a elements in a occur with bounded gaps. So, what does that mean which means to say that there exist some M; M in n such that a meets every interval of length m in n. So, such a set is called syndetic where the elements they appear with bounded gaps.

And now we have this result again I am putting that in terms of a theorem here. So, we have this theorem here. So, for the system X f the following are equivalent and the first condition is that X f is very strongly transitive and second condition is that for all epsilon there exist an n in N such that the backward orbit of x of level n is epsilon dense in x epsilon dense in x for every x in x right for every x in x.

So, here our n does not depend on a x basically depends on epsilon and the same n works for all x and the third thing which we have here is that if for every x in x and every non empty open U contained in X, this hitting time set N U x is a syndetic set we have this conditions for strongly transitive and I am not exactly going to prove this part, but one can think of this factor that when you are looking into a strongly transitive system what happens here is let finitely many images of U right covers the whole of X.

So, what happens here is you have just finitely many of this ns right which we happen to be just take epsilon neighborhood there let your U be an epsilon neighborhood of x right then there finitely many images will cover the whole of X and so, what you will get is that the backward image right finitely many will be epsilon dense in x and similarly you can think of this factor that you would look into N U x, right.

Now, what happens is since my x can be covered by finitely many images of U right. So, I could take up any epsilon neighborhood think of any epsilon ball just take any epsilon U to be any epsilon ball right then for that epsilon ball that U is open. So, finitely many image is of that would be covering the whole of x and hence you take any x right irrespective of what x you take up you take any x right it will lie somewhere over here. In fact, the entire orbit will be lying somewhere over here and. So, what you find is that this particular set happens to be a syndetic set.

So, x belongs to f n U happens to be a syndetic set because then your orbit of x is just moving around the finitely many aspect right just is moving in the finitely many balls right. So, you find that x lies somewhere over the other right because these are finitely many balls which cover the whole of x right. So, you find this to be a syndetic set.

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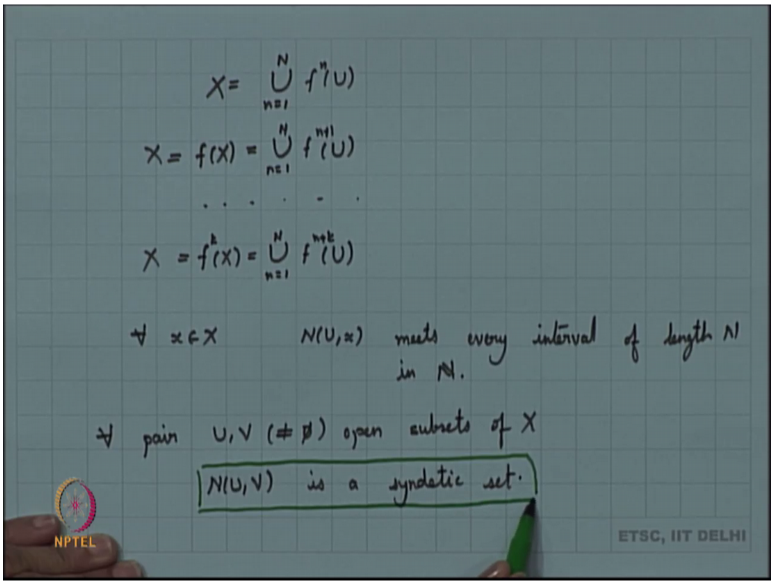

In fact, one can see it in a more better way supposing now I have x to be written as union of f n U say n going from 1 to n.

Now, what can you say about your f X, I can apply f on both the sides. So, this happens to be again I can say that this is union of n going from one to n f of n plus one U let me take fkx right this is also going to be union n going from one to n f of n plus ku. So, what can you say about your now we know that all of this is going to be x right this is also equal to x the system is surjective. So, this is also x, this is also x.

So, what happens here is that if I look into for every x in x you take any x here right what happens is what is N U x right N U x is going to be like maybe some n from here right, but then it has to be something from here it has to be something from here right then where does it meet then there is some kind of a bound here such that when you consider N U x right what are all the elements of n which come here then they come with bounded gaps because ultimately you just have something common here, right.

So, every interval in that case supposing this is n I can say that in a gap of length n right I am looking into gap of length n right. So, all the elements in N U x they will occur in the gap of length n. So, this set happens to be syndetic. So, all I can say is that N U x right meets every interval of length n in n. So, this happens to be a syndetic set and from this I can say something more that x is very strongly transitive that gives me that this is a syndetic set right. So, if I have any pair of open sets. So, for every pair U V of non-empty open sets what happens to N U V.

Supposing my x was in v right then this would have been contained in N U V. So, what can we say about N U V right N U V is also syndetic set right. So, N U V is a syndetic set this observation that this is a syndetic set right and this happens for a very strongly transitive system this observation is important. So, I am defining something else which we have not defined that earlier, right.

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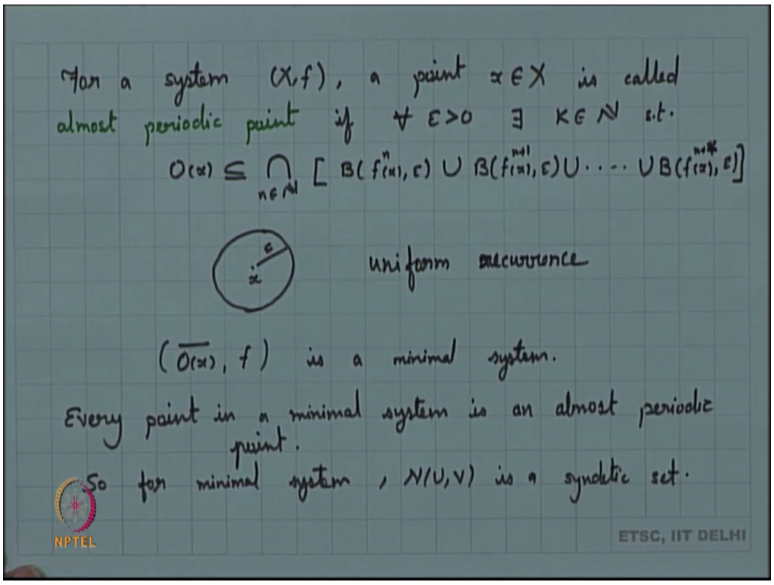

So, we defined that for a system X f a point x in X is called I want to specify this point we have not taken up this definition earlier is called an almost periodic point if and there are various ways of defining this we define this in a more geometric manner if for every epsilon greater than 0 there exist a k in n such that if I look into my orbit of x this orbit will be contained in right the intersection over all n in N what I am what intersection I am taking I am taking a ball of radius epsilon centered at fnX right union a ball of radius epsilon centered at f plus one x union a ball of radius epsilon centered at fn plus kx right this is the same k that we have taken up here.

So, if I take this ball right at these points in the orbit right k consecutive points in the orbit any k consecutive points in the orbit if we take an epsilon ball around them we would find that the orbit of x is completely contained inside that set and this is also sometimes called as uniform recurrence. So, say I am looking into some point x right then I know that if I have x right I am taking an epsilon ball around x.

So, you have this point x and you are taking an epsilon ball around x it is taking some kind of epsilon ball around x then you know that you are looking into x then you are looking in an epsilon ball around fx right and. So, on fk plus x and then again you what you find here is that some iterate of f right it will lie fx fkx right where is fk plus 1 x, right.

So, what you will find is that somehow it will fk plus 1 x fk plus 2 x, one of them will definitely come inside here right. So, what you find here is that there is; since there are finitely many balls right and we are dividing all these orbits completely over here then you will find that the set of all iterates of x right which basically not just that x is a recurrent point what we find is that the iterates of f which converges to x right they also come up with some kind of a bounded gap.

So, what happens here is my x is not a periodic point maybe it can be a periodic point all periodic points are almost periodic right my x need not be of a periodic point necessarily, but even if x is not a periodic point, but then there is a bound m such that if I start with x then there is one orbit between x. So, I start with x fx and so on up to fmX right one of the points would have definitely come inside this point then again we start the same cycle again one of them would definitely come up over here.

So, now if I am looking into supposing I say that since this is a recurrent point because something will definitely come here for every epsilon something is coming here. So, this happens to be a recurrent point. So, this is a recurrent point, but not only recurrent point this is a uniformly recurrent point because whatever comes the sequence that converges right the sequence of iterates of x that converges to x right this is occurring uniformly.

So, there is some kind of bounded gap which is coming up there. So, you find that this is a uniform recurrence. So, this kind of recurrence is a uniform recurrence and that is what is meant by almost periodic point. So, this is not a periodic point, but with bounded gaps right the iterate right I will should say bounded iterates of f will definitely lie close to x. So, x is not a periodic point, but it is an almost periodic point right although all periodic points are almost periodic points, but we do have almost periodic points without the point be periodic.

Now, very interesting to observe that supposing these particulars I have a point x such that this x is an almost periodic point then what can we say about the orbit closure of x. So, what can we say about this particular system x is an almost periodic point right and for an almost periodic point. If I am looking into its orbit closure because its orbit is any way lying over here right supposing I am looking into its orbit closure I am not going elsewhere right what happens to the set. So, it is a little bit it is you will need to think a little bit, but you can see that this is a minimal system and why this is a minimal system you take any close subset of orbit of x right you will not find it to be invariant take any closed subset of orbit of orbit closure it is not going to be invariant, right.

So, this is this system happens to be minimal with respect to this property of being closed and invariant and hence this will be a minimal system conversely you can say that every point in a minimal system happens to be an almost periodic point right. So, every point in a minimal system is an almost periodic point. Now this is one of the properties of minimal system that each and every point is almost periodic; that means, the each and every point is recurrent not just recurrent it is basically uniformly recurrent this concept is there only true for in the minimal system that every point happens to be uniformly recurrent.

And now if we try to see what happens for a uniformly recurrent. So, our system is uniformly recurrent now what happens in that case is that for a minimal system right. So, for minimal system now minimal system every point is uniformly recurrent or every point is almost periodic. So, you can think of this scenario here for a minimal system and what does that give you that gives you that for a minimal system this hitting time set N U V is a syndetic set.

So, this set is a syndetic set and this set is a syndetic set is one of the properties of minimality; what we find here is that this is basically a property of minimality, but for our very strongly transitive system also right we have the same property holding true although we know that very there can be we very strongly transitive systems which are not minimal right, but still we find that this property is basically holding for very strongly transitive system. So, this is basically a property of minimality.

So, is there any really any kind of relation between minimality and very strongly transitive. So, we try to see that in the next theorem. So, let me write the next theorem here.

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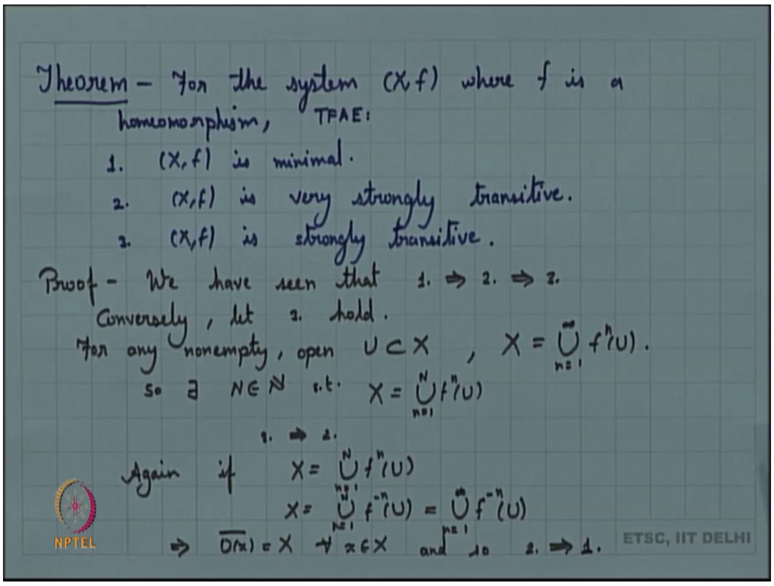

For the system X f where f is a homeomorphism, the following are equivalent the first condition that we want is X f is minimal the second condition that we want here is X f is very strongly transitive and the third system the third condition here is X f is strongly transitive what we want to say here is that if we have a homeomorphic basically for system our map is a homeomorphism. So, in an invertible system we have these 3 conditions to be equivalent.

And the proof is again not very difficult here. So, if we try to see this proof right we have seen that one implies 2 right implies 3 we know that part. So, conversely let us assume that 3 holds and what is the meaning of 3 holding is that for non empty any non-empty open U subset of x right we can write x to be the union of f n U n going from one to infinity, but f is a homeomorphism right. So, each of this f n U is open. So, what I get here is the fnUs they give me an open cover of x and my compactness of x this should have a finite sub cover.

So, there exists an n some integer n right such that x is written as one to n f n U and well. So, basically I have my 3 implies 2 and again if I assume that x is from 1 to n f n U, I again know that f is a homeomorphism here. So, I can continuously applying f inverse again and again right we are applying f inverse again and again I can say that x is same as union n to n is from 1 to n of f minus nu and that again I can think of saying that this is same as n going from 1 to infinity f minus nu that gives me that the orbit of x will be dense right because I know that x can be written over here. So, I take any point x right then there will be a U such that fn of x will belong to U and this is since this is true for any non-empty open U for any non-empty open U my x can be written in this form. So, if for every x in x right for every x in x and every open set U right there will exist an n such that fnX belongs to U right.

So, this gives me that the orbit of x is dense in x for every x in x and. So, my 2 implies one. So, when we have a homeomorphism these 3 concepts are the coinciding they are the same, but we know that in general these concepts are not the same right we have seen an example of a minimal system which is not very strongly transitive what about very strongly transitive and strongly transitive are they the same concepts we know that one implies the other that is very well clear from the definition, but are they the same thing do we have something to differentiate them.

So, let us try to see an example here now we are not getting into.

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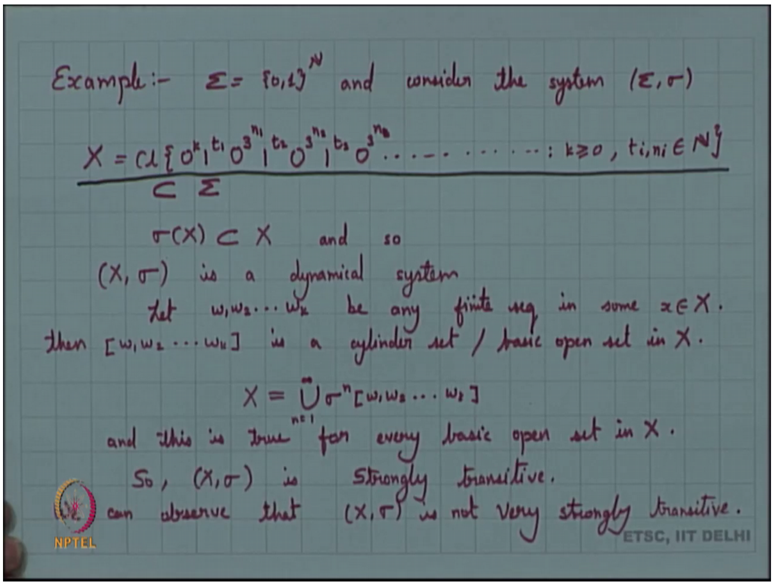

Further example we are just looking in our sigma to be set of infinite sequences of 0s and ones right and we consider the system same example that we have been working out, but now there is something that I would like to see inside this particular system. So, let me take the sequences I am looking into the set of all sequences of the form 0 to the power k then I have one to the power t 1 right, then we have 0 to the power 3 raise to n 1, then we have 1 to the power t 2, then 0 to the power 3 to the power n 2 one to the power t 3 0 to the power 3 to the power n 3 right and so on, I am looking into set of all sub sequences where my k is 0 k could can be 0 also; that means, I have nothing over here right k equal to 0 means it is not 0 to the power 0. So, basically means that I have nothing over here this is a empty word here and k is any number any finite thing. So, I can have any finite string of 0s, basically followed by this is a nice pattern I have something like one to the power t 1 0 to the power 3 raised to the power n 1 so; that means, the 0s that occur between 2 ones right they occur in powers of 3.

So, the number of 0s between any 2 ones occurs in powers of 3 I have this system and I my ti and ni, they can be any element in n, I am looking into this particular set of all sequences I want to look into the closure of all the sequences; that means, I am not only looking into these sequences I am collecting the limit points also. So, we look into the closure of this points and we call such a set to be x where I am looking into this closure in sigma. So, look into the closure of this set in sigma and we take x to be this closure in sigma.

Now, observe one thing what can you say about sigma x if any point is a limit point also right the initial part will have to be something of this form. So, any sigma x will be basically have subset of x so; that means, x is invariant under sigma right and. So, I can think of my x along with sigma right is a dynamical system now we are interested in looking into what kind of dynamical system have we obtained here.

So, we take any x in x or to be more easy to say that we take any cylinder set here right any basic open set here. So, any basic open set here would be of the form say I am looking into some sequences right should coincide with those sequences now whenever I look into any sequences that sequence will be of this particular form only any sequence will be of this particular form only.

So, we take say let w 1, w 2, wk be any finite sequence any finite sequence in say some x I want this finite sequence to occur in some x in x right. So, I am looking into finite sequence of this kind only then this is a cylinder set or I can say a basic open set this is a basic open set in x and I am taking that I start with this factor and I am I am trying to look into all unions. So, basically I am looking into all the images of all such points which started with w 1, w 2, wk and looking into the images of all such points, right.

So, we find that the union of sigma n of this particular set when I am taking my n going from one to infinity now thing of that what are all the points which are coming up right definitely all points of this form will come up now think of this here I am considering t 1 t one could be anything right n 1 could be anything. So, I am considering all these possibilities of n 1 and t 1 right. So, all can come up over here I mean infinite there then I am looking into the limit points.

Now, when I say that something is a limit point right it is a limit point of these kind of sequences then these point of sequences there will be say supposing say my y happens to be a limit point of such sequences then what you have is you have a sequence of some elements of this form which are converging to y. So, what can you say about y right the initial segment of y right would be something of this kind so; that means, when I am looking into this part right, I am including also including all the limit points of such sequences and we find that our x can be written as the union of all such things for every open.

Now, I am looking into this to be some basic open set right I can say the for every basic open set I can find that x is written as x can be written as union of n going from one to infinity sigma n of this particular to right and this is true for every basic open set now this is interesting since this is true for every basic open set in x, I can write my x to be the union of all such images of open sets right what we can conclude here is that this particular system will be strongly transitive.

So, x sigma now my query goes something else can I say that I can think of this to a finite level and say that all elements of x are contained here supposing I start with any word right now this word will have finite length fine. So, if I take a finite image here right since my ti ni can be anything right this is a gap of 3 in a finite way right a finite union right may not be equal to x right there will be some points which will be sort of there will be some points which would be getting excluded over here right.

So, if I look into this particular see this particular system right this particular system will not be very strongly transitive because we cannot stop it after some time we need that infinite iterations we cannot stop them at particularly finite. So, we can observe that very strongly transitive. So, we have an example of a strongly transitive system which is not very strongly transitive and we would be interested in looking into what can we say about this relation with other stronger forms of transitivity that we have seen.

So, let us look into this example once again. So, our example is the closure of these systems now think of this example what we want is that given any open set U and V right there should be finitely many. So, you can plug in finitely many say finite sequence of 0s and ones in between and you can take from an open set U you can go to an open set v what we find here is that this particular system is mixing the reason is that supposing from U to v I could go in one step right then I could immediately take another set also another point also right some iterate also some another point also such that I can go from U to v increase a one there right increase either one 2 once there or increase one more 3 more 0s there right. So, we could increase this particular form there right.

So, what we observe is that this particular system is weakly let us think of weakly mixing here this particular system is weakly mixing maybe that is enough for us here. So, this particular system is weakly mixing.

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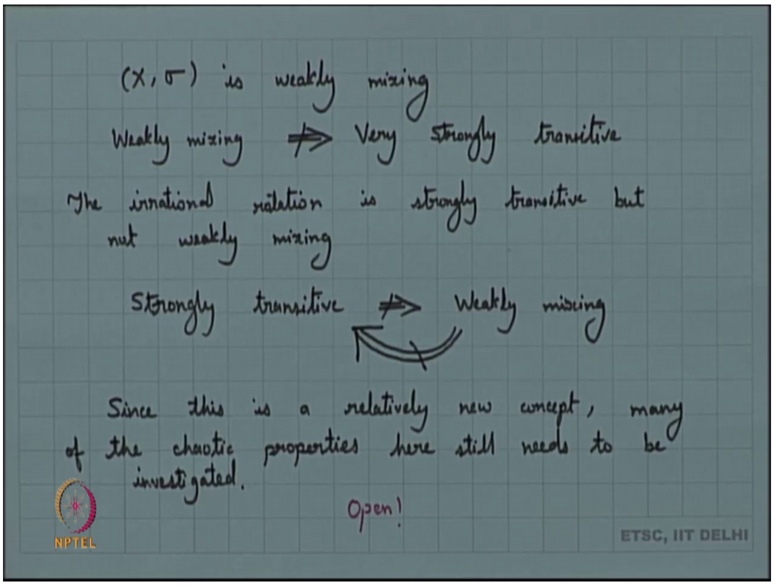

So, what do we have we now have a example of a weakly mixing system which is not very strongly transitive. So, in general we can say that weakly mixing does not imply very strongly transitive and if we look into the irrational rotation. So, the irrational rotation is strongly transitive, but it is not weakly mixing. So, what do we have here is that strongly transitive does not imply weakly mixing what about the converge part does weakly mixing implies strongly transitive what can we say about the converge part here think of that I know that if my system is a homeomorphism right f is a homeomorphism here system is invertible then my strongly transitive is equivalent to minimality.

So, supposing instead of taking a one sided shift right if I take the 2 sided shift we have already seen that the 2 sided shift is weakly mixing right 2 sided shift is weakly mixing, but it is a homeomorphism right. So, it can be strongly transitive if and only if it is a it is minimal, but the 2 sided shift is not minimal right. So, in general you are weakly mixing also is not strongly transitive. So, these concepts are again very distinct kind of concepts. So, strongly transitive does not imply weakly mixing.

So, you all you have is your mixing your weakly mixing your local eventually onto right all this concepts are interrelated where some kind of relations holds and some kind of like some kind of implications hold and some kind of implications do not hold right. So, in general this is sort of a very mixed up concept here there are other properties also of this stronger version of transitivity and since this is a very relatively new concept there is only something being studied there is still a lot of chaotic properties that needs to be studied here.

Since I do not want to waste much time or get into more details into this one we are more interested in looking into some more stronger forms or some more chaotic properties of system. So, all I would like to end up here saying that since this is a relatively new concept many of the chaotic properties still needs to be to be investigated. So, one can simply say that yes this particular line of thought is still open right and one can work on this further where I am not sure how and when something more will be done on these properties there are still other properties also which are studied up, but I have not collected everything in this particular lecture there is another thing which I noted was that we have studied we have taken up lot of examples and most of those examples that we are taking up is in the set of sequences.

So, we would like to give a little bit more thought to this set of sequences and try to look into the properties of the set of sequences right and maybe that will be the topic for our next few lectures. So, let us try to look into because almost all counter examples we are generating from that aspect. So, will try to look into that in the next class, but for this class we stop here.