Chaotic Dynamical Systems
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Lecture – 21
Strongly Transitive Systems

Welcome to students. So, today we will be looking into strongly transitive systems. Now for look into there that we again look into our assumptions. So, our assumptions are again the same.

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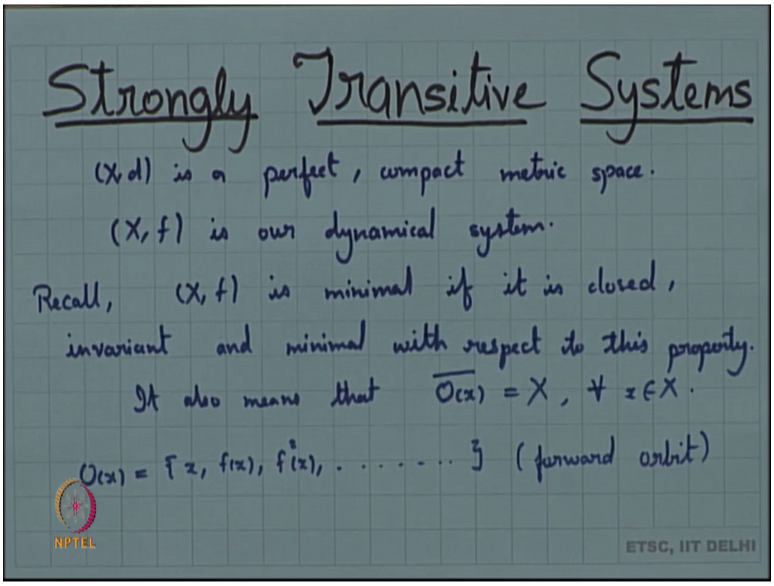

Our xd is a perfect compact metric space, and xf is our dynamical system. Now we recall that our system xf is minimal if it is closed, invariant and minimal with respect to this property.

It also means that you take any x and take the closure of it is orbit, then that is equal to x, right for every x in X. So, since we know that the system is minimal it is closed it is invariant and it is minimal; that means, it no proper close subset can be having the same property. So, we have this that the orbit is dense, right for every x in X the orbit is dense. We know this definition of minimality, we have seen this property of minimality.

Now, let us look into this once again. So, we again recall that what is our orbit of x our orbit of x was the sequence x fx f square x and so on. So, generally when we talk of orbits, we are basically talking of forward orbits; that means, we are always moving in the forward direction. But we can similarly move in the backward direction. So, we define the backward orbit, you define the backward orbit.

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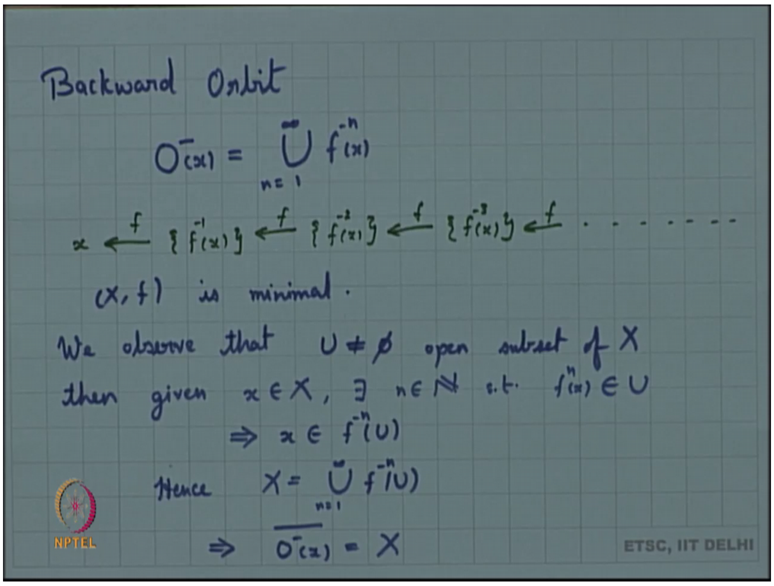

And the backward orbit is defined as we write o minus here to be the union of all f minus n x, where n goes from 1 to infinity.

Now, we recall here that our f need not be always an injective mapping. Since f is not an injective mapping, right f inverse of x need not be a singleton. So, your f inverse x or f minus nx in that sense will always be a non-trivial subset of x. So, these are all sets, and then we are looking into the union of all these sets.

So, backward orbit is basically the union of all these sets. So, what happens here is one can think of this factor, that you have an x here, right. And then there is a set mapped to x under f, and this is our set f inverse x. Then we have this set f inverse x here. So, you have a lot of points here f inverse x, right. And each of this point will be mapped, right. By some points this will be in x there will be some points in x, which will map which f maps it to all these points.

So, you have these points becomes f minus 2 x, right you have this set. And then similarly there is a set being mapped into it which is we call it as f minus 3 x and so on. So, ideally what we are looking into is our negative orbit means, we are looking into our backward orbit. We are looking into all this sets, right union of all these sets. Now again we are assuming that our system is minimal. So, our xf is minimal. Now what we observe that say u is a non-empty open subset of x, then for given n, now we know that the orbit of x orbit of every x is dense basically.

So, the orbit of this given x is dense right. So, there exist an n in the set of natural numbers, such that fn of x will belong to u. And this clearly means that x belongs to f minus nu. So, if I take a any point x in X, right. There exist an n such that x belongs to minus nu. So, all x in X will belong to some f minus nu, right. And hence we can deduce that x is basically the union of f minus nu. Where all n going from 1 to infinity. Now think of this fact x is the union of minus nu, right. That actually tells me that the negative orbit of x, right. I pick one-point x. I know that x is basically the union of all minus nu, right. Then that would tell me that the negative orbit of backward orbit of x, will be dense in x, right?

So, this clearly implies that if I look into the backward orbit of x, right that is dense in x. Now this is a very strange observation, because what we have seen is that in minimality we know that every forward orbit is dense. But it also implies that every backward orbit is dense. So, for minimal systems both forward orbits are also dense as well as backward orbits are also dense. And this leads to a very nice observation and we shall try to see this observation.

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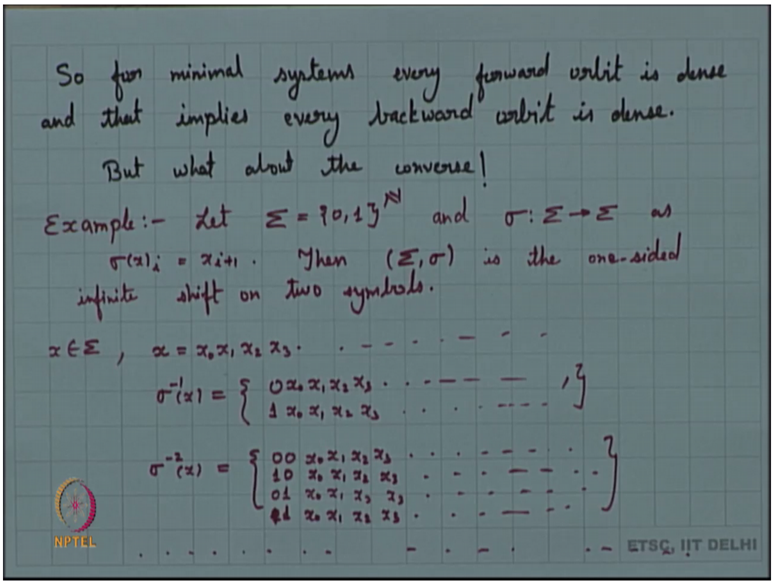

So, for minimal systems every forward orbit is dense, and that implies every backward orbit is dense. So, every forward orbit is dense implies every backward orbit is dense, but what about the converse? Suppose we assume that every backward orbit is dense. What can it say about the forward orbit? So, let us look into an example here. So, we start with an example here. Now again our example is the same on sequences of infinite sequences of 0s and 1s. So, let me take my sequences of 0s and 1 given the product topology. And then we have the shift map defined on it.

Then this is my dynamical system. It is basically the one-sided infinite sequence, shift on 2 symbols, right. We have seen this example earlier also. Now let us try to look into a typical point here. So, typical point here x would be of the form. So, x belongs to sigma, right. This would be of the form x naught x 1 x 2, right x 3 and so on. What can we say about sigma inverse x? What is sigma inverse x? So, basically, I am looking into all those points which are mapping into x after shifting. So, we just taking shifting, right. We are forgetting the symbol the first symbol we just shifting it up. And what we get as a result, right is basically x.

So, your sigma inverse x will be a set, and that set would be 0 x naught x 1 x 2 x 3 and so on. And similarly, I will have a 1 x naught x 1 x 2 x 3 and so on, right. I get these 2 points in the set sigma inverse x, right. Because both these points if I apply sigma I get back x naught x 1 x 2 I get back the sequence x. What does sigma minus 2 of x? Then again, I know that I am looking into all those points, which under sigma are being mapped to these 2 points. So, I get this sequence 0 0 x naught x 1 x 2 x 3 and so on.

Then I have a 1 0 x naught x 1 x 2 x 3, these are the 2 points which are mapping under shift to the first point here. And then I have the second point here, right 0 1 x naught x 1 x 2 x 3 and so on, and the 4th point is 1 1 x naught x 1 x 2 x 3 and so on. So, basically, I have this set which is sigma minus 2 x, right. I can continue this, right. Defining all basically defining the backward orbit of x. What do we get here?

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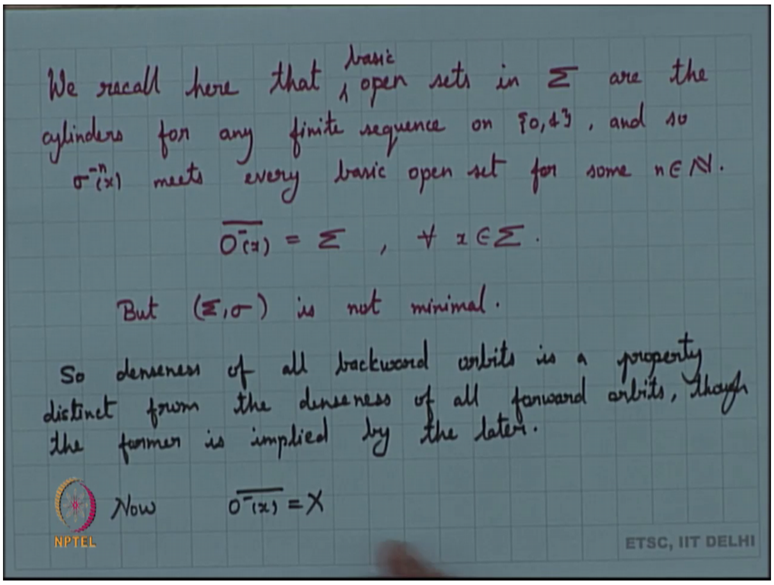

Now, we recall here not just open the basic open sets. These are the cylinders, right for any finite sequence. So, if I take any finite 0 1 sequence, right. And define the cylinder set that gives me the basic open set. And so, what we get here is; so, sigma minus n x will meet every basic open set for some n in N, right depending on what is the length of the word what is that finite sequence that we are taking right. So, depending on that sigma minus nx will meet every basic open set for some n in N. And that means that, we started with a typical point x and sigma.

So that means that, if I take the backward orbit of x, right it is dense, right. And sigma for every x in sigma. So now, we have a system in which all backward orbits are dense, but clearly, we know very well that this system is not minimal right. In fact, the sequence of 0s and the sequence of 1s they are the fixed points here right. So, this sequence this system is not minimal, but we know that the system, right. Is not minimal. So, denseness. So, what we can conclude from here is that denseness of all for backward orbits, right. Is a property essentially very much different from the denseness of all forward orbits, right?

So, denseness of whole backward orbit though the former is implied by the later. So, the former implies the later, but still we find that this happens to be a very distinct orbit it just happens to be a very distinct property basically now we want to look into some other property here. So, let us start with say negative orbit of x is dense in x, or maybe let me go back to the next page.

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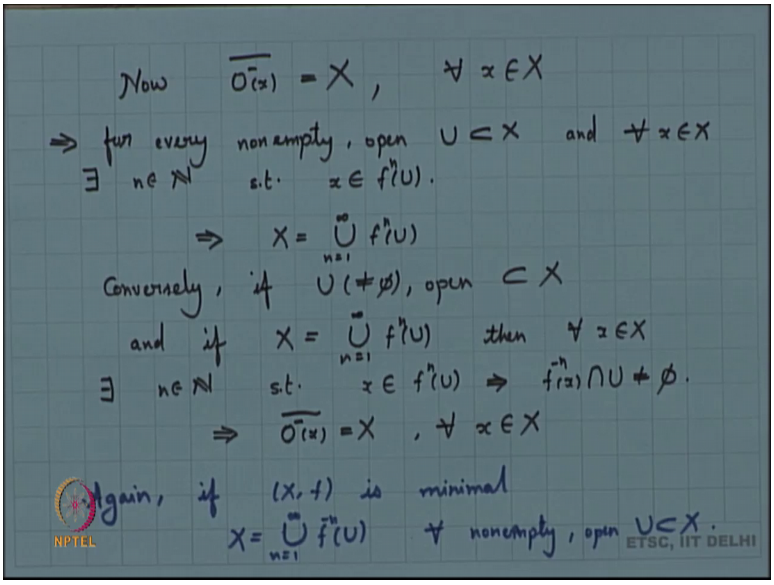

Supposing this is dense in x for every x in X. You start this property this is dense in x the backward orbit is dense in x for every x in X.

What happens in this case? So, this implies for every non-empty and for every x in X, there exist an n such that x belongs to f nu. Because there will be some n for a given x there will be some n, right. Such that f minus n of x will intersect u, right. Because this is dense, and that would mean that x belongs to fnu. Now this is 2 I have just taken 1 upon u, and for this 1 upon u and every x in X I find that there is an n such that x belongs to fnu for every x in X we have an n such that x belongs to f nu. And so, I can say that x is basically the union of all fnu n going from 1 to infinity. So, clear to all of you the backward orbit of a point for every x is dense, right. The backward orbit of every x is dense in x, right gives me that x can be written any given u x can be written as the union of fnu for all n in N.

And conversely if we say that if u is nonempty, and open subset of x and if x is written as the union of fnu for all I n going from 1 to infinity, then for every x in X, right. There exist an n in N such that x belongs to fnu. And that implies that this is nonempty. And this clearly tells me that now this is true for every x in X right. So, this clearly tells me and this is like we have fixed a u. So, we started with one u right. In fact, for any open u you can find an n, right. Given you can find n for every x in X you can find an n such that, f minus n of x intersection u is non-empty.

So, this says that the backward orbit of x is dense in x, right. For every x in X. So, essentially these 2 properties are the same. Now let us look into again this concept of minimality. So, let us again go back to our minimal system. So, again if your system is minimal, we have seen that x can be written as f minus nu, right. N going from 1 to infinity for every non-empty open u subset of x. You just looked into this property x, can be written as minus nu; for all. Now think of that we know that our system is compact, x is compact. And these are all open sets given one open set u, right. F minus nu is always open.

So, x can be written as a union of open sets. So, this forms an open cover right. And so, by compactness of x it should have a finite set cover right. And so, we can say that by compactness of x there exists an integer n in N such that x is the union of n going from one to n of f minus nu.

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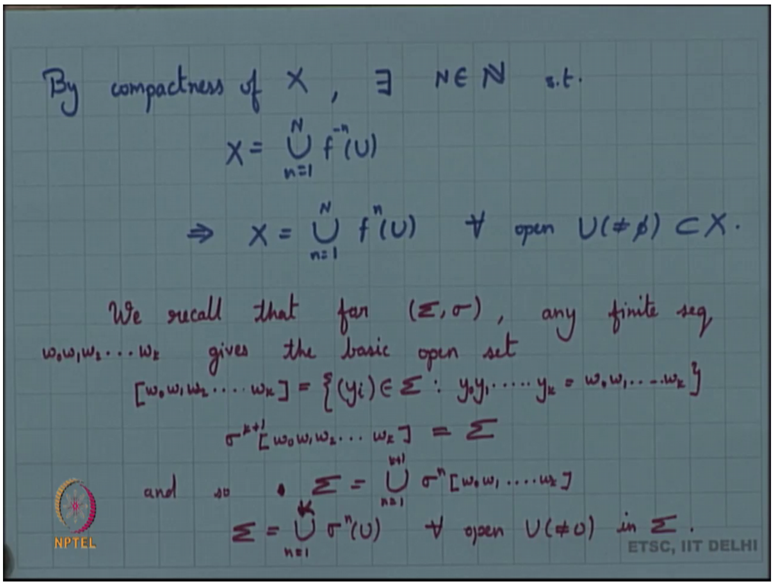

Now this is an interesting factor, I am writing x as a finite union of the backward images of u. So, I can conclude from this factor, that x can be written as a forward image of u, finitely many backward images, right. This is true for every u in U, right forward in many backward images.

So, I can say that for any even open set, right. X can be written as n going from one to n of fnu for every open of course, u is non-empty subset of x and this gives me another property of a minimal system that a minimal system can be described as a finite it is it can be written as a union of finitely many images of, right. Finitely many images of or finitely many iterates of u, right. It can be written as a union of finitely many iterates of u. Now let us try to see our example once again.

So, we recall that our system any finite sequence. So, I can take my finite sequences w 1 w, w naught w 1 w 2. So, wk. So, any finite sequence gives the basic open set which is basically our cylinder set, right. Defined as the sequence of all yi in sigma such that y 0 y 1 up to yk is same as w naught w 1 up to wk. So, this is basically this sequence is same as that sequence, right. And then the rest can be anything, right. That is what our basic open set is; we observe here that, since I am looking into all yi for which the first k plus 1 values are fixed, right. The first k plus 1 values are fixed here the rest can be anything.

So, what happens if I take my sigma k plus 1 of w 1 w w naught w 1 w 2 wk? What happens if I take sigma k plus 1 of this particular basic open set. We find that this will be whole of sigma, right. You get everything after that. So, this is whole of sigma. And so, we can deduce that since in fact, the k plus 1 the image is sigma itself we can deduce that your sigma can be written as union of n going from 1 to k plus 1, right of sigma n of this particular basic open set. And in fact, I can say that since this was my any basic open set, right. I can say that sigma is union of n going from 1 to k plus 1 or maybe some particular say capital K, right. For sigma nu, right. For every open u in sigma, right. K depends on definitely u.

But you are going to find that it has it is this is true for everything. And again, we know that this system is not minimal. So, minimality implies this property, but again this property is not same as minimal we find that, there is an example of a non-minimal system which also has this property. So, let us go back to again a definition that we had seen earlier.

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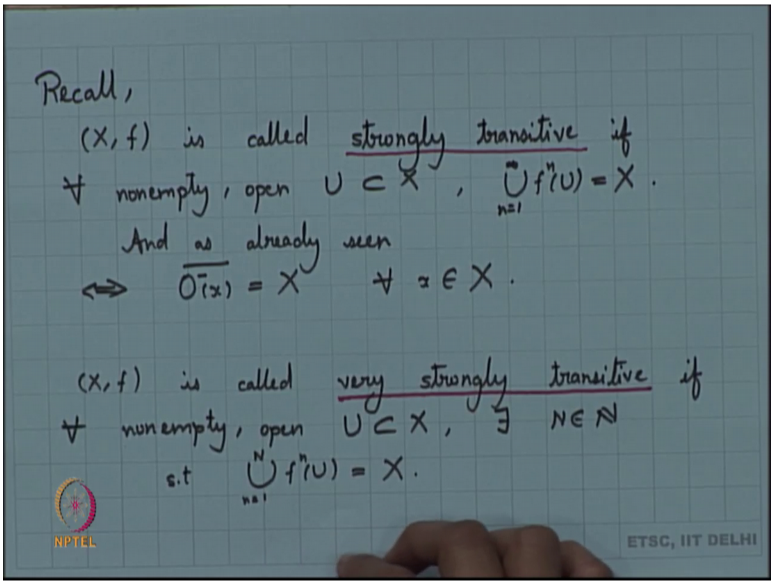

So, we again recall is called strongly transitive, if for every non-empty open u subset of x, we have union of f nu n going from 1 to infinity is whole of x.

So, this is basically our definition of strongly transitive, and as already seen this is equivalent to saying that the backward orbit of x is dense in x for every x in X. And we have also seen that this property is implied by minimality, but this is not minimality this is a distinct property. Now again we recall that your system xf is called very strongly transitive, if for every non-empty open u subset of x, there exist an integer n in N, such that union of f nu n going from 1 to n. So, this finitely many iterates of u covers the whole of x this is equal to x.

So, this is our strongly very strongly transitive. Now this is our very strongly transitives system. And what we have seen is look into the fact here is; that finitely many union of iterates is equal to x. So, this implies that infinite iterate infinite iterates is equal to x right. So, very strongly transitive is a stronger property than strong transitive. So, what we have already seen that our minimality implies very strongly transitive.

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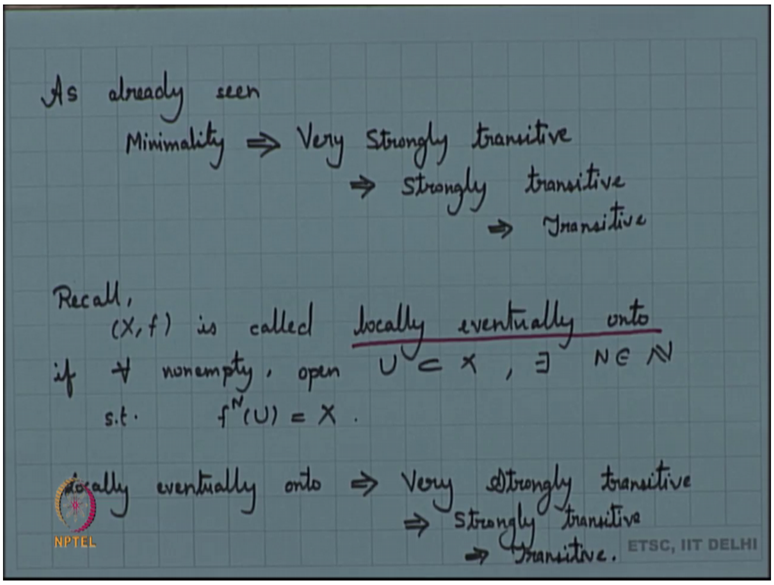

Very strongly transitive implies strongly transitive. And we have already seen that strongly transitive, right will imply transitivity.

So, this implies transitive. Because for transitive we want that you take fnu, right. You take the union of all this iterates you take the union of all these images fnu, then that should be dense in x right. So, transitivity same as that part. So, strongly transitive implies transitive. And we have also seen these definitions, we recall this definition, that our xf is called locally eventually onto, if for every non empty open u subset of x there exist an integer n in N such that fn of u I mean u itself expands to cover up the whole of x right. So, this is our property of locally eventually onto. And we know that if we have one single iterate which is covering the whole of x that would imply that the union of finitely many will cover the whole of x right.

So, what we have here is that locally eventually onto implies very strongly transitive. This implies strongly transitive, and this implies transitive. So, what we have seen is that this particular locally eventually onto, it is also something like a stronger property. All we had seen is our example.

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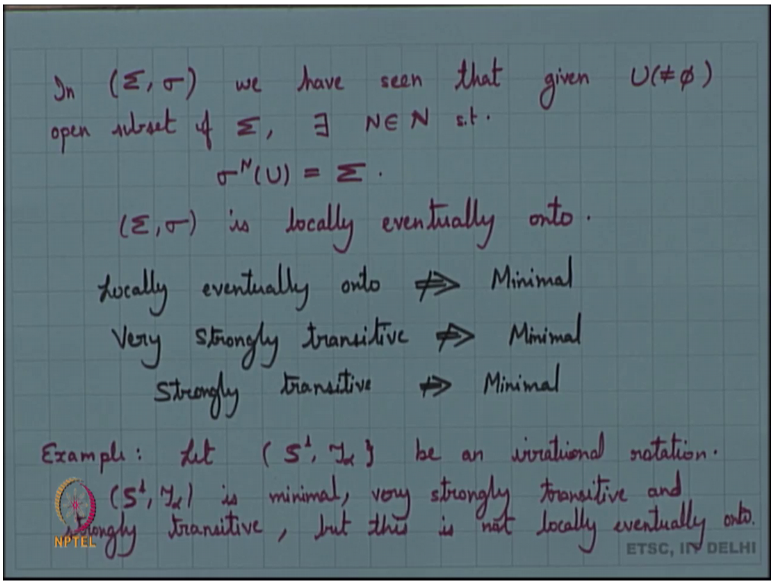

We come back to our example. So, what we had seen is that in our example, right. There exist an n in N, right. Such that I am finding that sigma n of u, right. Itself is whole of a sigma, we have just seen this example.

So, what do we know about this particular example. So, this is basically locally eventually onto, and definitely this example is not minimal right. So, we can say that in general locally eventually onto, does not imply minimal. Now since this does not imply minimal, right. We know that this system will always be very strongly transitive, this will always be strongly transitive. So, that tells me that very strongly transitive does not imply minimal. And we have seen this fact also that strongly transitive does not imply minimal. So, that means, these properties are very distinct from minimal.

But we have also seen that minimality implies very strongly transitive it implies strongly transitive. So, let us take an example here. Start with another example here. So, let us take our irrational rotation. We have already seen that the system is minimal right. So, this system is minimal. In fact, we can directly say that minimality implies very strongly transitive and intransitive right. So, or else you can simply say that take an arc, right. And what you find is that after finitely many times the arc is going to cover the whole of circle right.

So, we can simply say that this system is minimal this system is very strongly transitive and strongly transitive, but it is not locally eventually onto. This is an equicontinuous system, right. Where equicontinuous system we know that the length of all iterates of u is going to remain the fixed, right. You start with whatever is the length of the u, right. You find that the rest of the places that it length remains the same the diameter of u remains the constant under any iterate of t alpha.

So, this is an equicontinuous system, and hence you cannot find an iterate for which you will expand to basically cover the whole of s one, right. The whole of circle cannot be covered by any iterate of u right. And so, this is not, but this is not locally eventually onto. And so, we can deduce that minimality is a distinct property. So, minimal does not imply locally eventually onto.

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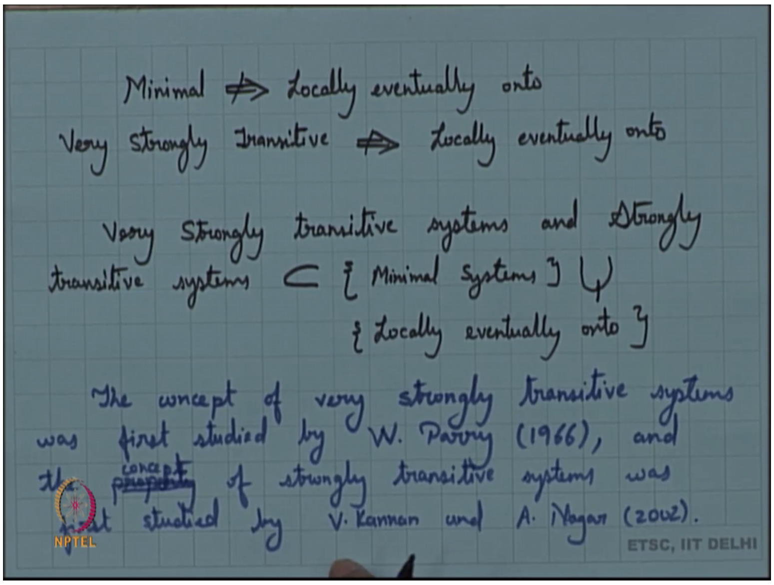

In fact, our very strongly transitive also does not imply and definitely strongly transitive does not imply locally eventually onto. So, we find that this very strongly transitive systems are very interesting cases because they lie in the intersection on these are 2 distinct cases minimality means every orbit is dense, right. And locally eventually onto means you take any open set it expands to full.

So, this some kind of expansion seen that, right. It expands to cover the whole of x. So, these 2 properties are very, very distinct properties, but they have a common intersection. And the common intersection is the systems which are very strongly transitive. So, we can say that very strongly transitive, a systems very strongly transitive systems, and strongly transitive systems they are basically contained, right in and I am looking into the class of all minimal systems, if I am saying that if you has a system is minimal it is very strongly transitive right.

Although these 2 systems are distinct right. So, if a system is minimal it cannot be if a system is minimal it cannot be locally eventually onto right. So, as such we do not have this part, but then you say that this is contained in this class, right. And this has a disjoint union with, right. The class of locally eventually onto systems. So, this is a disjoint union. Basically, these are lying in this 2 distinct classes, and then we know that locally eventually onto systems are also mixing systems right.

So, these are mixing systems these are weakly mixing systems. So, what is interesting is to note that whether we have these properties, right. Whether they imply weakly mixing they imply mixing, we know that they imply transitivity. So, what are all the other properties that the mix that they imply. So, we try to look into these properties. So, it is right now here I would just like to note that the concept of very strongly transitive was first studied I have W Parry, William parry in 1966. And the property of strongly transitive systems it is a weaker property. I shouldn't say property I should say basically the concept here because the concept of strongly transitive systems was first studied by v kannan and myself in 2002.

So, of course, is a long gap of getting from a stronger property to a weaker property. And then studying it is properties. So, the properties more properties of these concepts were studied by akin auslander and myself in 2016.

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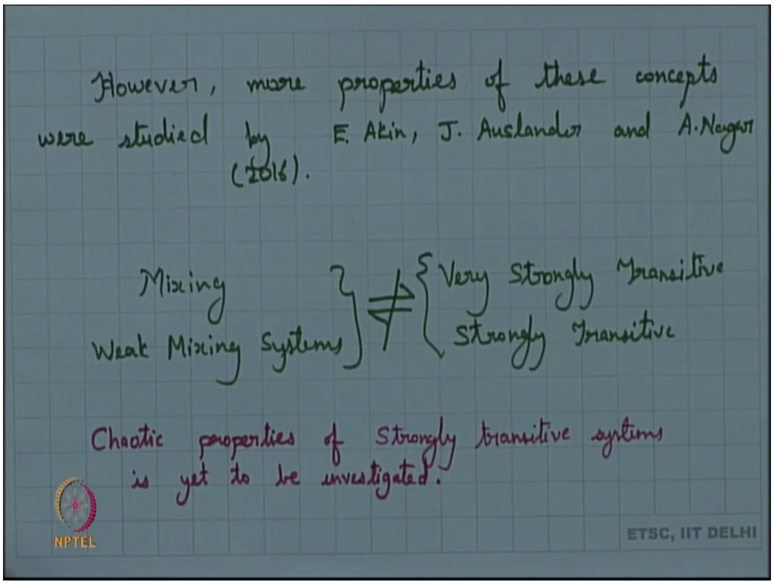

So, there are still some properties here, right which we have on one hand we have this system of mixing systems which we had just seen in the previous classes the mixing system the weak mixing systems. And on one hand we have this concepts of very strongly transitive. And we have this concept of strongly transitive.

So, we have 2 different distinct concepts here that we have seen which are the stronger forms of transitivity. And we are not sure what is the inter relation between them. So, this is again something which is not investigated. Of course, we will see in the next the next lecture we will be seeing that this 2 properties are distinct, but we see that this is not these are basically 2 different classes of systems. In what could be the condition under which this would imply this?

Or in general what could be the chaotic properties that we can get from strongly transitive systems, right? That is still open. So, well I would end up today this lecture by saying that the chaotic properties of strongly transitive systems is yet to be investigated. Yes, this is the spring of 2017, and these properties are yet to be investigated. But then there are some more properties some more interesting properties of strongly transitive systems. That is what we shall see in the subsequent lecture. So, this lecture, I just stop here.