Chaotic Dynamical Systems
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Lecture – 20
Weakly Mixing & Chaos

Welcome to students. So, today we will be continuing with the previous lecture, and we will be looking into chaotic aspects of weak mixing.

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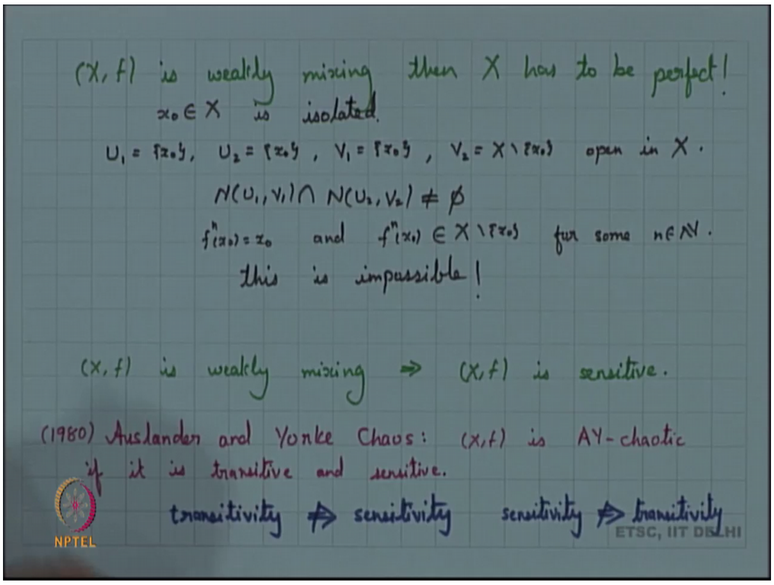

Now, when we start we always take up our xd to be a perfect compact metric space and x f our dynamical system. We recall that x f is weakly mixing if for non-empty open u 1 v 1, u 2 v 2 non-empty open subsets of x. We have the hitting time sets and u 1 v 1 intersection N u 2 v 2 to the non-empty. And as we had seen in the previous lecture; that if this basically means that x cross x under f cross f is transitive. And as we had seen in the previous lecture, it also means that the product of x n times, right. With the action of f product n times is also transitive.

So, today we shall be looking into more general aspects of more chaotic aspects of weakly mixing. But it is when we look into this fact it is very natural to note that; we start with a perfect compact metric space. So, we have not done that earlier. Why do we do it now? So, starting with the perfect metric space, we can just it is very simple to realize that if your system is weakly mixing there cannot exist any isolated points.

So, let us try to look into this fact. What I want to say here is that if xf is weakly mixing, and x has to be perfect. What do we mean by that? So, we try to take up supposing that your x naught in x is isolated. Now suppose x not in x is isolated. Then we know that x naught is isolated. So, it is both open and closed and singleton is both open and closed. So, let us take u 1 to be singleton x naught. I am taking the u 2 also to be singleton x naught. V 1 also let us take that to be singleton x naught. And my v 2 happens to be x minus x naught.

So, these are all open subsets of all, these are all open, right in x. Now since x f is weakly mixing, right. We are sure that the hitting time sets n 1 u 1 v 1 intersection N u 2 v 2 is non-empty. But what is the meaning of this being non-empty, right. That fn of x naught is x naught. And fn of x naught belongs to x minus x naught, right for some n in n. But that is impossible right. So, this is impossible. So, when way whenever we have a weakly mixing system it is a guarantee that our space has to be perfect.

Now, we also recall what we had seen in the previous class previous lecture; that if xf is weakly mixing, implants this is sensitive. What we had seen was something more than that it cannot have an equi continuous factor, right. We will try to relate this with some definition of chaos. So, in 1980 auslander and yorke had defined chaos. So, this is auslander and yorke chaos.

So, we say that xf is auslander yorke chaotic, if it is transitive and sensitive. In fact, in the same paper they had defined sensitive, and then along with transitivity they had said that this could be called this could this is something chaotic, where is this definition of chaos was later on used by rule takens in some other aspect in the aspect of differentiable dynamics. But this was definitely considered to be one of the older definitions of chaos. And if you look into Devaney's chaos, Devaney's chaos is much inspired by this aspect because that is what people started observing, but Devaney observed that there is also an element of regularity in chaos. And he had insisted that the periodic points be dense that should also be one of the conditions.

We have already seen that if you think of transitivity and if you think of periodic point being dense, then sensitivity becomes redundant, right. That implies sensitive, but auslander and yorke when they defined this fact. So, auslander yorke chaotic consists of 2 ingredients that is transitivity and sensitivity. And we have also seen this that transitivity, right does not imply sensitivity. And this is very clear because, we can think of the rational rotation, right. The rational rotation happens to be an equi continuous minimal system and since it is minimal it is transitive, but it is not sensitive.

So, transitivity generally does not implies sensitivity. And we also know that sensitivity cannot imply transitivity. We have systems you can just think of say, some kind of a tent map construction on 2 disjoint open interval a 2 disjoint intervals. So, you take 2 disjoint closed intervals, and define some kind of a tent map construction on them. Then we know that on an interval a tent map kind of construction will be transitive sensitive everything, right. And on the other hand, what you find is that on the other interval also closed interval the other a disjoint closed interval also will give you the same thing the system is sensitive.

But if I take the union of 2; that means, I am now looking into both of these systems, and both of the systems together, right. As one dynamical system, then we know that they both are invariant they both are 2 pieces of the same system. So, they are not transitive, but definitely they will be sensitive. So, the systems are sensitive, but they are not transitive. So, it is very easy to see that sensitivity in general does not imply transitivity. So, there is no redundancy basically in this particular definition. And auslander and yorke defined that a system means auslander yorke chaotic if it is transitive and sensitive. For us it is of interest to note that; if your system is weakly mixing it is transitive. We have already seen that it is sensitive, and hence it is auslander yorke chaotic.

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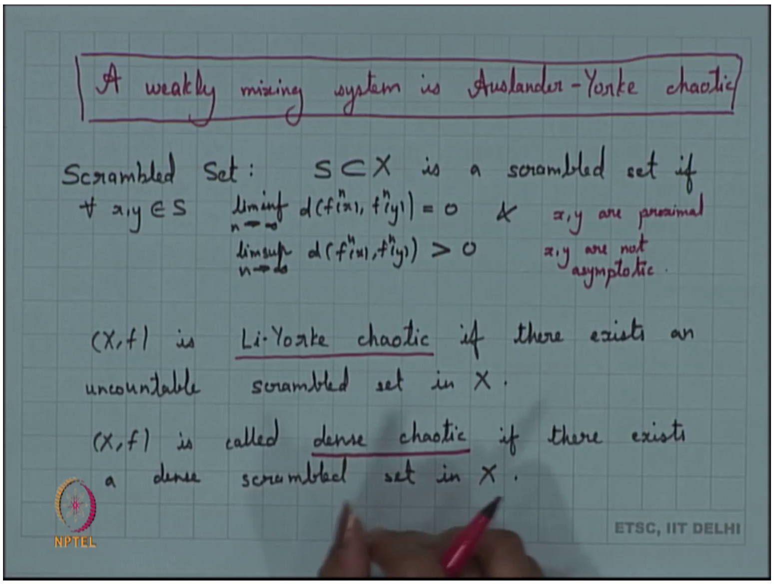

So, we just observe this part weakly mixing system. We also would like to recall today the concept of scramble set and Li-Yorke chaos that we have done earlier. So, let us recall the definition of scramble set. So, what do we mean by a scramble set? Now we say that s a subset of x is a scramble set if for every pair x y in s your lim inf as n tends to infinity and the lim sup is positive.

So, this is our definition of scramble set. And we recall again that your system is Li-Yorke chaotic if there exists an uncountable. So, if x has an uncountable scramble set and we say that the system is Li-Yorke chaotic. Now Li-Yorke chaotic basically tend at to be something like it is not sort of everywhere existing everywhere right. So, you can have sort of the scramble set maybe there is a portion of x which does not contain any part of the scramble set. But then it lies somewhere else, and the whole system becomes Li-Yorke chaotic because of that particular reason. So, people thought that it should be the property should be made no more global concept. So, they defined something called dense chaotic.

So, the system xf is called dense chaotic, if there exists a dense scramble set. So, this was the 2 things which was found that fine we could modify this definition a little bit. So, we need this concept of dense chaotic. Now we would be interested in looking into again scramble set, but before that let us look into another aspect of weakly mixing systems.

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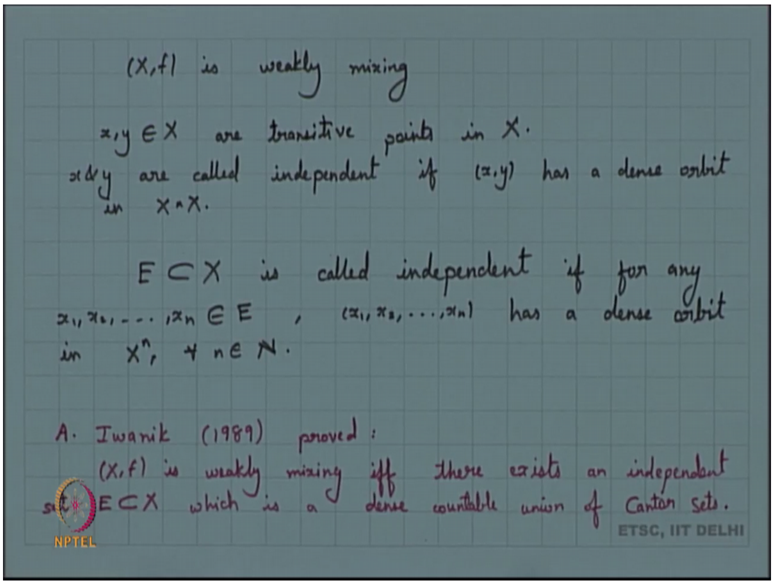

So, we recall here that and xf is weakly mixing. Now since f xf is weakly mixing we know that the n product for any n the n tuple, right: N product of x n product of f, right. That also happens to be transitive.

So, we think of 2 points, 2 transitive points in x. Now we know what is transitive point. We want to say that yes, the orbit of existence the orbit of y things. Now this x and y are called independent; i this tuple xy has a dense orbit in x product x. Now we know that it is not always possible that the tuple xy, right. For any given a any transitive point because if x is transitive fx is also transitive, but x comma fx need not always have a dense orbit, right. In x cross x. So, it is not always possible that any 2 transitive points, right for any 2 transitive points the tuple will be here dense orbit in x cross x.

So, we are calling this points this transitive points we call them to be independent, if this tuple happens to be a dense orbit in x cross x. Now we want to take slightly more generalization of this concept. So, we say that E subset of x is called independent, if for any I take x 1 x 2 x n belonging to E. This tuple x 1 x 2 x n, right has a dense orbit in x to the power n. So, since we know that we are looking into weakly mixing system right.

So, we will have tuples of we will have tuples forming the transitive point. So, this tuple forms a transitive point in x to the power n. And this is true for every n in N. So, no matter what when you take up right. So, your set is said to be independent if you have this distinct points x 1 x 2 x and in E, such that this tuple has a dense orbit in xn. Now, it is always possible that in weakly mixing systems you will find such dense you will find such independent sets right. So, the existence of independent sets is definitely true, but we have something more here that. So, a iwanik in 1989, he proved this result if x f is weakly mixing iff, and only iff. So, this is if an only if condition there exist an independent set which is a dense countable union of cantor sets. Actually, topologically a countable union of cantor sets is called micelles key set, but we are not looking into we will we will not get into too much of mathematics here.

. So, we can just the think of that we have dense countable set of dense of dense countable union of cantor sets, and we know that cantor sets happened to be uncountable perfect sets. So, we have such a set which is an independent set, whenever xf is making mixing. We will not getting to the proof of basically this result. Because anyway the proof would take a very, very long time to discuss. So, we will try to look into this particular result and try to accept that whenever we have a weakly mixing system, we have a dense set of independent points. So, that means, we have too many transitive points anyway we know that the set of transitive points is always dense, but more than that we have transitive points which are independent, right. That also performs a dense set.

Now, let us try to understand this independent set once again. So, we try to see what happens when 2 points are independent; so let x and y be independent.

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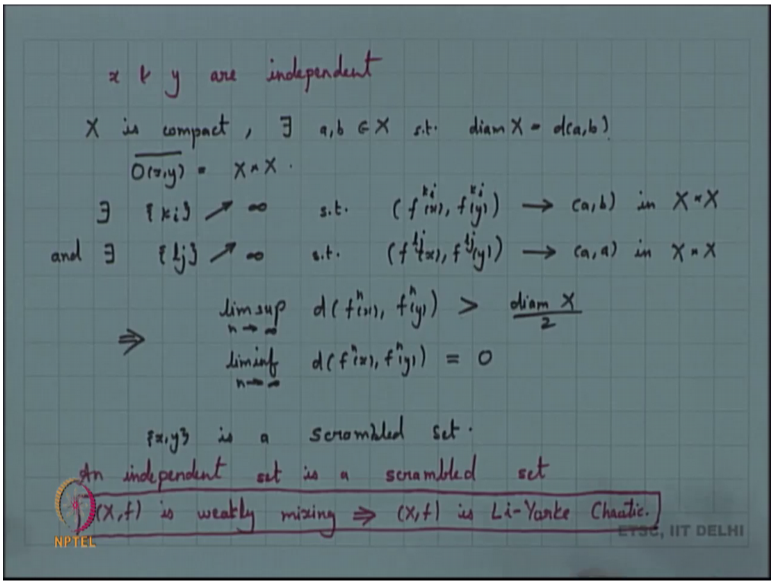

Now, what happens in that particular case when x and y are independent. Now, our x is compact. So, definitely it has some kind of a finite diameter, right. Since x is compact and since it has a finite diameter I can think of some points a and b in x, right which such that the distance gives you the diameter of x. So, there exists a and b in x such that I can say that diameter of x is nothing but since the distance between a and b, right. We can find out these things.

Now, my x and y are independent; that means, x y has a dense orbit. So, orbit of xy, right is dense in x cross x. And then we can say that since the orbit of x y is dense in x cross x I have some sequence. So, there exist a sequence say I am looking into ki, increasing to infinity. So, I have an increasing sequence ki such that if I look into f ki of x and f ki of y, right. Now this orbit is dense.

So, this basically is converging to the point a b, the point a b belongs to x cross x, right in x cross x and there exists a sequence lj, right. Again, it is an increasing sequence increasing to infinity such that I have f lj of x f lj of y, right. This is converging to the point a a in x cross x and this is possible because my xy has a dense orbit, right. Since it has a dense orbit, right there will be a sequence of it, right. There will be a subsequence which converges to each and every point. Now what does that what does this observation tell us?

Let us look into the first observation a b; that means, if I am looking into the lim sup of the distance between the orbits of x and the orbits of y, right the look into the lim sup. Then the lim sup is at least greater than diameter by 2, right. It is greater than diameter by 2 right. So, that gives us that the lim sup. So, this basically implies that the lim sup is n tends to infinity. And we look into the next observation, right. That tells me that x and y are proximal, right. Because they are converging to the same point that there is a subsequence under which they are converging to the same point.

So, what we have is that the lim inf of n tends to infinity is 0 which means that my xy, right. Is a scramble set. Now think of that any 2 independent, if I take any 2 points which are independent, right. The mutual independent, then we find that this pair is a scramble set. So, what can we say about our independent set, right. That will be a scramble set. So, an independent set is a scramble set, and what does that mean? Well, in terms of our weakly mixing system we know that in a weakly mixing system we have an uncountable independent set weakly mixing system. We have an uncountably independent set which tells us that in a weakly mixing system we have an uncountable scramble set right. And so, xf is weakly mixing implies x f is Li-Yorke chaotic.

Now, you just want to mention here is that Li-Yorke chaotic is a very, very weak condition for chaos. So, almost you will find at all almost all properties that imply Li-Yorke chaotic. But then weakly mixing implies Li-Yorke chaotic. So, weakly mixing certainly turns out to be a much stronger property then Li-Yorke chaos. More than that it also implies dense chaotic because we have a dense scramble set, right; the set of independents existence. So, it is dense chaotic. So, weakly mixing has nice chaotic properties.

Now, I would like to again recall this definition again scramble set. So, we look into the points x and y.

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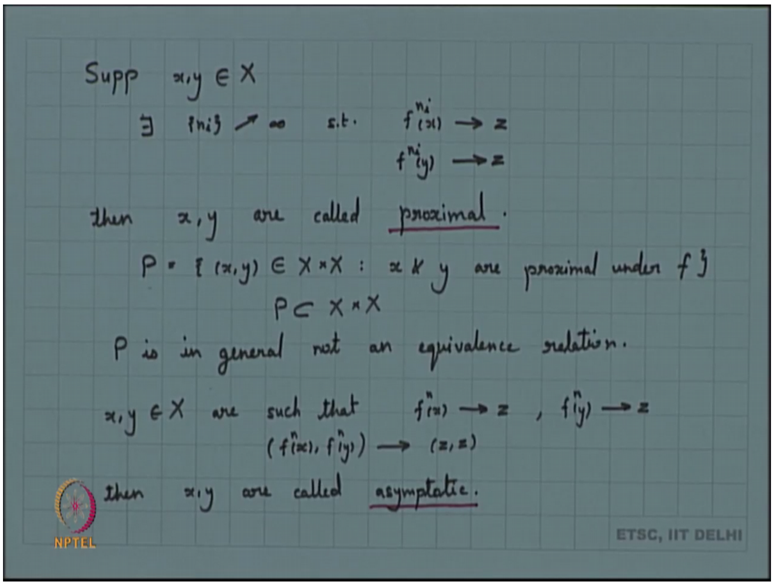

So, suppose now x and y are in x, and supposing what we find is that there exists a sequence say ni, right. Such that f ni of x is also converging to the point z, and f ni of y is also converging to the point z. Supposing both of them are under the same sequence, right. Both of them are converging to z. Then we say that x and y are called proximal when we think of x and y to be proximal, right. I can define now a relation on say x cross x. So, we define this relation. So, we define this relation p, right. Which is basically a relation on x cross x. So, this is the set of all x y belong to x cross x, such that x and y are proximal under f. This happens to be a relation. So, p sub set p is a subset of x cross x and this is a relation, right. 2 points x and y are related if they are proximal. So, this is a relation.

Now, what do we know what do we know about this relation. Is it a close relation? It is a close relation, say reflexive relation right, but in general it is not a transitive relation, right. This not an equivalence relation in that case, right. Because this is not transitive x and y could be proximal y and z could be proximal, right. That does not mean that x and z could be proximal.

So, this is in general not an equivalence relation. I am looking into this relation not being an equivalence relation. Of course, there are different ways some study to look into when your proximal relation can be any equalence relation. But that is not related to what we want to do here, but there is another concept here now supposing now for the same x and y. So, if my x and y in x are such that my fn x is converge into z. And fny is also converging to z.

So, both fnx and fny are converging to the same point, which basically one can think of that in terms of a tuple that if I take fn x, and if I take fn y if I take this pair, right. This tuple is converging to z comma z. Supposing this happens, then we said that x and y are asymptotic. Now we look into these 2 definitions, right. We look into proximal and we look into asymptotic. And now let us again go back to our scramble set. So, we again go back to the scramble set here.

So, a scramble set this is a scramble set, right. If your lim inf is 0, what does that mean? Your points x and y are proximal right. So, x and y are proximal, and what is the meaning of saying that lim sup is positive? Lim sup is greater than 0; that means, see the fact here is that the limit is not 0, right. What we want to say here is that the limit is not 0, because lim inf is definitely 0, but lim sup is positive; that means, the limit is not 0. If the limit would have been 0, we would have said that the points are asymptotic. But the limit is not 0. So, we said that x and y are not asymptotic.

So, what we have here is that for a scramble set any 2 points are proximal, but not asymptotic. And that is what we call them as a Li-Yorke pair.

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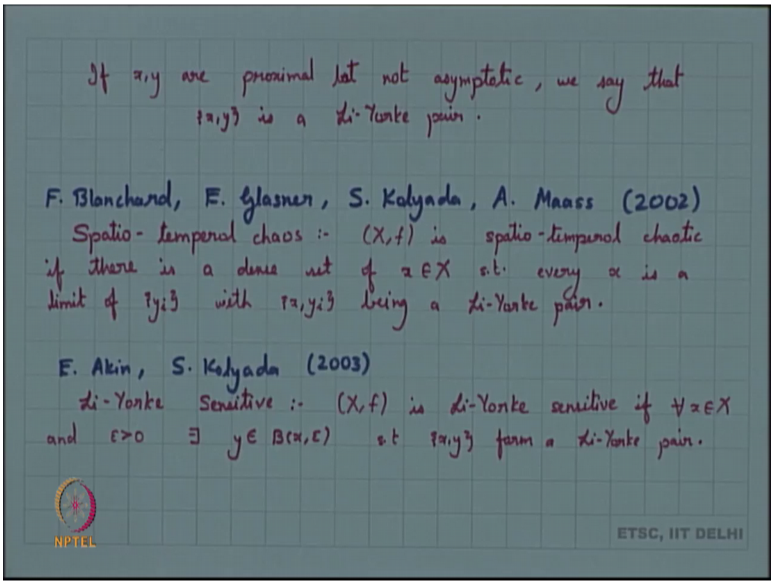

So, we said that if x and y are proximal, but not asymptotic. We say that this pair xy is a Li-Yorke pair. So, 2-point scramble set. So, this is a Li-Yorke pair. Now we are interested in looking into the on this existence of proximality, existence of Li-Yorke pair, right. One has lot of implications which we shall see. It is good to note that in a weakly mixing system you have a dense set of independent sets.

Now, when you talk of a independent set what do we what do? We mean here by independent sets we know that they are any 2 points there, formally Li-Yorke pair right. So, any put 2 points are proximal there. So, we can say that this set of this proximal relation in x cross x that also happens to be dense. Because you have a dense this tuples, right. Of all points in the independent sets they are dense, right in a x cross x. So, you find that for a weakly mixing system the proximal set is a dense set. We are more interested in looking into this concept of Li-Yorke pair, and that is what is one of the definitions? There is another definition of chaos, right. We will write it over here. So, we had blanchard, then glasner kolyada and maass.

So, in 2002 they defined something which is called a spatio temporal chaos. So, what do we say what one, when do we say that the spatio temporal chaotic. So, we say that xf is; if there is a dense set of x in x, such that every x is a limit of a sequence yi with this xyi this pair being a Li-Yorke pair. So, this is some kind of another generalization to your Li-Yorke chaos, right. That you have Li-Yorke pair, but then what you have is that you can find a Li-Yorke pair almost everywhere, right. In the whole x cross x, you can find Li-Yorke pairs, and that is what they called as spatio temporal chaos.

Now, further we have again something more interesting than this, where we had akim and kolyada. So, in 2003, they defined something which is called Li-Yorke sensitive. Now what do we mean by Li-Yorke sensitive? So, we said that xf is Li-Yorke sensitive, if for every x in x and positive epsilon, there exists a y in a ball of radius epsilon centered at x such that x y forma a Li-Yorke pair. So, that is what is li-york sensitive? And it is very clear that whenever you have say since whenever you have Li-Yorke sensitive; that means, you have sensitivity because sensitive means that you are trying to take, right.

Your basically, your orbits of x and y to close by point x and y their orbits are going apart, right. That is what you get in sensitivity, but Li-Yorke sensitivity is something more it says that you have something which is forming a Li-Yorke pair and now when we talk of a Li-Yorke pair, right. Since we are saying that they are not asymptotic we can always think of them to be say not delta asymptotic right; that means, their orbits are going at least delta apart right.

So, they are not delta asymptotic, and that is what gives this concept of Li-Yorke sensitive. And then again, the same set of people, right. Went ahead to prove something more. That you look into this concept of spatio temporal chaos, it is the same as the concept of Li-Yorke sensitivity.

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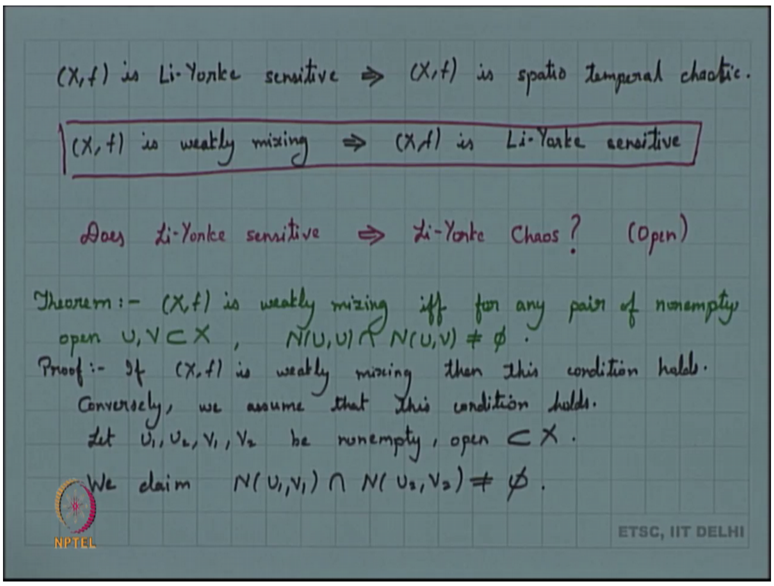

So, the vendor had to prove that x f is Li-Yorke sensitive, implies that it is spatio temporal chaotic. And then they went further more to prove that if xf is weakly mixing, that implies that it is Li-Yorke sensitive.

So, this this is another chaotic property of weakly mixing, that it gives you Li-Yorke sensitivity. Now there is an open question related to this factor, that does Li-Yorke sensitive imply Li-Yorke chaos. Question is still open, one can think on that particular line. But let us note one fact here that already when standing intervals, we had seen that in intervals, right. A 2-point scramble self implies an uncountable scramble set. So, this is true in intervals right, but for a general x whether it is true, we do not know.

So, this is still a conjecture given by them I think we do not still do not have an example which says that the system is Li-Yorke sensitive, but it is not Li-Yorke chaotic we do not have an example even to the state. So, this question is left open it should have a proof, but there is no proof existing here. So, we are still not sure we are sured that weakly mixing implies Li-Yorke chaos, right. We know that weakly mixing implies Li-Yorke sensitive, but as such in between these 2 there is no relation formed out.

So, we now look into something else. And we have observed that when we define transitivity we defined it in terms of 2 open sets, when we define mixing we are defining it in terms of 2 open sets. And when we define weak mixing, right we start with 4 open sets, why do we need to do that? Why can not we just have some condition on weakly mixing which can be given in terms of just 2 open sets. So, we have a very nice result here, in that case, and that result is something like this. That your x f so, we think of this as a theorem xf is weakly mixing if and only if for any pair of non-empty open u and v subsets of x the hitting time sets N u u intersection N u v is non-empty.

So, we are trying to give a characterization of weakly mixing in terms of just 2 open sets, and the proof is simple a good idea. So, if x f is weakly mixing then this condition holds, right. It says a unit just a part of the definition. So, this condition definitely holds if xf is weakly mixing, what is important here is to realize the converse part. So, let the condition; so assumed that the condition holds. So, conversely, we assume that this condition holds. Now what does this condition tell us? It tells us that n uu intersection nu v is non-empty, which basically also implies that whenever you have open sets u n v, your N u v should be non-empty looking into this condition, only says that whenever you have to open sets u and v your nuv is not empty.

So, this is definitely assuming transitivity here right. So, you are having this nuv is non-empty. So now, we want to prove that this is weakly mixing we prove that this assumption holds and we will prove that the system is weakly mixing. So, we start with our usual definition of weakly mixing. So, we start with n 1. So, let u 1 u 2 v 1 v 2 be non-empty open sets in x. And our claim is N u 1 v 1 intersection N u 2 v 2 this should be non-empty, right. That is what our claim is.

So, we try to again prove this using this particular condition.

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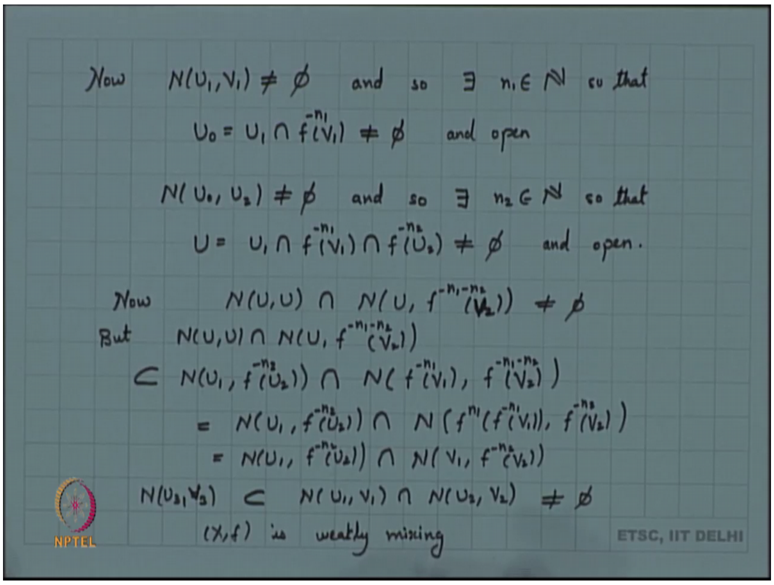

So, we say that N u 1 v 1, right. Is non-empty. And so, there exist an n 1. So, that your u naught which I assume as u 1 intersection f to the power minus n 1 v 1, right this is non-empty. So, this is a non-empty open set right. So, this is non-empty and open. And now for this particular u naught, right. We observe that N u naught and u 2 will be non-empty; since N u naught u 2 is not empty. So, there exist an n 2 in N. So, that I am looking into the set which I call it as say u which is my u naught. So, I am writing my u naught to be u 1 intersection f to the power minus n 1 v 1 intersection f to the power minus n 2 u 2, right. This happens to be non-empty and of course, it is open.

Now, I know that this is u is open. And now I am looking into this particular condition now I am applying that condition again. So, I have 2 open sets here I am looking into the open set u, and I am looking into the open set now we have not done anything with v 2, you have to write u 1 v 1 u 2 they all come up here we are not done nothing with u 2 yet. So, we see that f minus n 1 minus n 2, right. Minus n 1 of minus n 2 of v 2 happens to be an open set.

So, with this open set u and that open set v f minus n minus n minus n 1 minus n 2 u 2 we starts with the fact that. So, we know that this is non-empty right. So, n u u intersection N u f minus n 1 minus n 2 of v 2, right. We know that this is non-empty by our condition, but what is the set n u u intersection N u f minus n 1 minus n 2 v 2. We try to analyze the set. Now what is nu u. So, it is basically the hitting time sets and some point of u, right is hitting some point of u again. Now u is basically the intersection of u 1 and specific contained in the intersection of u 1 and f minus N u 2.

So, I can say that this would be contained in N u 1 and f minus n 2 u 2. And then what is this particular set. So, then we have u here, and u is basically the hittings basically a subset of f minus n 1 v 1. So, I can say that this is this nu f minus n 1 minus n 2 v 2 is contained in the hitting time such that you would observe with f minus n 1 v 1, right. And f minus n 1 minus n 2 v 2, but if I look into this hitting set, right. It is same as I have N u 1 f minus n 2 u 2, right. Intersection n now f minus n 1 v 1 hitting f minus n 1 minus n 2 v 2 is same as f n 1 of f minus n 1 v 1, right. And f minus n 2 v 2 this is the same thing. And we know that this would be same as N v 1 v 2.

So, we say that this is same as N u 1 f minus n 2 u 2 intersection N v 1 f minus n 2 v 2. Now think of that we recall here what we had proved in proof of furstenberg intersection lemma, right. That if this holds true, then that would mean that there exists open sets u 3. So, they then take this to be an open set right. So, you can take this to be an open set u 3. So, let v 3 u 3 be or I can simply say that we can think of this set my giving us an open set u 3 this set giving us an open set v 3, right. And then what we have is that n of u 3 v 3, right. Is contained in so, I am not taking much space here. So, n of u 3 v 3 is contained in N u 1 v 1 intersection N u 2 v 2.

So, from this point on we just reduce the proof than for furstenberg intersection lemma, and we prove that n there exist open sets u 3 v 3 such that N u 3 v 3 is contained in N u 1 v 1 intersection N u 2 v 2. And that means that this intersection is non-empty which means that this intersection is non-empty. And that means that your system is weakly mixing.

So, your weakly mixing also can be given in the aspects in terms of 2 sets of course, there are some conditions here in in this particular condition for our case it definitely holds true it is not hold true as such in general, but for our case definitely holds true.

Now, we want to look into a little bit of what recurrence thus weak mixing give. And we have already seen what recurrence transitivity gives some definitely weak mixing will give us the same recurrence.

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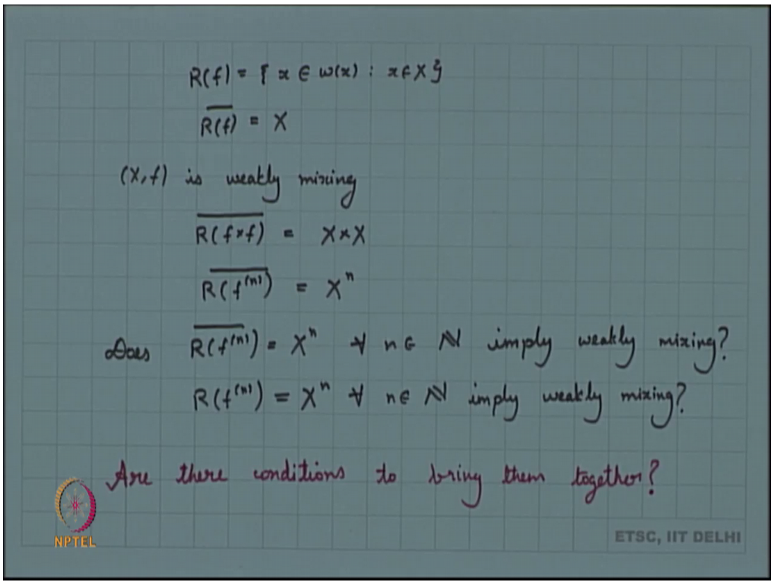

Now, we had seen that you take the set of all recurrent points in f. So, basically this is all the set of all points such that x belongs to omega x right. So, and our omega x is the closure, right. The orbit it gives us the limit points of the orbit closure of x. So, we know that the recurrent points of f, right. And we have seen that for transitive if f xf is transitive then this recurrent the set of recurrent points is dense in its.

Now, what happens if our xf is weakly mixing? If xf is weakly mixing; that means, x cross x f cross is f, right. That particular system is transitive. So, that means, if I am looking into the recurrent points of f cross f in x cross f, right. That will be dense, so that means, this closure is x cross x. And we know that 2 implies n right. So, all I can say is that the recurrent points of fn, right; is dense in x to the power n. So, that means, if I take tuple basically I take any n tuple. Now what does that mean? Whenever you take any n tuple, right the set of all n tuples that are recurrent in xn happens to be dense.

We would like to see we would like to note that in weakly mixing system, right. We know that this is dense, but is this dense going to imply weakly mixing or in other words, right; does, now let me go a little bit further. I do not want that the set of recurrent tuples should be dense. So, what we want to say is that what happens if R f to the power n all tuples are dense. Or all tuples are recurrent right. So, this is like all tuples are recurrent. So, if this happens for every n in N, right. Does that imply weakly mixing? What is a relation of weakly mixing with this aspect?

So, in general we know that we do have examples where this does not imply weakly mixing, mixing does not imply that. But the open question here is that are there conditions to bring these properties together. Then we still do not have an answer here at least as of now this is spring of 2017, we do not have an answer here.

So, that is all that we had to do for today. We will take up something else in the next class.