Chaotic Dynamical Systems
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Lecture – 18
Stronger forms of Transitivity

Welcome to students. So, today we will be looking into stronger forms of transitivity. Now what is our assumption? So, as our assumption today we are going to take our x d to be a compact metric space.

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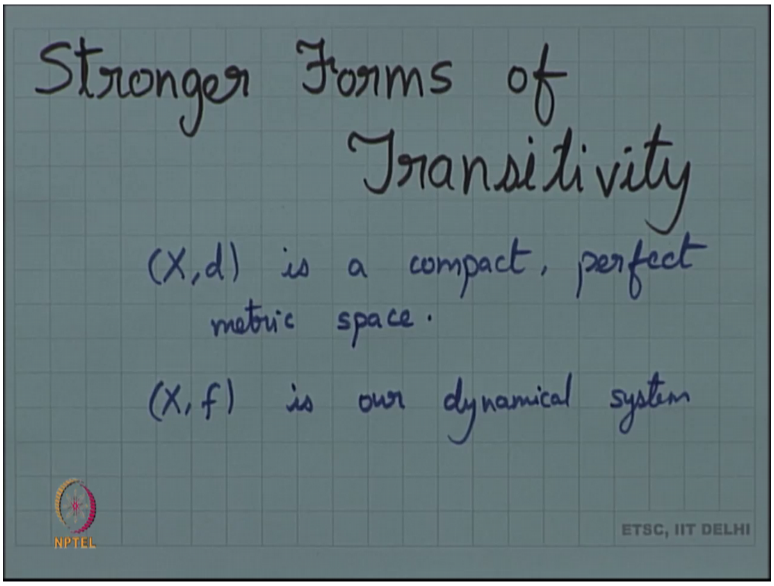

And it is not just a compact metric space, we want this metric space to be perfect also. So, that we are avoiding any kind of isolated points. And this is our compact metric space, and we want our x f is our dynamical system.

Now as we have previously also mentioned that we see that transitivity is one of the stronger forms of chaos, we would like to see what are the stronger forms of transitivity, because there are lot of chaotic properties attached to it. So, we want to basically interrelate all the chaotic properties with transitivity, and that is one of the reasons why we want to look into these stronger forms of transitivity. So, we quickly recall.

So, let us just quickly recall these concepts.

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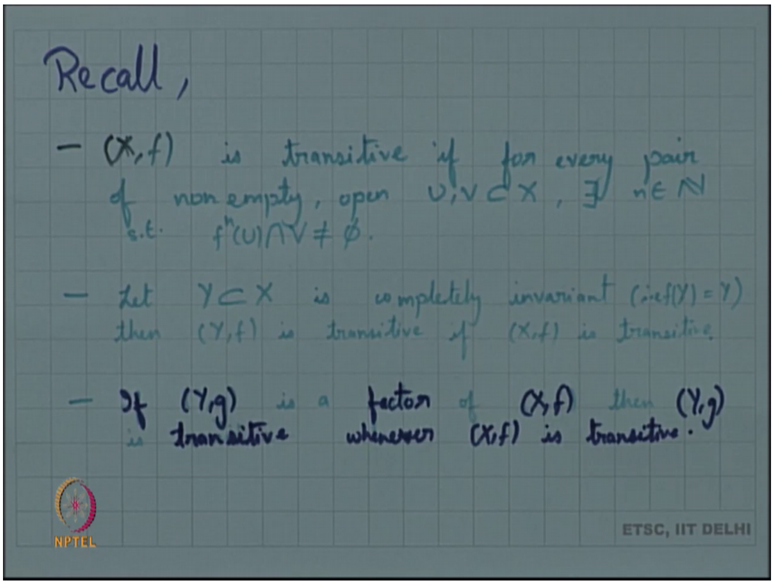

So, the first concept that I would like to recall is the concept of transitive, right. So, we all know that x. So, we all recall that this x f is transitive, if for every pair of non-empty open subsets U and V in x, there exists an integer n such that f n U intersection V is non-empty. So, we recall quickly the definition of transitivity that we say that the system is transitive, if whenever we have 2 pairs of we have a pair of non-empty open subsets U and V. Then there will be some point in U, right. Whose np trait will land up in V.

So, this is our concept of transitivity. And again, as a property of transitivity, we had seen 2 concepts that if our y is a subset of x now, I am not putting up any conditions and y. So, y is a subset of x and this y is completely invariant, we already know what is the definition of completely invariant. So, this y is completely invariant; that means, your f of y is y.

Then if I consider the system yf, right. This system is transitive if x f is transitive. So, if x f is transitive we can take any completely invariant subset of x. And if we restrict f to that particular subset, we can say that that particular system will be transitive. You can easily see this in case if you have some periodic points, right. You take the set of all periodic points that happens to be completely invariant subset. And then you find that this system of periodic points will be transitive.

The next we have is if y g is a factor of x f, then y g is transitive; I think the spend also does not work is transitive whenever x f is transitive.

We have that a factor. So, if my yj happens to be a factor, right. If this is a factor of x f, then y g is also transitive. We have seen this properties right. So, we just had a quick recall of these properties. And now we want to look into something more. So, if we try to look into that something more, right. What comes up to our mind is that is this closed under products? Now we have already seen that this is closed under subset, right. This is closed under factor what happens to the property of transitivity is it closed under products?

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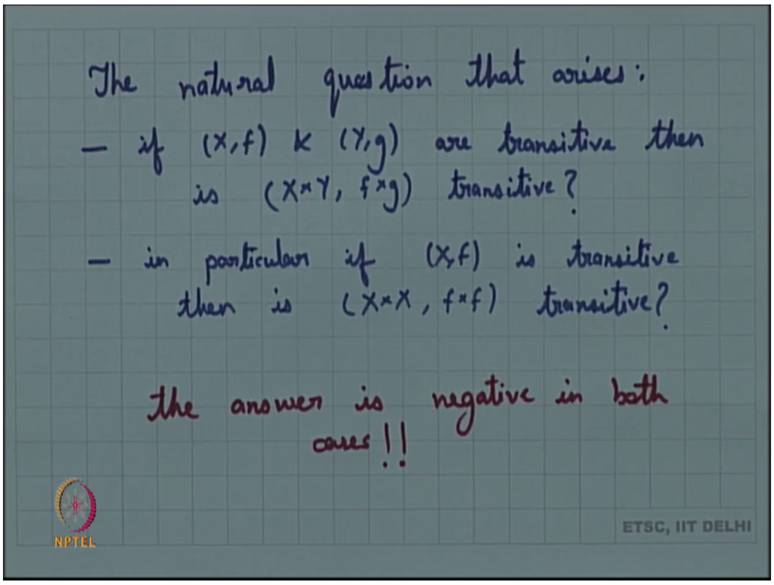

So, the natural question that arises here, the natural question that arises is if I have my system x f and my system y g, right. If these 2 systems are transitive, then is this system I am taking this product x product y and f product g acting on x product y. So, if we have this system x f and y g to be transitive, then is the system xy f cross g is this product system transitive. Now this is one of the questions that we have. And in particular we can take up this question. So, I can say that in particular, if your system x f is transitive, then is this system x product with itself and f product with itself is this product system transitive.

Now, these are some natural questions which come up, and what we observe here is that, the answer is negative in both cases. So, why is the answer negative will try to look into some examples.

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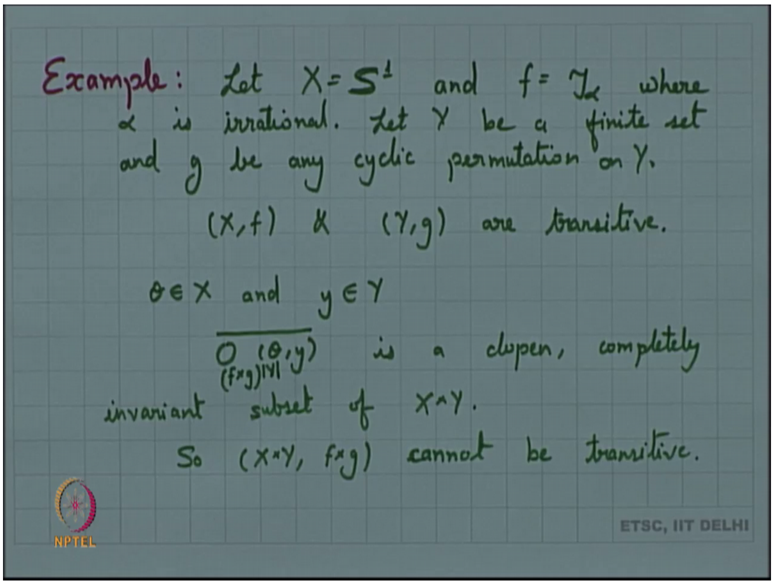

So, will we look into some examples here. So, let me take my x to be S 1 and my f to be T alpha where alpha is irrational; that means our looking into the irrational rotation. Now I am making my y to be a finite set. And g be any cyclic permutation on y. So, it is very clear that both the systems x f and y g, right. These are transitive systems. So, both the systems are transitive.

Let us look into what happens to this system right. So, we take a theta in x, and we take since our y e y, right. We fix a theta in X and a y e y, then we look into this orbit of theta and y, right. This product system we look into the orbit of theta y under now I am looking into something else I am looking into f cross g to the power mod y. And let me look into the orbit of this under f cross g; that means I am taking f to the power mod y and g to the power mod y.

On these particular points on this particular system x point x y and will take this closure. If I take this closure, we know that basically y is a finite set right. So, every point here is a sec right. So, open set. So, what we find over here is that this particular orbit is orbit closure is a clopen completely invariant subset of x cross y.

So, we find that this particular orbit is it clopen completely invariant subset of x cross y. And hence your f course y cannot be transitive on x cross y right. So, this system, right cannot be transitive. So, we have that this system. The product system f cross y is not transitive, will also see that the system actually does not give transitivity also.

So, let us try to see that start with anything let us try to see the system x f.

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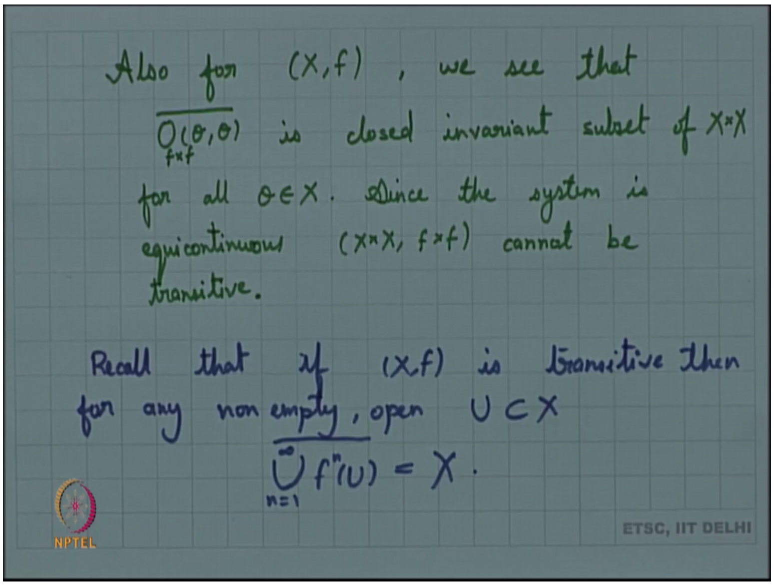

Also, for x f we see that if I am looking into this point theta, right. I am looking into against a taking the same theta I take the point theta I take it is orbit, orbit under f cross f, right. And I am taking the closure of that. Then this is definitely a closed invariant subset of x cross X. And this is a closed invariant subset of x cross X, we started with one theta, but does not matter we can start with any theta right. So, this is a closed invariant subset of x cross X, for all theta in x.

Now there is a another property that we know about this irrational rotation, and that the irrational rotation is equi continuous, right. The system is equi continuous. So, since the system is equi continuous, this system x cross f and f cross f this particular system cannot be transitive, the simple reason being if I take product of 2 arcs, right. One arc may be on the second quadrant and one arc on the 4th quadrant, then there will be no theta which takes the orbit, right. You cannot find up something the same iterate you will find some point over there, right.

So, I am taking one here, one here, one here, one in the first, one in the third and we were we cannot find this to be transitive, will discuss this more in details later. So, this is not transitive; that means, that what we find here is that in general product of 2 transitive systems need not be transitive. But we are interested in something more. So, let us observe some more properties here. So, we take this property, and let us look into this aspect we again recall something else, we recall if x f is transitive, then for any non-empty open U subset of x. We have this union n going from 1 to infinity fn U closure, right. That is equal to x.

So, this union of all images of U right; happen to be dense in x. We have seen this property, but let us look into one example again here. So, let us go back to this example.

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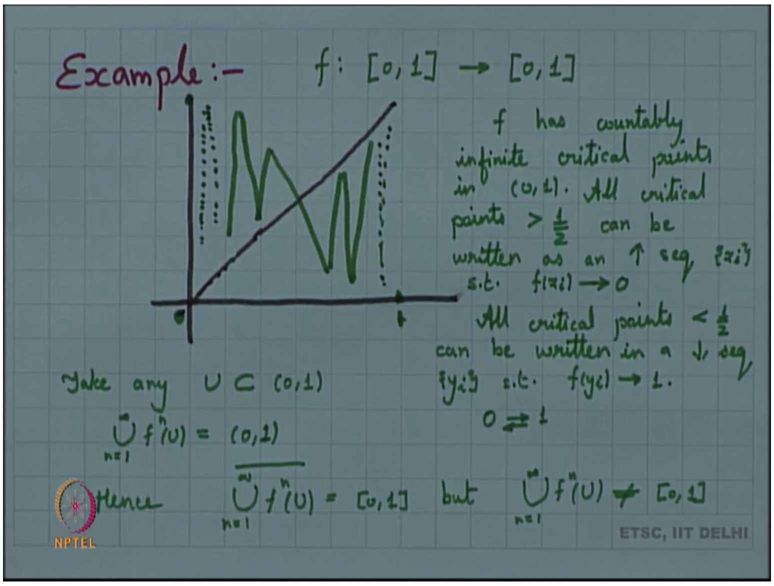

Now, I am defining f from r to r or maybe I am just defining f from close 0 1 to close 0 1. I just draw the graph of f. So, let me just particularly draw the graph of f. So, what is my graph of f looks like here? Maybe I can think of this graph of f. Say, and here at the point one it takes the value 0, and at the point 0 it takes the value 1.

So, let me take this particular graph let us try to understand this graph. So, what is my graph doing here. So, my graph takes up say f has countably infinite critical points in this open interval 0 1. So, understanding this graph is important f has countably infinite critical points in 0 1.

Now all critical points which are greater than half, now half is not a critical point, but all critical points greater than half, can be written as a increasing sequence increasing sequence say xi, right such that your f of xi is converging to 0. If you look into the critical points which are greater than half, they are in increasing sequence, right. And this increasing sequence is such that f of xi is converging to 0. And we know that xi is converging to 1, but f of xi is converging to 0.

So, we have our critical points in that particular manner. If I look into all critical points which are less than half can be written in a decreasing sequence, let me write that decreasing sequences yi, right such that f of yi is converging to 1. So, basically my yi is converging to 0, but my f of yi is converging to 1. Clearly, we know that 0 is mapped to 1. And 1 is mapped to 0. So, these 2 are periodic points. So, this is basically my map f. If I try to look into this particular map f, right. We take any U contained in open interval 0 1.

If I take any U contained in open interval 0 1, what we find is that the union of n going from 1 to infinity fn U, right will be basically equal to 0 1, because the point 0 is not taken up by anything inside 0 1. The point one is not taken up the value one is not taken up by anything inside 0 1 right. So, we find that this is open interval 0 1. So, what we can deduce from here is hence my union is just dense, right. It is closure is 0 1 the whole space x, but if I take my union. So, n going from 1 to infinity fn U this is not equal to closed interval 0 1.

So, what do we deduce from here? So, we deduce from here that in a general for a non-empty open U subset of x.

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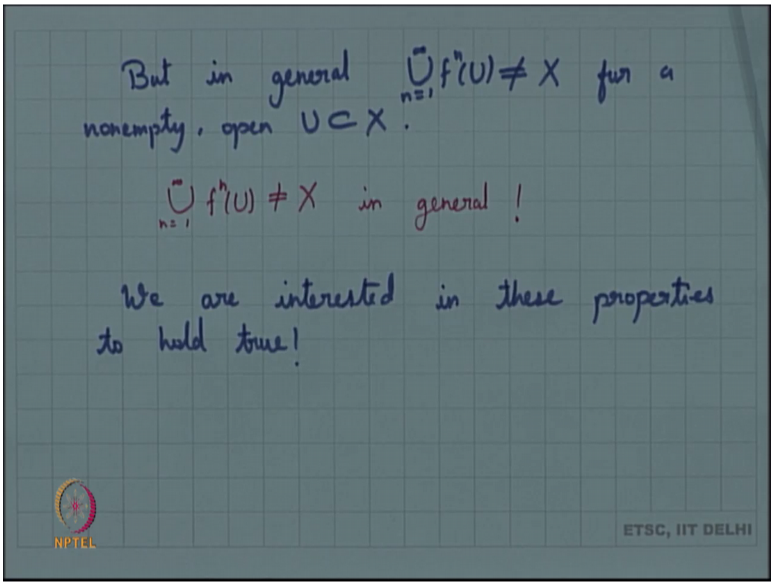

But we find that so, what we find here is that this is lacking right. So, were property of transitivity is lacking. So, here if we look into what we had seen here, right, what we find over here is we can simply say that here that transitivity, what we observe here is that transitivity is not closed under products.

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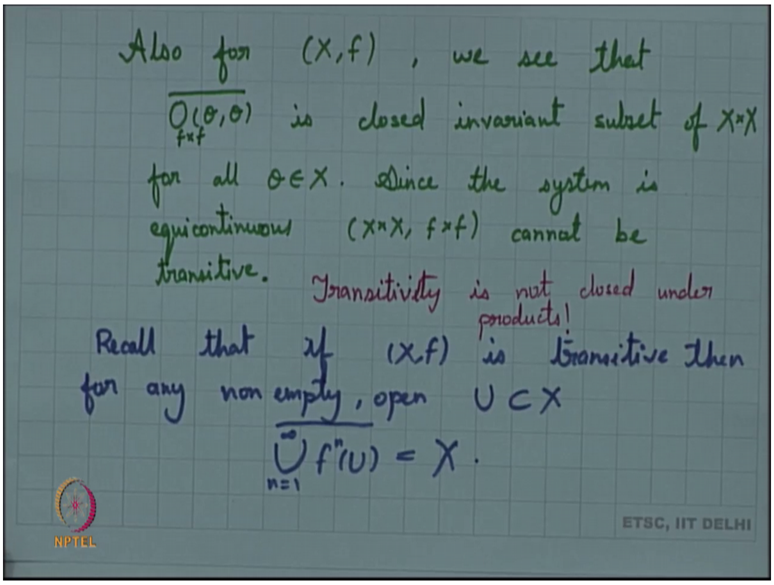

This is our first observation transitivity is not closed under products. And our second observation that comes up here is that union of n going from 1 to infinity f n U is not equal to x in general. So, that means, the property of transitivity is lacking, right. We can say that the property of transitivity is lacking in some particular case in some manner, but we would like to have these properties also.

So, supposing these are some properties characteristics which we want to have for our transitive system. So, what do we have here? So, we would like to define. So, we would like to take this now it is very clear that in mathematics. If you want certain characteristics to happen to hold to, but you are not able to find it to be holding in some particular case you take it as an assumption, and take it as your definition. So, we will try to implement these properties and to include these properties we would try to take up some (Refer Time: 23:35).

So, let us try to look into some case here. So, we would like to go we are interested in these properties to hold true. And so, we shall go into some stronger definitions of transitivity. So, keeping in this mind we define some stronger forms of transitivity.

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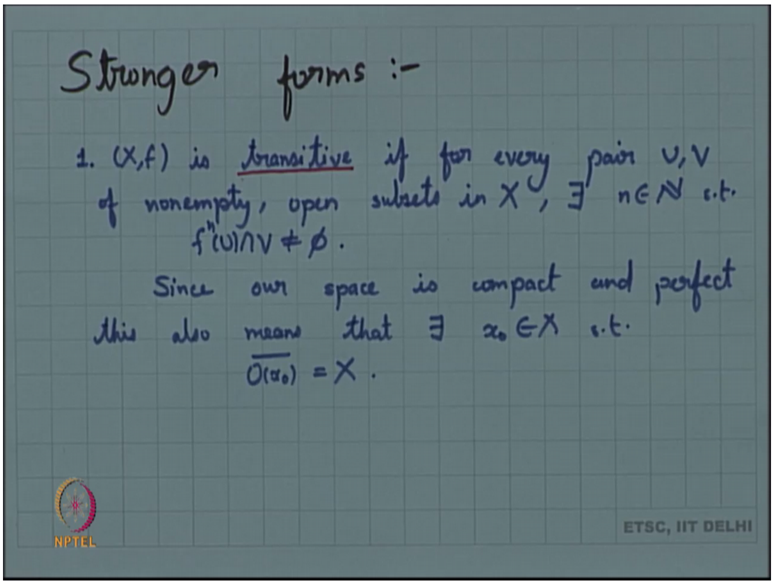

I would like to recall here that my first definition is definitely the definition of transitivity. So, I want to recall here that x f is transitive, if for every pair U and V of non-empty open subsets every pair U and V of non-empty open subsets in x; there exists an integer n E N such that fn U intersection V is non-empty.

Now, this is our property of transitivity this is our definition of transitivity that we have taken up. And as we have assumed that our spaces compact and it is a perfect space, right. This also means that. So, since our space is compact and perfect. This also means that there exists a point x naught in X such that orbit of x naught is dense in X there exists a dense orbit. So, this is the property of transitivity that we have taken up, but we are interested in the product to be transitive, right. We know that it is not true in general.

So, what we try to do is we take up another definition for it. So, let me just emphasize that this is our transitive, right. The definition of transitive, we take up another definition. So, the next property that I am defining here is x f this our system is called weakly mixing if my product is transitive.

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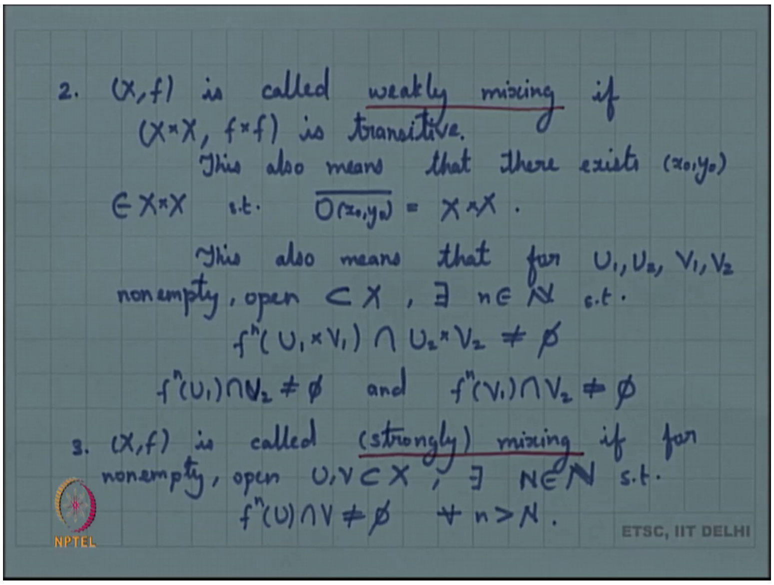

In general, that may not be true. So, I am saying that fine if the product is transitive we call the system to be weekly mixing.

So, the product here this system is weekly mixing if my product x cross X f cross f if this product system is transitive. Now again going back to the fact that our system is compact, right. The system is perfect right. So, this also means there exist say x naught y naught in X cross X such that, the orbit of x naught y naught is dense in X cross X. So, this is a similar definition, we have the product to be transitive. We say the system to be weakly mixing. Look back into our open side definition because that is what we had come up to. So, what does that mean in the product system?

So, this also means that for U 1, U 2, V 1, V 2 and non-empty open subsets, right. In X there exists an integer n E N such that fn of U 1 cross V 1 intersection U 2 cross V 2 is non-empty or in other words I can say that I fn of U 1 intersection U 2 is nonempty. And fn of V 1 intersection V 2 is non-empty. So, we find that if I take 2 pairs, right. Of open sets then this happens to be non-empty.

So, this is our property of weakly mixing, but we are not satisfied with this property we want something more. So, we try to look into some stronger property here and that stronger property can be given as. So, x f is called now I want to write here strongly, right. Although in general the word strongly is generally omitted. So, this is called strongly mixing or just mixing, if for given pair non-empty open U and V, right. Subsets of x there existed N belonging to N such that fn U intersection V is non-empty, right for every n greater than this particular N.

So; that means, that from some point and onwards, you will always find n a point in U such that it is nth orbit lies in V. So, there is some point definitely and U which starts going, right. Whose orbit starts hitting V after some more. So, we find this system. So, this system is something which we call as strongly mixing. Now what is this strongly mixing system? If we try to look into this fact here, look into the fact. Supposing we have pairs U 1 and U 1, U 2, V 1, V 2. Supposing we have these pairs of open sets, we have a strongly mixing system. Then we know that after some time we can find n 1, right. Such that f n of U 1 intersection U to write is non-empty; for all n greater than n 1 and again, we can find them n to such that fn of V 1, right. Intersection V 2 is non-empty for all n greater than n 2.

So, if I take in general something to be greater than n 1 and n 2 maximum of n 1 and n 2. Then we find that give have an n such fn of U 1 intersection U 2 is non-empty and fn of V 1 intersection V 2 is non-empty. So, what does that mean? Our strongly transitive system will our strong our strongly mixing system will be weakly mixing also. Now look into the fact what is the meaning of weakly mixing. The weakly mixings means that your product x cross X f cross f is transitive. Now if I look into my system x fi can always realize my system x f to be a factor of the system just take the projection right. So, the system x f is always a factor of the product system, right.

So, what we have here is we know that if the product system is transitive any sorry, if the system is transitive any of it is factor is going to be transitive. And hence if the system is weakly mixing, right. Then that would mean that your system is transitive.

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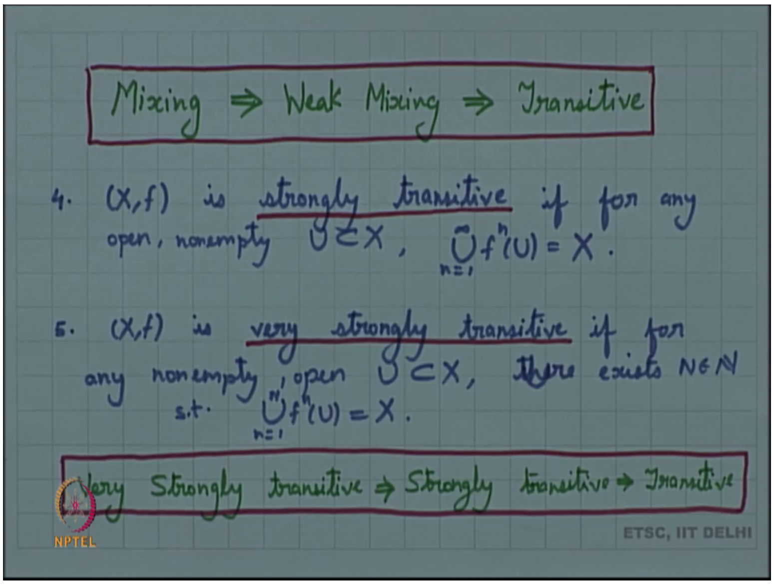

So, we can observe from here that mixing implies weak mixing and weak mixing implies transitivity. So, in general this is a bigger system right. So, we find that this particular system is a bigger system. So, we have taken care of the product case now. And we have 2 stronger versions of transitivity here.

Now, what we are interested in looking into us, we are interested in now that open set condition. So, we try to see the 4th definition. So, we say that x f is strongly transitive I am sorry, we do not have any words here. So, known you words here. We call it strongly transitive, if for any open non-empty U in X. This union in going from 1 to infinity f n U is equal to x, if the union itself instead of being dense. If the union itself becomes equal to x, then we say that the system is strongly transitive.

Now, I am not satisfied with that. Because y should find images, right. Fnu now one thing we note that here, that my U is open fu need not be open. So, this is not all our union of open sets because continuous image of open sets need not be open. So, this is not all union of open sets, but what if I am interested in not taking all this infinitely many, right. I am just interested in the finite case. So, we are interested in something still stronger. And again, we have a loss of proper words here. So, we say that x f is very strongly transitive if for any non-empty open U subset of x. There exists in n E N such that union n going from one to n fnu is equal to x.

So, this gives me something which is called very strongly transitive. Let us very easy to see that if the system is very strongly transitive it is also going to be strongly transitive, right. And if your system is strongly transitivity is definitely going to be transitive because this is whole equal to u. So, definitely it is dense right.

So, what in general we have here is that our very strongly transitive, implies strongly transitive, implies transitive. And we have this new characterization here. Now we observe that we have here mixing, right and if we recall the definition of mixing. The definition of mixing said that fnu was always intersecting V. So, whenever you have a pair U and V you get an n such that fnu is intersecting V for all n greater than N right.

So, we are interested in something more stronger than this part, and we also have. So, we are interested in something more stronger, then this very strongly transitive part that we have taken up. So, we define another definition. We take up another definition here. And I write the definition may be the 6th definition here.

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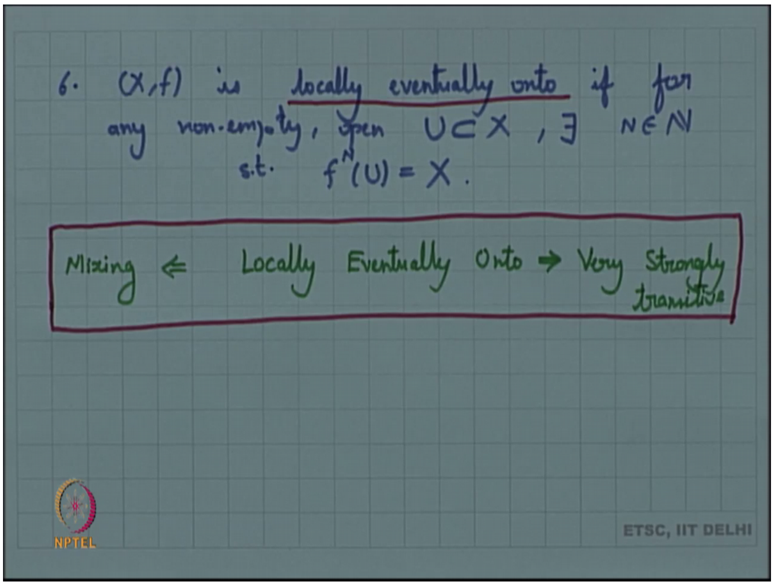

So, I am saying that x f is locally eventually on to; if for any non-empty open U subset of x, there exist some n in n such that f n of U the n th image of U itself is equal to x.

So, it is basically locally; that means, locally on any U write becomes eventually onto the system becomes eventually on to. So, this is called locally eventually on to. So, my system is locally eventually onto if this particular property holds true. And now if we try to see my locally eventually on to you find that locally eventually on to this f n U is equal to x right.

So, it will definitely imply my very strongly transitive, right. If we recall the definition of very strongly transitive right; that means, the union of finite images is whole of x, but since the one the nth image itself turns out to be whole of x. So, definitely image fine it becomes. So, locally eventually on to, right it implies very strongly transitive. So, what we find here is that locally eventually on to. So, if I say what is might this is my locally eventually on to, it implies very strongly transitive.

Now let us again look back to the definition of your local eventually on 2. It says that fn of U, right. When there is a n such that fn of U is equal to x. Now since fn of U is equal to x right; that means, if I take any open set V, right. This fn of U is going to intersect that. Now if I take any n to be greater than n or any k to be k greater than n then definitely my fk of U is also going to be x. Because x is my whole system, right. So; that means, I can say that if the system is locally eventually on to; it means that for all n greater than n, right. Fn of U is equal to x. So, that means, if I take any open set V, right. Then that would mean that my fn U intersect V for all n greater than n, which also means that my system should be mixing.

So, if my system is locally eventually on to it is also mixing. So, this also implies mixing. So, I have another characterization here of strongness that this basically locally eventually on to implies mixing. And it also implies very strongly transitive. Now how was this related to chaos? So, will try to all see these properties, the chaotic properties of this transitivity. One thing is clear if you would feel that maybe this is like a lot of definitions we have not looked into examples here.

Because we will try to take up 1 by 1 each property. And then try to see some examples and say some related properties and how they are related to chaos. Especially we are interested in not all properties of that particular say characteristic, but we are interested in some chaotic properties of each characteristic. So, will look into that aspect, will look into examples at that point of time. But I would like to just recall one example to say that yes, we do have systems which are locally eventually on to. Say for example, take the tent map right.

We have already seen this we have already seen this idea that what happens in the tent map write you take any open set U. Then you find that this U expanse in such a manner that one point of U will be taking the value at some n, right. You will find iterate n such that at that particular value n this U 1 point will take the value 1 and one point will take the value 0 right.

So, what happens is at that particular n? This whole U, right is expanding to become 0 1. So, fnu happens to be equal to 0 1. So, our tent map is a system, right. Was tent map is a very nice system which happens to be locally eventually on to and since it is locally eventually on to. It is having all the other characteristic, right. It is very strongly transitive it is strongly transitive it is mixing it is weak mixing it is everything. So, this is one aspect that we have seen. So, today we have just seen the definitions and will later on see the chaotic properties of each of these systems. So, I hope this is clear to all of you.