Chaotic Dynamical Systems
Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture – 15
Transitivity

Welcome to students, today we will be looking into this concept of transitivity. Now what is transitivity if we try to look into that among all the properties which gives us chaos transitivity turns out to be the strongest property.

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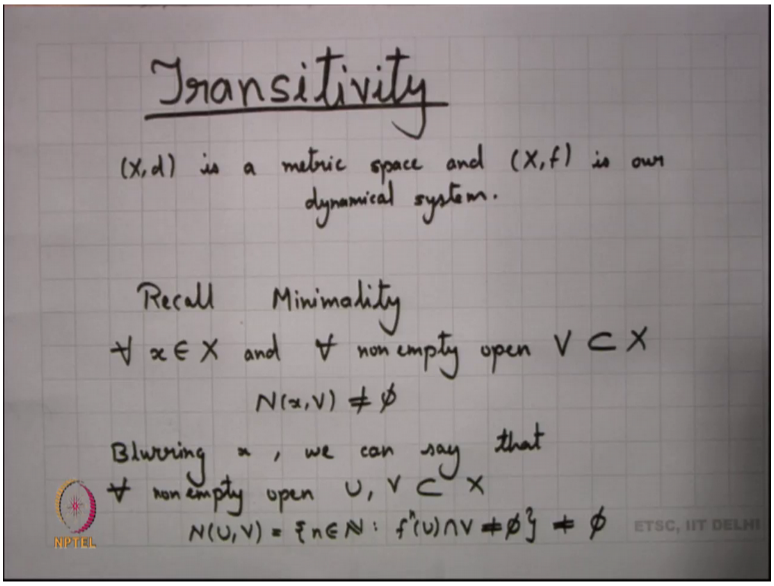

So, many times many other definitions of chaos are implied by this particular property and this is a global property for the system. So, as a motivation for transitivity, let us now see what we are trying to look into. So, for us right now we have this as a metric space our x, d is a metric space and this pair x, f is a dynamical system.

Now, let us try to recall something which we have already studied and so, we recall this concept of minimality; now what is minimality? As you all seen what does that mean that whenever I have an x in x. So, I take for every x in X right and for every non empty open V subset of X, I have this hitting time set N xv right to be non-empty. Minimality basically means that every orbit every point has a dense orbit right since every point as a dense orbit; that means, the orbit of x right hits every non empty open set v. So, we get Nxv to be non-empty.

Now, we want to actually sort of look into a little bit generalization or abstraction of this concept. So, if we try to generalize this concept or maybe I am trying to approximate this concept, supposing now we are not looking into our point x, but instead of looking into a point x, we want to look into say some neighborhood of x. So, we are saying that we have blunt the point x and the blurring of the point x is coming out to be the neighborhood say U of x. Now what happens when we are looking into the neighborhood U of x, we again want that the neighborhood the orbit of the neighborhood tries to hit every open set or maybe the orbit of the neighborhood is dense.

So, try to if will try to blur this out right. So, I am just trying to blur it up, we can say that for every non empty open U and V. So, I have 2 pairs of open sets now U and V, for every non empty open U and V subset of X, I want N U V the hitting time set which basically turns out to be the set of all n in N such that fn U intersection V is non-empty I want this particular hitting set to be non empty. So, we are just trying to sort of blur out the concept of minimality right we just trying to approximate the concept of minimality and the approximation of this concept of minimality results into this particular property that we have the hitting time set N U V to be non-empty and this is something which we call topological transitivity.

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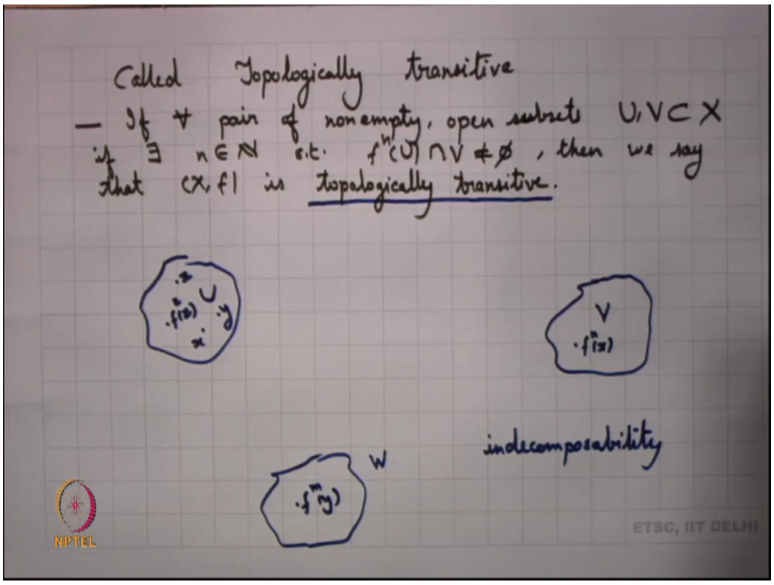

So, this concept is what is called topological. So, let us try to take the formal definition here. So, if for every pair of non-empty open say subsets U and V of X, if there exists an n in N such that fn U intersection V is non-empty then we say that the system x f is topologically transitive.

Now, as you would have guessed it right that this property turns out to be a global property, like you cannot say that it happens at this particular region of x or it happens at that particular region of x, this is a global property and if I try to see that I am having some open set U here right I have another open set V here right this says that there is an n for which fn of U intersects v; that means, that U has some particular point x right for which fn of x is in V right.

Now I could very well think of my pair of open sets to be unw where my w is somewhere not near V at all its W here right then that would guarantee that for this pair U and W, I should have some m for which fm of U intersects w is non-empty; that means, I should have some point y here right for which fm of y should be here right and I could think of the pair U with itself right once we think of the pair U with itself, then I can say that there should be some k such that fk of U intersection U is non-empty; that means, that I should have some point z here right such that for which my fk of z is also there in U.

So, if I try to look into this particular property right this particular property and this happens for every open uv right every pair of open sets uv in the space. So, this property gives some kind of in decomposability of the system. I cannot divide the system into 2 parts right such that one part is not interacting with the other part right this property says that every part of x interacts with every other part of x. In fact, from every part of x there is a point right whose orbit is going to some other part to every other part of x. It need not been now its quite possible that the point that visits uv need not be the same as the point that visits w right it could be distinct, but it says that yes every part intersects interacts with every other part.

So, this is our concept of topological transitive and this has been studied in various names. In fact, this was first studied by Birkhoff.

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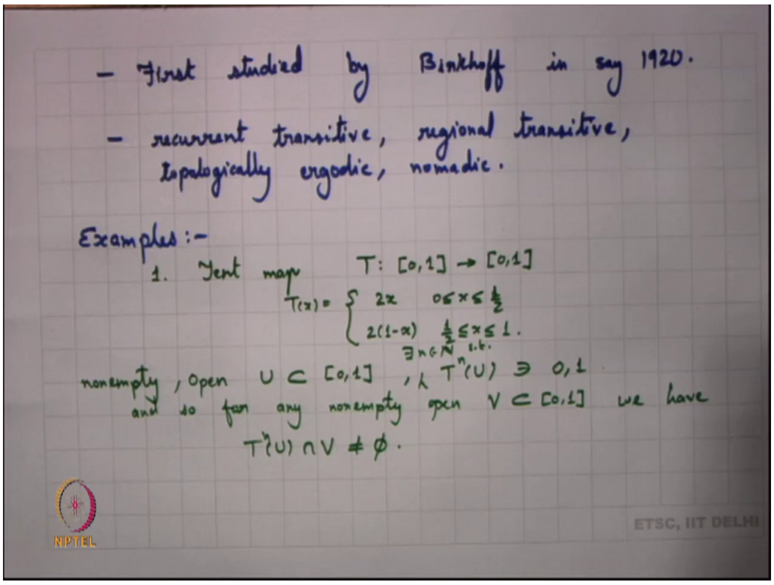

I think we all know who Birkhoff is right we have all studied how he was the student of poincare right basically he followed poincare and he has done a lot of work in the like initiated the work of discrete dynamical system. So, this was first studied by Birkhoff in say 1920 and this property has lot of other names also. So, there are other words also which describe the same thing. So, many people use the word recurrent transitive for it. So, this is called recurrent transitive and you will know why we call it recurrent transitive soon.

So, this is also called regional transitive, there is a concept of ergodicity as seen in terms of measure theory and many call it topologically ergodic its very rare, but in past people did use this word nomadic for this purpose, which describes the nomadic nature of orbits here. So, this concept is also called nomadic. Now we are interested in looking into this global aspects of this property and let us start with looking into some examples.

So, let us try to look into some examples here, now the first example that I can think of we are not going to describe new examples here, but let us try to look into some examples that we already have seen earlier. So, we look into this example of tent map; that means, I am looking into the map T from 0 1 to 0 1 right described as Tx equal to twice x for 0 less than or equal to x less than or equal to half, and it is twice 1 minus x for half less than or equal to x less than or equal to 1.

Now, we I am also we have studied this particular map and after studying this particular map we seen we actually we have seen this concept that there exist an n right. So, you I start with any open set U of course, I want it non empty. So, let me say that this is non empty right, we have seen that what happens is that this U expands right under the action of f under the action of T it expands such that Tn of U contains both right. So, Tn of U contains both it contains 0 and it contains one right we have already seen this in terms of periodic points, but this is the concept that we are looking into.

So, for given any non-empty open U right there exist an n. So, I should say that there exist an n in N right such that Tn of U contains both the points 0 and 1. Now since it contains both the points 0 and 1 right I can say that it would basically be containing the entire stuff here right and. So, for any non-empty open V subset of 0 1 right we have Tn U intersection V is non-empty so; that means, the tent map is open and there is another example which again we have seen that part, but I would like to look into this topology of this once again.

So, the second example that we have studied is the example of the shift space.

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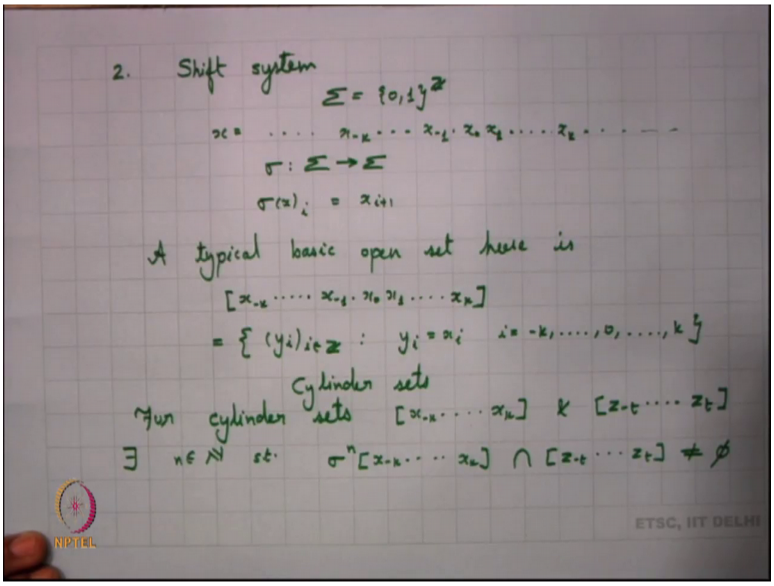

So, I should say the shift system; what is our shift system let me describe it again. So, this is our bigger sigma which happened to be 0 1 to the power z right with the product topology and we know that we have a metric over here right which says that basically which says that 2 things are very close by right if the middle portion right if sequences they coincide. So, typical sequence here is of the form and we studied the map over here right there were shift was basically this map right choose the right shift which can be defined as sigma x of I is basically xi plus 1.

Now, we have we have seen that in this product topology we said that. So, that 2 neighborhoods are close by right. Now basically I can say that 2 points are close by right if they agree on the middle part. Now what are the basic open sets or what are the open sets here in shift. So, since this is product topology we can think of basic open sets here. So, you can say that a typical basic open set. So, I am writing this in terms of course, I am expressing this as a cylinder, all sequences right I can think of all sequences yi i belongs to z such that. So, basically in this portion yi is same as xi. So, it takes the same value as this xi right otherwise I am looking into all those sequences which satisfy this property.

So, this is a typical basic open set here and we call this as a cylinder. So, these are basically called cylinder sets. Our property of transitivity says that you take any 2 pair of non-empty open sets, there should be an n large enough such that one intersects such that n some n such that after some time or basically the n th image of the first open set intersects the second open set.

So, as well this could be true for a basic open set also. So, we say that here look into the shift system, supposing now I am fixing this up right. So, what is this set; that means, now what is the restriction on yi. So, if I am looking into this set the restriction on yi is that only in this portion it should be same as xi right in the rest it can be anything right. So, if we if I if I want to see what happens to the left of it and what happens to the right of it, it could be anything right the sequences could be anything right, but only in the middle portion it should be same as this part.

So, let me we take 2 basic 2 cylinder sets. So, for cylinder sets I am naming the cylinder sets. So, I am naming the cylinder sets x minus k up to xk and z minus t up to zt right naming the cylinder sets. So, what happens for these 2 cylinder sets? Now I know that this zt z minus t zt need not overlap with this at all even if it overlaps right we do not care about it just does not overlap with it at all let us assume that part then what happens here is. I am looking into a typical sequence which belongs to the cylinder set right then what happens is after certain times right it could assume any value. So, it could also assume the values z minus tz t somewhere.

So, for some say n greater than k, because I am shifting this midpoint right. So, for some n greater than k I could have that this particular thing this, but there is a sequence here right whose middle portion is this part, but after n instance right if I look into the n th image of it then its middle portion would be same as this point and that n could be not greater than k, but I wanted greater than k plus t right. So, for some n greater than k plus t because if am considering them to be non overlapping right then for some n greater than k plus t right I could have that this there is a sequence right which agrees on this in the middle part, but after n instances n it agrees with this particular thing on the middle part.

So, there exist in n in n such that sigma to the power n of this cylinder set minus kk right intersection is non-empty right and. So, this system is also topologically transitive I would like to give one more example here which turns out to be a very simple example, supposing I take my x to be just these points x 10 right and the map is such that f of xi.

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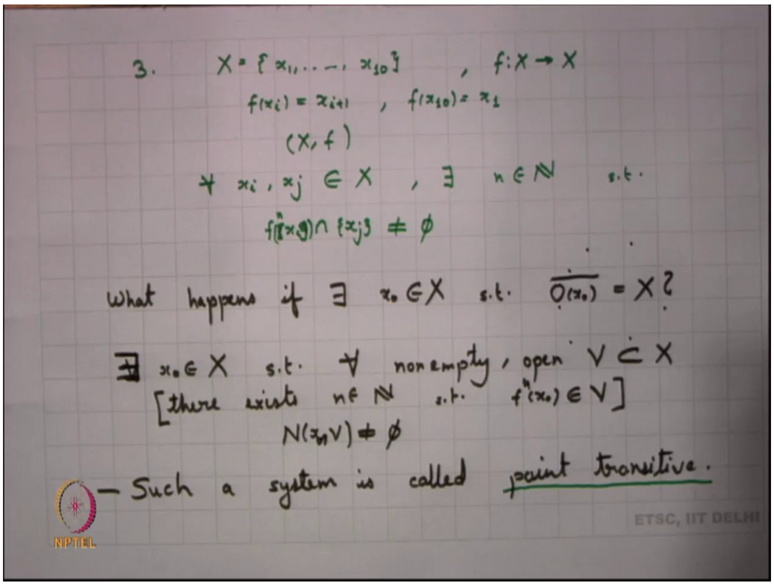

So, I am defining this map from X to X as f of xi is equal to xi plus 1, it will be true for everything other than I equal to 10.

So, we define that f of x 10 is x 1. Now think of this this is just a periodic orbit if I look into the system X f its just a periodic orbit and since this is a finite set right I can think of this to have the discrete metric right. So, every singleton is open here, can I say that for every xi and every xj right there exists an n in N such that now I am looking into this part fi of this singleton xi intersection this singleton xj right is non empty so; that means, this is also a topologically transitive system. So, any periodic orbit you just have an orbit right that turns out to be topologically transitive. After looking into this example and especially this example gives us a motivation what is this example doing right my x 1 is moving to x 2, my x 2 is moving to x 3 moving to x 10 back to x 1 and so on.

So, it is just moving in some kind of n orbit right it is just moving in an orbit. Now I could have something moving in an orbit right even for an infinite set need not be a finite set. So, that gives some kind of a motivation right what happens if there exists an x. So, what happens some x naught in X right such that orbit of x naught I am looking into its closure happens to be whole of x, what happens in that particular case. So, my orbit of x naught is same as x; that means, what I am looking into the fact here is that for every x naught in. So, there exists sorry not for every there exists an x naught in X such that for every non empty open V contained in X right ok.

So, there is some x naught in x such that for every non empty open V in X right there exist an n in N right such that fn of x naught belongs to V right or I can replace this statement by saying that N of xv the setting time set is non empty. So, sorry x naught V is non-empty so; that means, that this is another aspect now if such a thing happens right we call the system to be point transitive.

So, such a system is called point transitive, now we would like to see what happens is there any relation between point transitive and topological transitive. So, let us try to look into that aspect here. So, I have point transitive on one hand.

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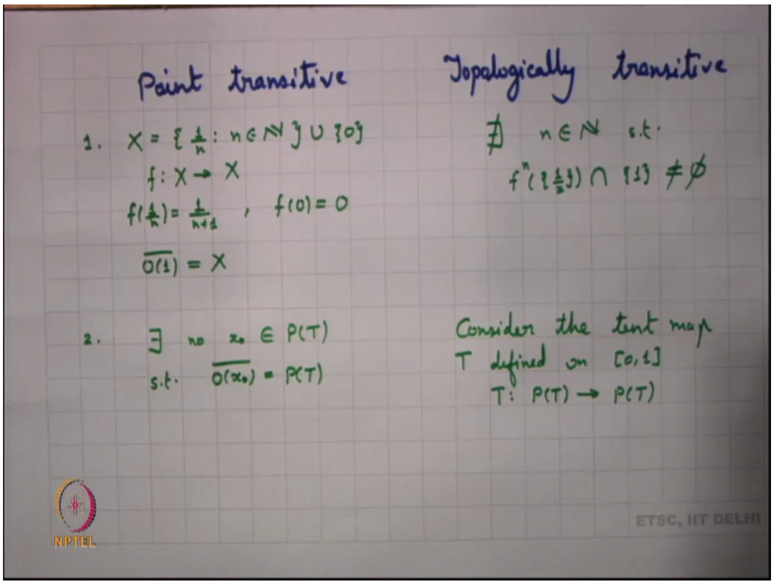

And on the other hand I have topologically transitive, is there any relation between the Y. So, let me take an example here say I look into this particular example. So, I think we have perhaps seen this example earlier also. So, my x happens to be the set of all 1 upon n such that n belongs to N right union 0 and I define my f from x to x right such that what is f of 1 upon n is nothing, but 1 upon n plus 1 and f of 0 is 0.

So, it look into typically I look into this particular example right then what we find here is that the orbit of 1 if I take its closure is whole of x so; that means, this particular example is point transitive. Now is it topologically transitive? So, we find that there exists no n this system X f is point transitive, but it fails to be topologically transitive. On the other hand let me take another example here. So, other example here I can take off to be say again I am going back to my tent map right let us go back to the tent map and let consider the tent map.

Now, I am taking the restriction of this tent map because I know that tent map on T has a dense set it has lot of periodic points right it has periodic points of all periods. So, let us take T restricted to. So, I am taking this T restricted to periodic points of T. Now let us look into this particular system, now we know that right we know that in tent map right given any open set q right. You start with some periodic point at one stage right you will be able to get a periodic point which reaches any other neighborhood.

So, this system is topologically transitive I am restricting it to the periodic points I am not looking into the other points just take care of all the periodic points. So, supposing T is defined on only the periodic points here, then my system is this is topologically transitive right, but have an dense orbit is there any point which has a dense orbit no all the all orbits here are finite right and none of them will be the whole of P T because we know that, this is happens to be an infinite set right.

So, none of them its closure can be whole of P T right. So, we find that this is not topologically. So, there exists no x naught in P T that is the periodic points of T such that orbit of x naught closure is whole of P T. So that means, that these 2 concepts though we say that of course, we are calling them transitive right they are distinct concepts. There are systems which are point transitive, but there are not topologically transitive, there are systems which are topologically transitive, but not point transitive.

So, let us try to see can there be any relation here. So, let us try to now look into more examples of point transitive system, we have not looked into the examples of point transitive system. So, let us go back to these examples once again or maybe I will push up another example here. So, let us look back 2 examples once again.

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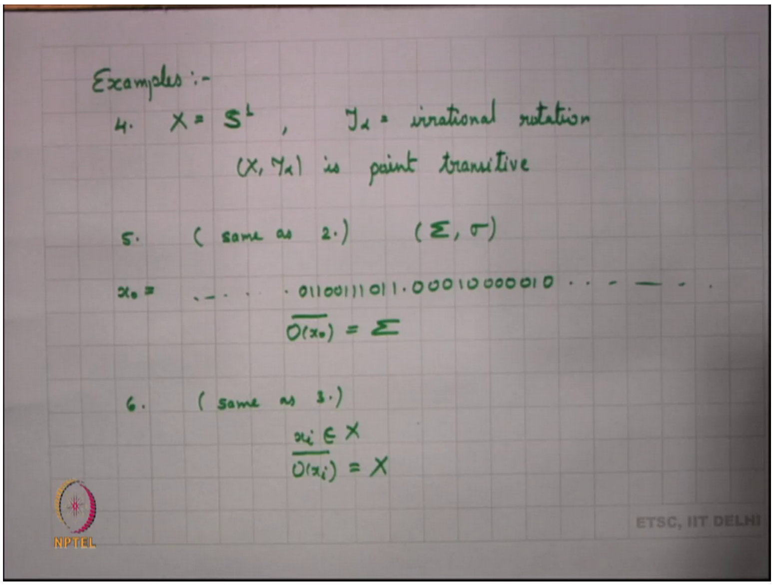

Let me call this example 4 right let me take my x to be S 1 right a unit circle and my tau alpha right to be the irrational rotation.

Now, we know the system is minimal right minimal means there for every x the orbit of x is so; that means, there is at least we have one x right. So, this system is point transitive right. So, this is point transitive, what are the other examples that you can think of. So, let me go back to example 5 which is same as the example 2 that is we have seen, that was our shift space right. So, we are going to this example once again, and we have already seen that in this particular example right there exist a point x naught right which happens to be; what did we start with right. We have already seen that we have a point x naught here such that if I look into the orbit of x naught closure right this the whole of sigma because every block occurs here at some particular point right and so, this has a dense orbit.

So, this shift space again happens to be point transitive and again if I look into my sixth example which is same as example 3 we considered. So, our example 3 was a set of 10 points right which is basically just a periodic orbit. And now we know that just a periodic orbit is a minimal set just take a may periodic orbit it happens to be a minimal set right. So, this is also what I know is that I take any xi belonging to x right, then we can say that the orbit of xi right. In fact, the orbit itself is the whole of x right the orbit is whole of x. So, the orbit closure is also whole of x right.

So, this is again system of point transitive. So, we do have examples where we have this concept of topologically transitive right coinciding with point transitive. So, is there any general rule which says that the 2 concepts can be equal or are there any circumstances are there any conditions under which the 2 concepts can be considered to be equivalent and for this we have the theorem. So, let us look into this theorem now.

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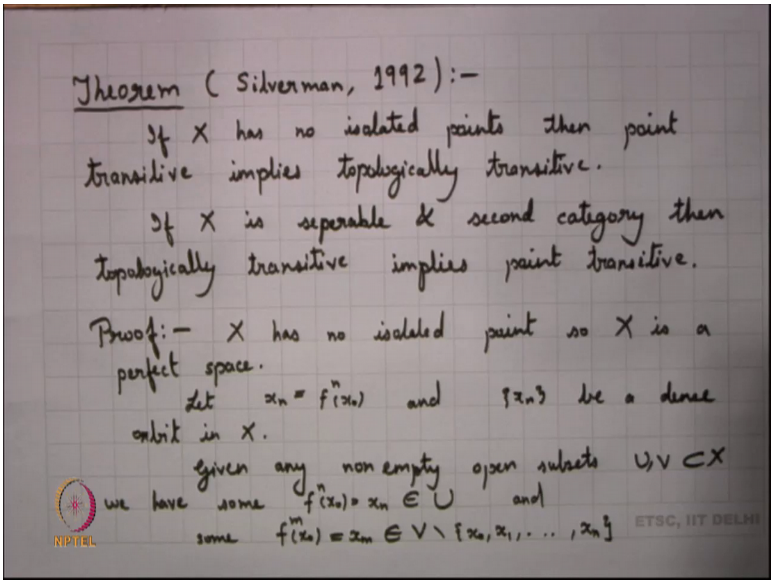

When this theorem is due to silver man proved in 1992, what does this theorem say?

So, this says that if x has no isolated points, then point transitive implies topologically transitive. On the other hand if x is separable and second category, then topologically transitive implies point transitive why is this theorem important because in many cases right it is just enough for us to look into some particular point, which goes from one region whose orbit goes from one region to the other region right.

So, that is enough for us many times, but why is it important to look into point transitivity because point transitivity would also say that fine what happens is if your point whose orbit is range happens to be isolated right, then you can never come back to itself. So, you do not get this concept of coming back to itself or this concept of recurrence, which comes up which is usually seen in topologically transitive. So, that is not seen over here. Whereas, what happens in the other cases many times it is enough for us to identify a point whose orbit is dense. Once you identify whose point is orbit is dense it is enough.

So, for the concept for proving theorems in topologically transitive or anything that leads to transitivity it is enough, it becomes very helpful for us to look into the concepts and if this 2 concepts are equivalent. So, this is basically what we have as theorem and let us now look into the proof of this part. So, the proof here for the first case happens to be very simple right.

So, now my x has no isolated point what does that mean. So, x has no isolated point; that means, that all points of x are limit points right. So, x is a perfect space right let me say this since the system is point transitive, let me say that let xn be say f to the power n of x naught right and this sequence xn right be a dense orbit. So, I my system is point transitive and also I know that my x is a perfect space. So, I am starting with a dense orbit in x, but then since the orbit is dense right given any open non empty open. So, given any subsets U and V of X we have some say x fn of x naught which is basically my xn right belonging to U, because this is a dense orbit xn is a dense orbit right. So, this is a dense set.

So, in U there will be some xn and some fm of x naught which is my xm belonging to V I am not taking just V. I am taking V minus the set x0 x 1 so; that means, that whenever I take open sets U and V right since this is a dense orbit there is some n for which fn xn belongs to U and there is some m for which xn belongs to V minus these 2 points. So, one thing is sure right we can say that our m has to be greater than n.

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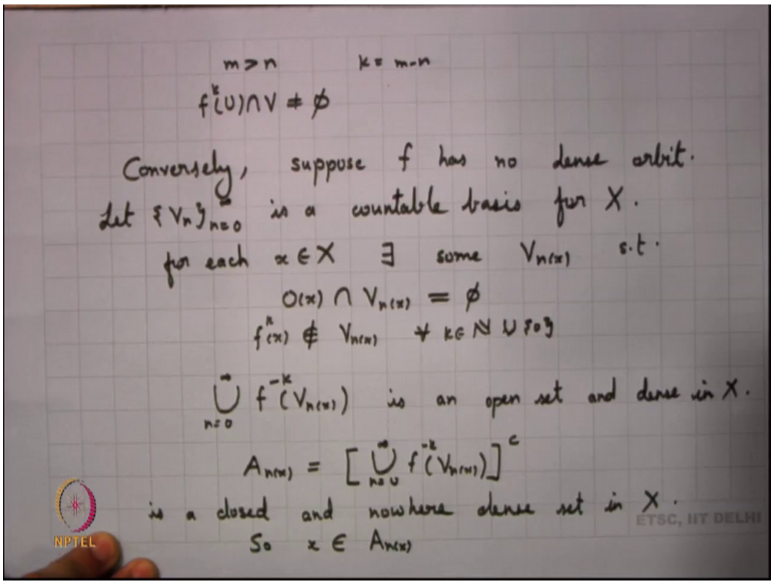

Now, m is greater than n. So, I can define my k to be equal to m minus n; then what do we know here is fk of U intersection V is non-empty.

So, given any 2 non empty open sets U and V right I do get a k in n, such that fk of U intersection V is non-empty. So, the system happens to be topologically transitive. Let us now look into converse part now my system is topologically transitive, and I just suppose that f has no dense orbit. Now my system is separable, it should have a countable basis. So, let my Vn where n goes from say I am talking of 0 to infinity right is a countable basis my f has no dense orbit right.

So, for each x in x. So, I am starting with an x in X I know that the orbit of x is not dense right since the orbit of x is not dense, there is some basis where which does not contain any points in the orbit of x right. So, there exist some Vn and x right there exists some now indexing this my x there exists some Vn x such that the orbit of x intersection Vn x is an empty set right.

The orbit does not come up to be an x at all right there is no point of the orbit of x which comes up to Vn x or which or there is no fk of x right which is lying in vnx; that means, that fk of x does not belong to Vn x right for every k in n does not belong to Vn x at all and I can say that n union 0, I can always choose this part that my x is also away from that part right.

So; that means, this never turns out to be this factor. Now what happens to my I am now looking into Vn x. Now this Vn x exists for every x in x right. So, now, what happens here is I am looking into this set f minus k of Vn x. So, I choose this Vn x I take f minus k of Vn x and I am looking into this union as n goes from 0 to infinity. My system is topologically transitive. So, I start with any open set right there will exist some n for which fn of that open set intersects Vn x. So, if I look into this particular set right this particular set is an open set which is dense in x right.

So, this is an open set this is an open dense set of x, because my f happens to be topologically transitive so; that means, that if n or maybe some fk of any open set will intersect this part will intersect Vn x. So, this turns out to be an open and dense set in x and now I want to look into I am defining another set An x to be the complement of this set. So, I am taking this union n going from 0 to infinity f minus k of Vn x I am taking the complement of this set.

Now, we know that the complement of the set will be closed nowhere dense right. So, this An x is this complement, and this is a closed and nowhere dense set in x right. Now interestingly the way we have defined Vn x right we had defined Vn x such that fk of x does not belong to Vn x for every k in n union 0. So, basically we want our point x also to be not in Vn x.

Since x is not there in Vn x right no point of the orbit of x is in Vn x so; that means, my x should belong to the complement of this union, and hence I can say that x belongs to An x for every x in x right. So, x belongs to and this is true for every x in x right. So, what do we say what do we get here.

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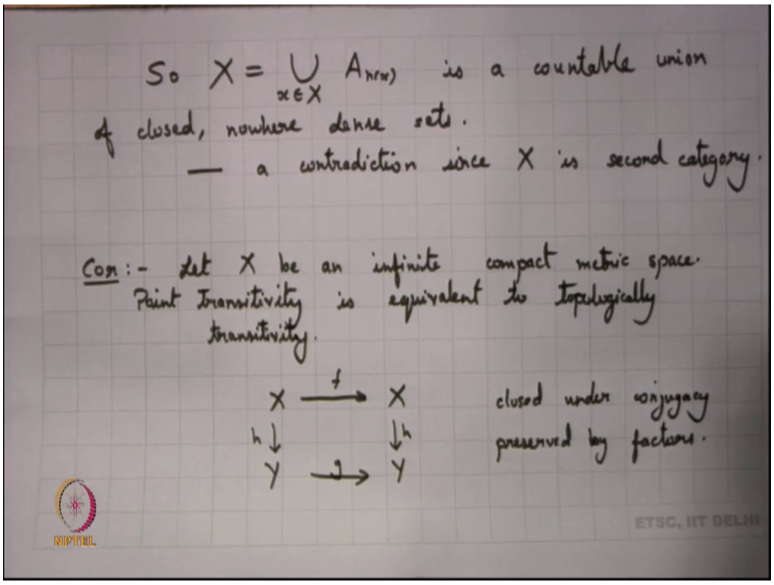

I can say my x is the union of all x in x An x right because each x belongs to An x right so; that means, x is the union of closed.

Student: No.

Nowhere dense sets in x, but what is my x second category right x is second category. So, x cannot be written as a union of close nowhere dense sets right and these are all and what is the Cn x the Cn x turns out to be countable because we started with the countable basis right. So, each Vn x right when we start with the Vn x, these are all countable basis. So, this is also countable. So, this is a countable union I can say that x is a countable union which is basically a contradiction right since my x is second category right. So, this is a contradiction and what does that mean? We came to a contradiction why did we come to a contradiction, because we assume that there are no dense orbit right f has no dense orbit and; that means, that there exist a dense orbit right and so my system is point transitive.

So, point transitive. So, now, if we look into this corollary, let me look into a corollary to this one let X be a compact let X be an infinite compact metric space. In fact, we do not need infinite here because for finite case also this would work out let x be an infinite compact metric space, then we know that a compact metric space right a second category since I am talking of a metric space I can think of that to be separable right. So, when I am looking into a infinite compact metric space right point transitivity is equivalent to topologically transitive. Point transitivity and topologically transitivity these are 2 equivalent concepts here.

Now, there is something more that we can think of transitivity and that is what happens under conjugacy. So, if we try to look into conjugacy supposing I have 2 systems right, I have my system X f that is one system, I have another system Y g right I have another system. Supposing now I have a homeomorphism here h what happens in that case? I can say that if x is transitive right if the system is transitive, X f is transitive then the system Y g will also be topologically transitive I am just looking into the definition right.

So; that means, that my transitivity is a dynamical property because it is preserved by conjugacy. So, if my h is a conjugacy right then the transitivity of the system X f right is equivalent to the transitivity of the system yg. On the other hand supposing h is not a conjugacy supposing it is just a factor map right. So, my Y g happens to be just a factor of X f what happens in that particular case. So, then in that particular case what we have here is that the transitivity of Y g will imply the transitivity of X f will imply the transitivity of Y g, but we are not sure because then my X f is happens to be an extension of Y g we are not sure whether the transitivity of Y g will imply the transitivity of X f.

So, topologically transitivity is basically closed under conjugation and it is also preserved by factors right. So, if X f is now topologically transitive it its factor will also be topologically transitive. So, it is preserved by factors. So, this is closed under conjugacy and it is preserved by factors. And then there are some equivalent conditions of transitivity. So, I am stating that as a theorem here.

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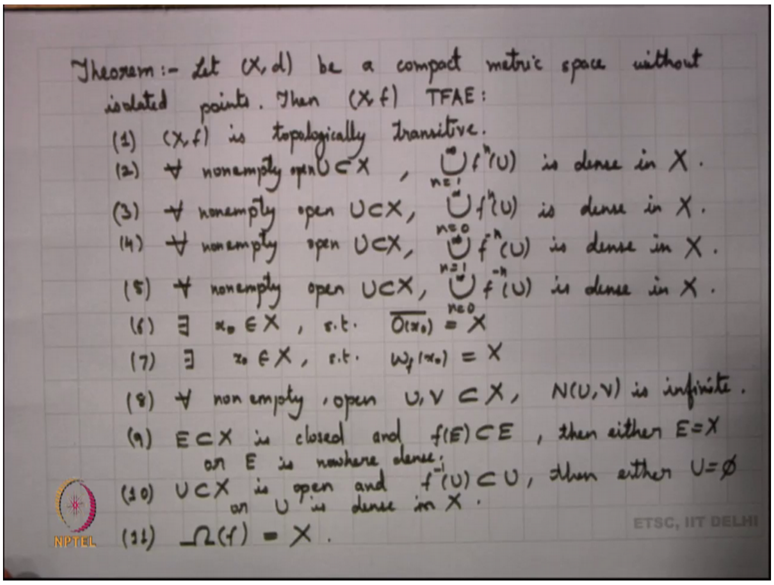

So, let me take X d to be a compact metric space without isolated points, then from a system X f the following are equivalent and what are this equivalent properties.

So, my first thing is that this system is topologically transitive right this would say that for every non empty U open U subset of X, If I take this union n going from one to infinity fnu. So, this is dense in x. In fact, I can restate this by saying that for every non empty open U subset of X, I am taking my n going from 0 to infinity fnu means I am include in my U also inside then this is dense in X.

The equivalent concept here and fourth property here goes that for every non empty open U subset of X, I am taking my n going from 1 to infinity and here I am taking f minus nu right then this is also dense in x, my fifth property is just a restatement here for every non empty open U subset of x, I am taking this n going from 0 to infinity f minus nu this is dense in x. My sixth property says that there exist in x naught in x right such that orbit of x naught closure is whole of x point transitive. My seventh property which is coming up from this part says that there exists an x naught in x such that now I am looking into the omega limit set of x naught that omega limit set is whole of x.

My eighth property which basically comes up from all these facts right says that for every non empty open U and V contained in x the hitting time sets N U V is infinite. My ninth property again comes from this aspect only it says that if I have E to be a subset of

X is closed and f of E is a subset of E; that means, E is invariant then either E is the whole of X or E is nowhere dense.

And the tenth property which again comes from this fact only says that if my U is a subset of X is open and is negatively invariant I have f inverse U to be contained in U then either my U is empty or U is dense in X its an open dense set in x. And my eleventh property which follows from all this right is that if I look into the non-wandering sets of f right this is basically non wandering points of X this is the whole of X right the non wandering set is whole of x.

So, I think there is no more space to keep all the 12, 13, 14, 15 properties, but we have this equivalent properties and I am leaving this as an exercise for you. So, today we stop here.