Chaotic Dynamical Systems
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Lecture – 13
Recurrence

Welcome to students. So, today we will be looking into the concept of Recurrence. Now we are basically interested to understand the chaos underlying within dynamical systems, and there are many related terms there are many dynamic terms which are important which play a role in trying to recognize the chaotic properties of the system. So, what we are today going to look into some of these underlying properties.

Now, recurrence means reoccur. So, we are looking into how the orbits can reoccur at in certain circumstances or at certain places. So, we want to look into that concept and the first thing that we can see over here is the concept of limits, limits of the orbits. So, we start here today, when we start this part our x will always be a metric space.

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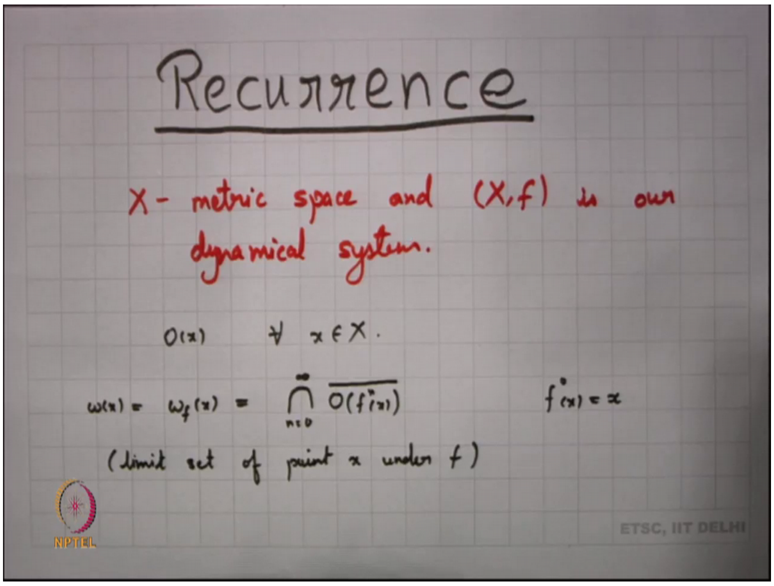

And for us our dynamical system is the pair x f. Now as we all know we are always interested in looking into the orbits of x, right for every x in x. Now one of the ways of studying the orbits is to study what are the limit points of the orbits.

So, we start looking into so, we define a concept here. I am saying that this is my omega f x which essentially means that I am looking into the orbit of all fnx. In fact, not orbit I am looking into the orbit closure of all fnx. And I am taking their intersection as n varies from 0 to infinity, when we take the convention that f 0 of x is x. So, basically f to the power 0 is the identity. Now this is called the limit set of the point x under f, and if the f is clear we drop the f from here. And we say that this is just omega x, right. Which is the limit set of x and that is basically the intersection of all orbit the intersection of the orbit closure of all points in the orbit.

So, with this definition, we try to look into some kind of examples. So, let us look into some examples here.

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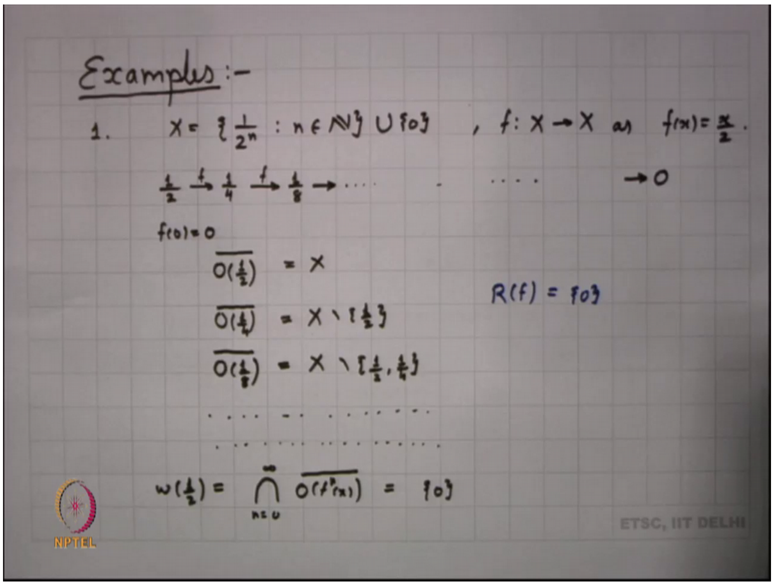

Now, the first example that I would consider here is my x to be equal to the set of all 1 upon 2 to the power n such that n belongs to N union 0. Now in that case my space is a compact metric space, and I define my f from x to x s f of x equal to x by 2. So, this gives us a dynamical system. And if I try to look into this system, what we find here is; that in this particular example in this particular system my 1 by 2 is being mapped under f 2 1 by 4, 1 by 4 is being mapped under f 2 1 by 8, right and so on.

What happens here is that ultimately this converges to 0, my 0 is a fixed point here right. So, what we have here is f of 0 is equal to 0. So, 0 is a fixed point here and if I look into now my orbit of f. So, if I look into what is my orbit of half. So, my orbit of half consist of all this x except the point 0. So, if I look into the orbit closure of half this is the whole space x. If I look into now, what is f of half, then f of half happens to be 1 by 4. So now, looking into the orbit closure of 1 by 4, and if we look into the orbit closure of 1 by 4, I retrieve all the points of x except the point half right.

So, I get this to be x minus half the singleton half. Now if I look into f square of 1 by 2; that happens to be 1 by 8. So, if I look into the orbit closure of 1 by 8, I basically retrieve the whole of x back excepting for 2 points 1 by 2 and 1 by 4. So, I can continued in this manner, right finding the orbit closures of all points in the orbit of 1 by 2. So, ultimately what I get is what is the limit set of half. So, the limit set of half is nothing but yes, as we have defined it is an intersection of n going from 0 to infinity orbit of fnx whole closure and that turns out to be this is the intersection. So, you get the singleton 0 here.

So, the limits at here is only singleton 0. Let us look into another example. So, we start with our second example here.

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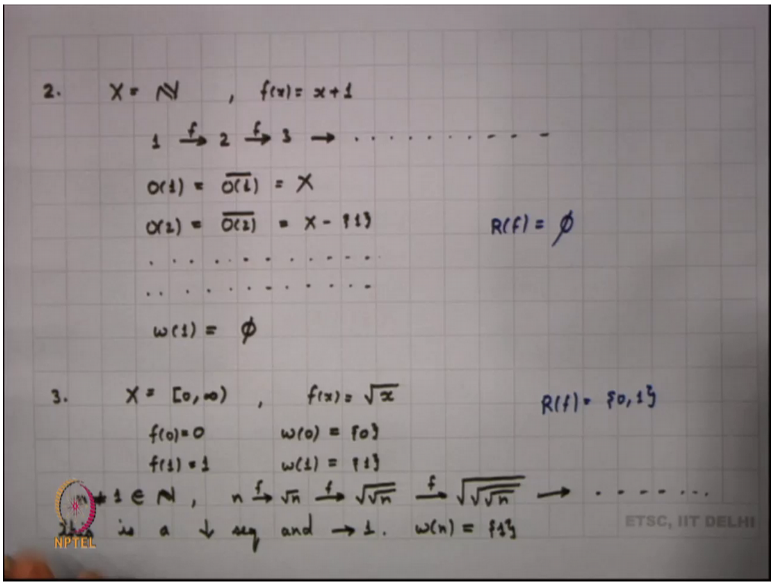

Now, my second example here I am looking into x to be my set of natural numbers n, and I define my f to be x plus 1 now it is very clear here that if I look into now my point one, right. My point one is mapped under f to point 2, which is mapped under f to 3 and so on. Our space is not compact right. So, what happens here is when I am looking into orbit of one. It is same as orbit clousure of 1, right. It is same as orbit clousure of 1.

So, my orbit of one happens to be same as orbit closure of 1 and which is my whole of x right. So, this is whole of x what is my and again if I look into what is orbit of 2, right. You would find that the orbit of 2 happens to be again the orbit closure of 2, which is x minus singleton 1. Well we can derive the orbits of rest of the point also, right in the orbit of 1. What is the limit set of one here? I am looking into the intersection of all orbits, right. It will be the empty set right. So, the limit set here happens to be the empty set.

Let us look into one more example. Now here I am looking into my x to be equal to say 0 infinity. So, I am looking into the positive real line. And I define my f to be fx equal to root x. Now we very well know here that 0 is a fixed point here, right. F 0 is 0, and if I look into the orbit of 0 it is always going to be equal to 0 itself. Now the question here is what is the omega limit set of 0. I am looking into the intersection of orbit closure, right. The orbit closure of 0 is singleton 0, right. The orbit closure of the any orbit basically because it is a fixed point right.

So, the omega limit set here is only singleton 0. Now let me look into say sum again I know that f of 1 is 1. And so, similarly I can say that the omega limit set of one, right is going to be just singleton 1. Now let me take n to be equal to not equal to 1, right n belonging to the natural numbers. What happens in that case? I know that I get an n here the orbit of n happens to be n, right. Then n is being mapped to root n, right. This under f is being mapped to root of root n, right. And this under f is again being mapped to root of root of root n and so on.

What do we know about this particular sequence? Now you get a sequence here, what do we know about this particular sequence? We find that this is a monotonically decreasing sequence, and it tends to 1 right. So, this is a decreasing sequence, and it converges to 1. So, what can you say about the limit set of any n here. This will be just singleton 1, right. So, the limit set is singleton 1. So, if I take one the limit set is singleton 1 if I take any n any natural number which is not equal to 1 we again get back the limit set happens to be singleton 1.

What happens to the remaining points? Maybe you can think about it right. So, this is our next point again we note that here our spaces are not compacted. Let us again go back to a compact space.

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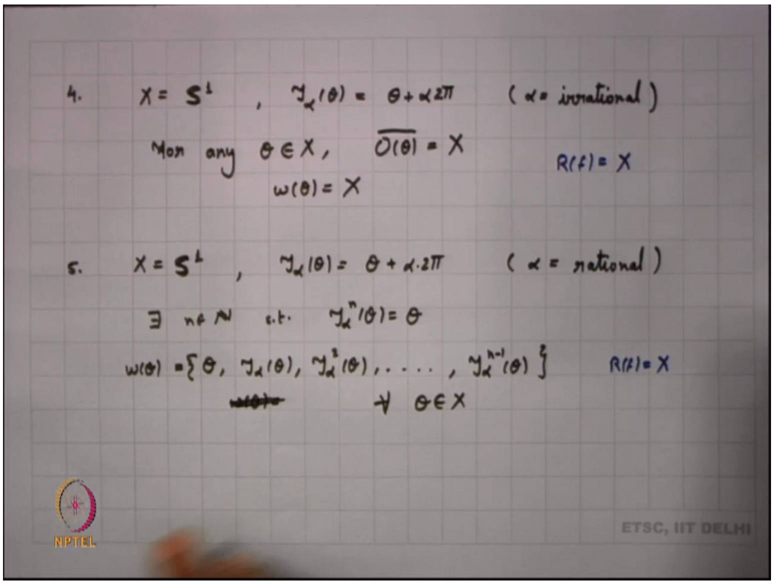

Compact metric space now, and this is something which we have already discussed. So, I am looking into x to be my circle s 1. And here we are given a mapping tau alpha of theta is basically theta plus 2 pi alpha, right. Alpha multiplied by 2 pi. And let me now take my alpha to be irrational. So, this is an irrational rotation.

Now, what happens under these circumstances? What do we know when we have an irrational rotation. You take any theta the orbit of theta is dense. So, what we find is that for any theta my orbit of theta closure is whole of x. What can we say about the limit set of theta? What will happen to the limit set of theta here? It will be the whole of x right. So, for any theta the limit set of theta is whole of x. Now interesting part here is what happens when I am taking the rational rotation.

So, let me take my x to be again s 1, my tau alpha theta equal to again theta plus alpha times 2 pi; when now I am taking my alpha to be equal to a rational number. Now what happens in this case? We have already studied rational rotations, right. What happens in case of rational notations? Now what is the orbit of a typical theta? It is finite, right, because any theta is a periodic point right. So, what we know is that there exist an N such that tau alpha to the power n of theta is same as theta. Now this is very interesting part, because now what we have here is if I look into my theta, right. I get a theta, right in the orbit of theta, I get a theta then I get tau alpha of theta, then tau alpha square of theta, right; which goes up and up on tau alpha to the power n minus 1 theta. And then tau alpha to the power n of theta is again theta.

So, what we get is the orbit of theta we get only these points. So, what can we say about the limit set of theta? What I get is I always get a finite set, the orbit closure is just a finite set. And the orbit closure of any point in the orbit of theta is also the same finite closure right. So, my omega theta is nothing but basically, I should say maybe not here maybe I am saying here, that the omega theta here happens to be just in this particular set and this is true for every theta right. So, this is true for every theta belong to x. So, this gives us some very interesting observation. This examples gives us some very interesting observations. So, let us try to look into some proposition now.

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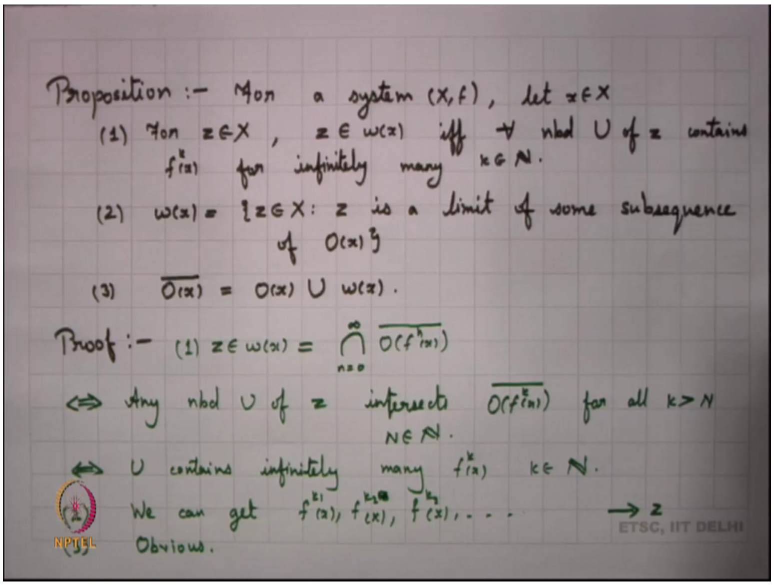

So, we start with the proposition here. So, for a system x f let x be a point in x. For some point z in x, right. We say that z belongs to the limit set of x, if and only if for every neighborhood U of z, or I should say that if and only if every neighborhood U of z contains fkx for infinitely many k in n. So, the next statement here is I can write my omega x to be equal to the set of all z in X such that, z is a limit of some subsequence of the orbit of x. And the third part here I have is that the orbit closure of x consists of 2 parts. The orbit of x union the limit set of x.

So, let us try to look into the proof here. Now what happens if my z belongs to omega x, right? We know that by definition omega x is the intersection of all n going from 0 to infinity orbit of fnx whole closure. So, that means, when I am whenever I take a neighborhood of x my since my z belongs to each of this part, right. Whenever I take a neighborhood of z, it will intersect the orbit of fkx for all k greater than some n. So, that means, any neighborhood of x. So, I should say this is basically this double implies, right; that any neighborhood of x neighborhood U of z sorry, any neighborhood U of z intersects the orbit closure of fkx, right. For all, right what does that mean? Again, this is same as saying that, right U contains infinitely fkx, right. For k belong infinitely many fkx for k belongs to N.

So, this is the proof of our first part. What can we say about our second part? We know that omega x we want to say that omega x is nothing but it is the set of all x in X such that all z in X such that z is a limit point of some subsequence of orbit of x. So, if I look into my second part, now supposing I am looking into this set of all z in X such that, z is a limit of some subsequence of orbit of x. Then we definitely know that it will be there in the limit set right. So, the right-hand side here is a subset of the left-hand side. All we want to see is that the left-hand side. If I can show that the left-hand side is also a subset of the right-hand side we are done.

So, what happens if I take any point z belongs to omega. All I know is from the first part is that any neighborhood will contain infinitely many fkx, k belongs to N. Now we can take this as our observation and you can say fine let us pick one fkx right. So, we take a neighborhood of z from this omega x. And we know that any neighborhood U will contain infinitely many this. So, I can always say that, let me take because we are working in a metric space. So, it becomes easier. So, let us take a say an open ball of radius 1 centered at z since that will contain some point here, I can call it fk 1, right. I can call it f k 1 x.

Now, we take again a ball of radius half, that will also contain some point in the orbit of x. And when we look into that some point in the orbit of x, we can always because there are infinitely many, we can always choose our k 2 to be greater than k 1. So, it will also contain fk 2 x, right. And likewise, we can again keep on taking balls of radius 1 by 3 1 by 4 and so on. So, we can get, right fk 1 x fk 2 x sorry, right fk 3 x and so on. And we know that since we are taking balls of decreasing radius. This is a sequence which is converging to z.

So, basically, we can say that yes whenever I am talking of a point z in the limit set of x there is a subsequence in the orbit of x which is converging to z. And the third part is very trivial, right. We know that this omega x is a intersection of orbit closure of x. So, it is definitely omega x is a subset of orbit closure. Orbit is also subset of orbit closure. So, one part is very simple, right. What happens to the other part? You take any point in the orbit closure, right. It is either a point of the orbit or it is a limit point of the orbit right.

So, the third part is nothing but that this is obvious. So, whenever we take a limit set, right. The limit set has this certain properties. Now let us try to look into something more of which we can gain from here. And we come back to our examples here right. So, let us come back to our examples here. We just look into the example of the rational rotation and the irrational rotation. And for this examples we have seen that we for the irrational rotation we find that the omega limit set of every point is the full x, and for a rational rotation we have seen that the limit set of every point happens to be a finite set right.

So, we look into these things, we think of these observations. And now we come up to another proposition. So, let me write down the preposition here.

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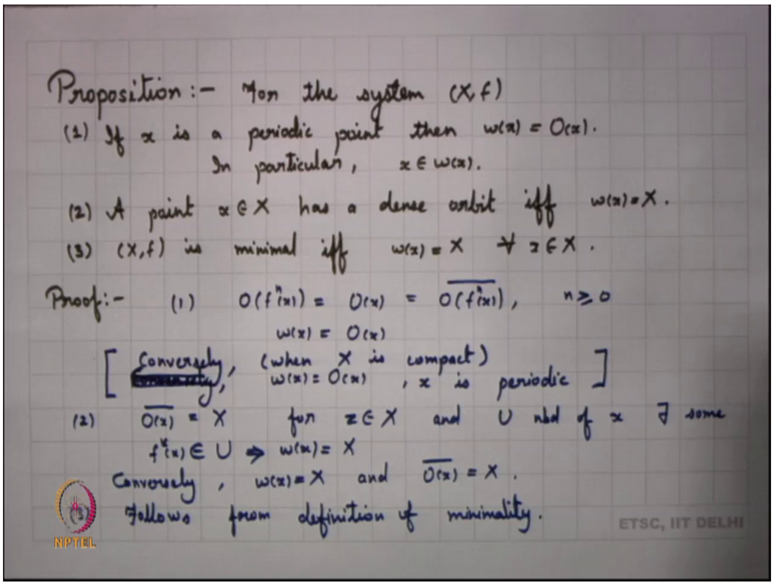

So, again for the system if x is a periodic point, then the limit set of x is same as the orbit of x. In fact, I want to add something more in particular what we have is that x belongs to omega x. Let us look into the second part here. I want to say that a point x has a dense orbit, if and only if the limit set of x is whole of x. And the third point is very clear which we one can easily derive from the second one.

I want to say that xf the system is minimal, if and only if omega x is equal to x, right for every x in x. So, let us try to look into the proof of this part, which I know that these are very, very simple observations. So, let us try to look into the proof here. Now let us try to look into the first part here. We note here when x is a periodic point, right. Then the orbit of fnx is same as the orbit of x. And this is same as the orbit closure of fnx, right. For every n in N whenever n is greater than equal to 0, right this is 2.

So, that means, when I am looking into the limit set, right. This is anyway going to be the orbit of x. So, if x is a periodic point, then definitely the limits that is the orbit of x. What happens conversely? What happens in that case? I just have the orbit of x, what happens if the orbit is infinite. Let us look into that part conversely now. Since I have written conversely let us look into that part conversely. I had something in mind now what happens in this particular case. I am now assuming conversely, right. When x is compact I am assuming what happens when x is compact, if my limits that is orbit of x. I know that orbit of x cannot be infinite, because if it is in finite it should have a limit point, right. If it is infinite it should have a limit point.

So, if conversely, I can say that if x is x is compact, then the orbit is same as the orbit of x if and only if my orbit of x is finite, right. What happens in that case now? I am looking into the orbit of x, right. My limit set is same as the orbit of x, what does that mean now? That means, the all points in the orbit of x reoccur here, right. The basically reoccur fine, because you keep on getting them not just that it is finite, you keep on getting every point in the orbit again, again and again, right. Which means that x is periodic.

So, this is an additional thing definitely it is not there in our statement, but this is an additional thing which I had in mind we need to discuss that. So, let us look into the second part. Now I know that orbit of x is dense right. So, I am looking into orbit of x I am looking into it is closure that is equal to whole of x, what happens in that case? So, if I take any open set U, right. I am taking since orbit closure of x is whole of x; that means, whenever I take any open set U right.

So, I take any z in x, right. For z belongs to x, right. And U and open neighborhood of x, what do we get here? There exist some point some orbit here, right. Some point of the orbit lies over here since the orbit is dense. So, there exists some fkx in u; that means, what I started with any point of x, any neighborhood of x contains a point in the orbit of x which would give me that the limit set will be equal to, right. Limit set is whole of x, because take any point take any basically you take any neighborhood, right. You will find some point of the orbit in that. And so, the limit set is whole of x.

What happens conversely here? What do I mean by saying omega x equal to x, we very well know that omega x is basically a subset of orbit closure. Omega x is a subset of orbit closure right. And so, orbit closure is x. The third part is still simpler, right. Again, it follows from this factor that all my points are dense orbit x is minimal means all points are having a dense orbit, since every point has a dense orbit, right. All the limit sets will be whole of x. And all of this points have limit set equal to x I can use the second part to say that x will have a dense orbit.

And when this is true for every x in x; that means, every x in X has a dense orbit. And so, the system is minimal right. So, this I can follows from definition of minimality right. So, this was again an easy proposition to look into. Now we can still after making this observation, we can still make now some more observation on the limit set. So, what do they know about the limit set we know that the limit set is orbit of x, when a x is a periodic point. We know that the point x has a dense orbit then the limit set is whole of x, and for minimal sets every point has a has a limit set which is whole of x.

We also know that for a limit set, right. We know what are the characterization of limit set. So, let us now look into what exactly do we mean by a limit set. So, we start with a limit set here.

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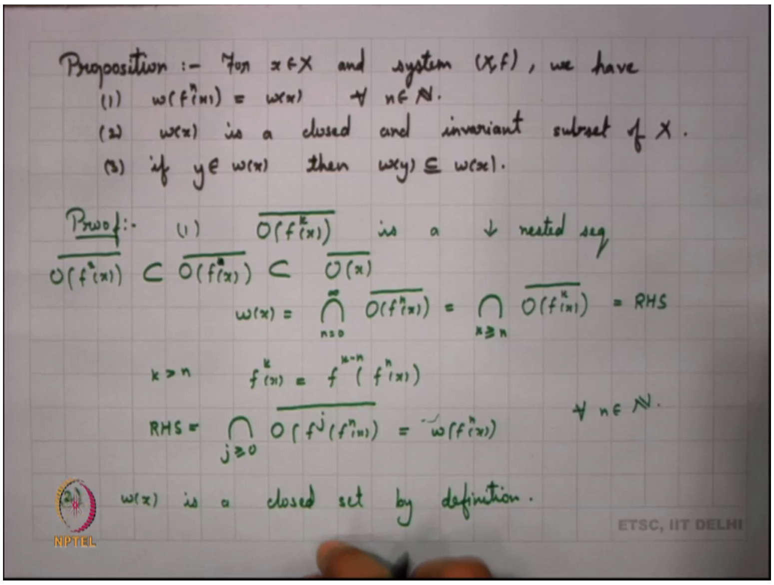

So, we start looking into this particular proposition here. So, for x in X of course, we have our system xf we have; so, what do we have? So, the first thing we have is that if I take the limit set of any point in the orbit of x. It is same as the limit set of x. And this is true for every n in N. The second point we have is that the limit set of x is a closed in invariant subset of x.

And the third thing that we have here is; if y belongs to the limit set of x, then the limit set of y will be a subset of the limit set of x. It could be equal also, but it will be a subset of the limit set of x. So, we try to now again look into the proof here. And the proof here happens to be simple. So, what happens, we are now looking into the limit set of fnx. Let us look into the orbit closure of fkx. So, we find here that the orbit closure of is it is just an simple observation. That if I take the orbit closure of fx sorry, just look into orbit closure of fx.

Its going to be a subset of the orbit closure of x. Now I can say that in extended one way back this happens to be the orbit closure of f square x will be a subset of the orbit closure of fnx. So, what we find here is that this orbit closure of fkx, right is a nested sequence. So, decreasing next nested sequence, right is a decreasing nested sequence. And so, what can we say here? My omega x is nothing but it is intersection I am looking into this as n going from 0 to infinity, right. Orbit closure of fnx, but this is the same as I am saying that let me take the intersection of all k greater than equal to n, right. And the orbit closure of fkx. This happens to be the same quantity here, because again we are looking into a nested sequence.

Let us observe another thing; supposing I take my k to be because I have taken my k to be greater than n. So, let us suppose that k is exactly greater than n not equal to n, but exactly greater than n. What happens in that case? So, I know that my fkx, right. Can be written as f of k minus n times fnx. So, ultimately my point fkx lies in the orbit of fnx, right. I am just trying to look into that part, right. That this comes from I can derive this from fnx. So, if I look into my rhs here, let me call this as my rhs here. So, if I look into my rhs here, what is my rhs equal to here? Because I had this factor, I can say that this intersection can be seen as j greater than equal to 0, right; orbit of fj of fnx whole closure.

And what is that equal to, now think of that this is the same as my limit set of fnx, right. This is the same as the limit set of fnx. Our limit set of x is same as the limit set of fnx means some observations here; and hence when that is what we started out to prove that the limit set of fnx. So, if you take any point in the orbit, right. If we start. So, it is basically whether you start from stage one, or you start from stage n, right. You get the same limit set. So, the limit set of fnx is same as the limit set of n, and this is true for every n because we had chosen our n n was some arbitrary stuff here right. So, this is true for every n in you could start with any stuff here, right. This n here was just anything.

So, we look into now our second part, right. And what is our second part our second part says that omega x is a closed and invariant subset of x. A simple observation is that omega x is an intersection of closed sets. So, it is definitely a closed set right. So, omega x is definitely a closed set. What happens further? Now I want to say that this is an invariant subset of x. What makes it invariant? Let us look let us go back to maybe some time back we had looked into this part. That if I take the orbit of x, right. That is an invariant set.

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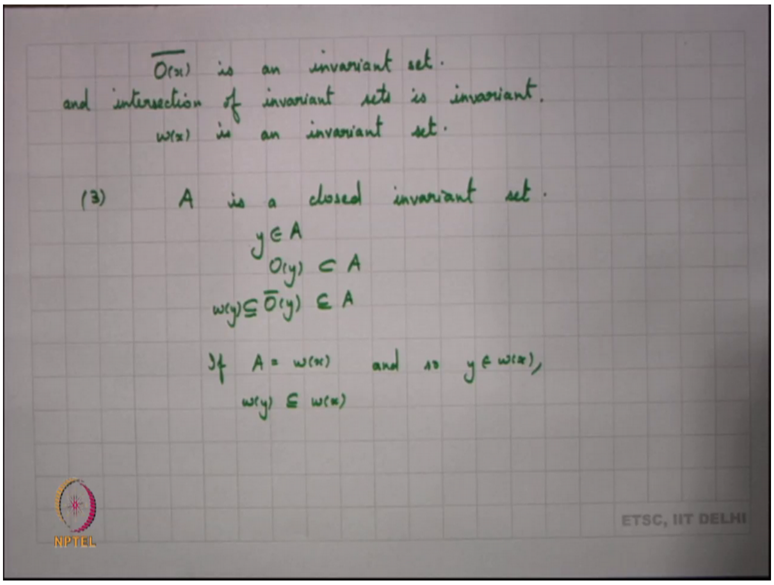

So, orbit of x is an invariant set. And so, my orbit of x closure is also an invariant set. Now orbit of x is an invariant set orbit closure is an invariant set. And what can we say about intersection of invariant sets. The intersection of invariant sets is also going to be invariant, right. An intersection of invariant sets is invariant. So, the intersection of invariant sets is in invariant and hence, when our omega x is invariant right. So, omega x is invariant.

We look into the third part. Now a third part we recall that, that was what happens if y is a point in the omega limit set, then or in the limit set basically then my limit set of y is a subset of limit set of x. So, let me take a to be any closed invariant set. So, let me take a is a closed invariant set. Let me take y to be an element of A. Then we know that orbit of y will be a subset of A because a is invariant right. So, orbit of y is a subset of A. And in fact, we find that the orbit closer of y is also a subset of A is closed, right. And what can you say about the limit set of y now?

The limit set of y is again a subset of the orbit closure of y. So, the limit set of y is a subset of orbit closure of y, and that is an subset of A. So, you take any closed invariant set you take any point y in it is closed in variant set, it is limits set will be contained in A. And we just observed that the limit set of x is a closed and invariant set. So, if a equal to omega x. Then definitely we know that omega y will be a subset of omega x by the above observation, right. For any y belonging to omega a right.

We have this observations for the limit points of orbits. We are now interested in looking into what is exactly recurrence. Limit sets are very important here, and that is why we studied them first. So, let us now go to the definition of recurrence. So, on to clearly push up what do we mean by a recurrent point.

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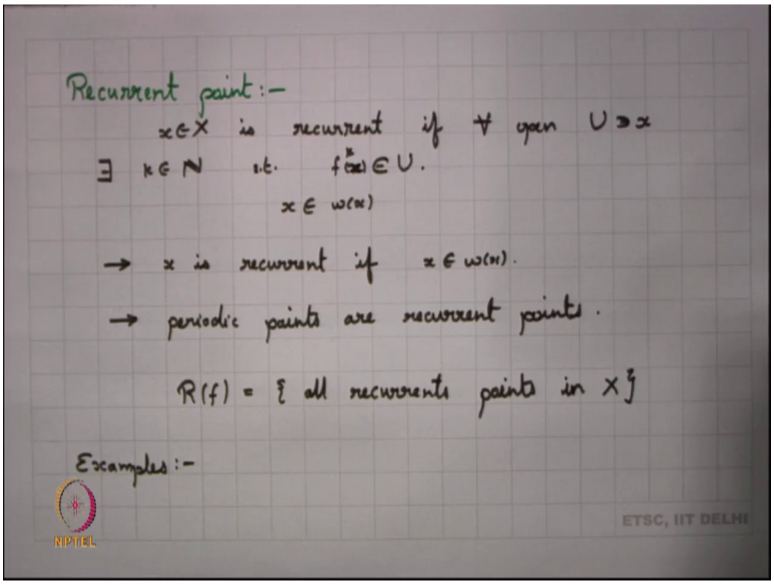

We start this definition, right. Just take this point x in X of course, if a system is xf. So, you take this point x in x. This is recurrent if for every open U containing x, there exist some k in N, such that fkn belongs to U. I want to look into this part again, right. What do we mean by saying that there is an open set U containing x, for an open set U containing x, there exist a k in N such that fk and x fkx belongs to U. What is the meaning of that?

It basically means that.

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X belongs to omega x right. So, this very part means that x belongs to omega x. And so, I can say that x is recurrent, right; if x belongs to omega x right. So, our simple observation here is x is recurrent if x belongs to x. So, one of the again one of the properties which we had seen here is one of the lamas we had seen that if you have a periodic point, right. Then the limit set is the whole orbit. And in particular the periodic point belongs to it is limit set. So, one thing is clear we have a clear example in hand periodic points are recurrent points right. So, in our next observation comes here is that periodic points are recurrent points.

So now we are interested in looking into what are all recurrent points. Now see there is this recurrent point concept is a little bit independent of what is your in the sense. That it is basically you can collect it anywhere in X recurrent point occurs anywhere in x. So, we say that we define we denote this term as rf, right. R of f because it depends on what your f is right. So, this is basically the set of all recurrent points looking into the set of all recurrent points in x.

So, let us go back to some examples now. We try to look into some examples here. And instead of defining some new examples, right. Let us try to look into the examples that we had already seen here. So, let us go back to our examples. So, we go back to our examples here. And let us try to look into the recurrent points for these examples. So, what is rf here in this case, right? Happens to be just singleton 0; so recurrent points the set of all recurrent points here is nothing but it is singleton 0 nothing else is recurrent, right. Because nothing comes back to itself, right. No orbit comes back to it very close to the point itself.

So, here the recurrent set is 0. What is a recurrence set in this case? Nothing comes back to itself at all, right. So, this is the empty set here. What is recurrent here, right? The recurrent set here is just singleton 0 and 1, right. Let us go back to one more example that we did here, right. Let us go back to this example. What is a set of all recurrent points here? The whole of x, right X is recurrently; and if I again look into the recurrent points here. We know that every point here is a periodic point right. So, again here this set of recurrent points is x. With this observation it is very simple to look into some something more. So, we again defined something more here. And that is something more is this set now we define another set N x U.

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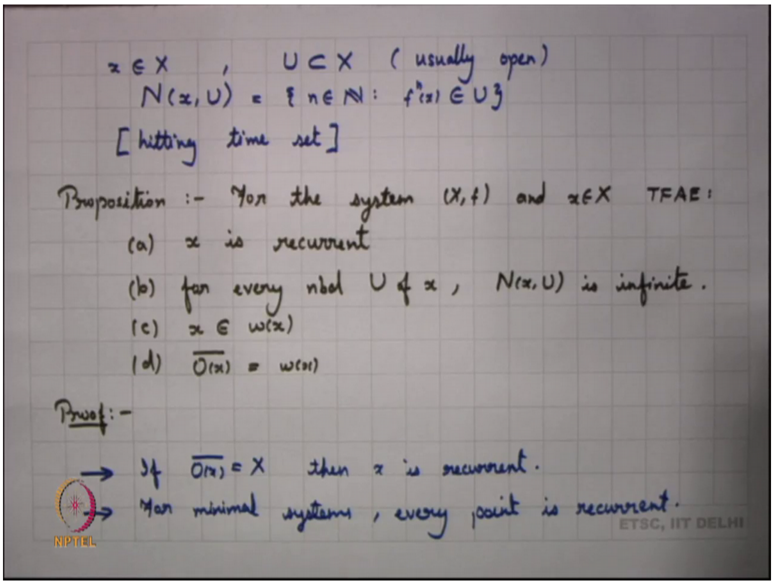

So, given any point x. So, x is any point in X and U is any subset of x usually we take it to be open we usually take this to be open. So, I define nxu to be the set of all n in N such that fnx belongs to U. Now you can think of this like a set U, and then I am looking into the orbit or I am trailing the orbit of x. And I am seeing that how many times; what are all the instances at which the orbit hits U, right.

So, basically this is also called a hitting set hitting time set. So, because what are all the times when it hits right. So, this is the hitting time set, right. And I defined this hitting time set as the collection of all basically I am looking into all those instances n when fnx belongs to U. Now it is very interesting to see what happens to x and U in that particular case. But here we will simply make some we will simply look into some kind of proposition here. Sometime to define a proposition here, by looking into this aspect now this is very clear to you that if I try to look into all the hitting time set.

So, for example, we can look into the example of irrational rotation right. So, we find that what happens to this hitting set hitting set turns out to be infinite for a periodic point we know that we just get this periods, right. We just get the multiples of n back right. So, we try to look into this particular proposition here, for the system xf and a point x in X the following are equivalent. So, what are all the equivalent statements? So, first is that x is recurrent point. The second part is for every neighborhood of is infinite. The third part is which we have already seen that x belongs to the limit set of x. And the 4th part here is the orbit closure of x is limit set of x.

We look into the proof here. And then you will just discuss the proof here. Now just try to look into this part that what happens if x is recurrent. X is recurrent means every neighborhood contains one point then, right. And we have seen that what happens in that case we can say that x, is basically a point of the limit set of x, right. And by one of the prepositions we had seen here is that then there will be; in any neighborhood there will be infinitely many points. So, you take a neighborhood of U of x you will find infinitely points of the orbit there. And that would mean that the hitting set happens to be infinite right.

So, basically, we can say that this is equivalent. All we need to look into is this part, the last part right. So, if I look into the last part of all I want to say is that the orbit closure of x is whole of is basically the limit set of x. Now we know that x is in the limit set of x. We know that the limit set of x is closed and invariant right. So, the orbit of x has to be a subset of the limit set. We already know that the limit set is always a subset of the orbit closure of x, right. And hence they are basically equal right. So, for a recurrent point it is very it is very desirable to have something as a recurrent point and we find this concept.

Another observation which I want which we have already seen here, but I want to make this observation here is that if x has a dense orbit can we say that x will be recurrent. If x has a dense orbit can we say that x is recurrent? The orbit of x is whole of x, right. Then we know that omega x is also whole of x right. So, we have seen that this also means that omega x is whole of x omega x is whole of x; that means, x belongs to omega x right. And so, x has to be recurrent. So, if orbit x equal to x then x is recurrent; which means that, if a point has a dense orbit then it is always a recurrent point. And so, far minimal systems, what happens for minimal systems? Every point is recurrent.

Now, there is a very small difference between we know that all periodic points are recurrent right, but all recurrent points need not be periodic for example, in a minimal case we do not have periodic points right, but every point is recurrent. So, there is a small difference between periodic points and recurrent points, but we will try to see that they satisfy almost the same property, and that is what we shall see in the next class.

So, today we stop here.