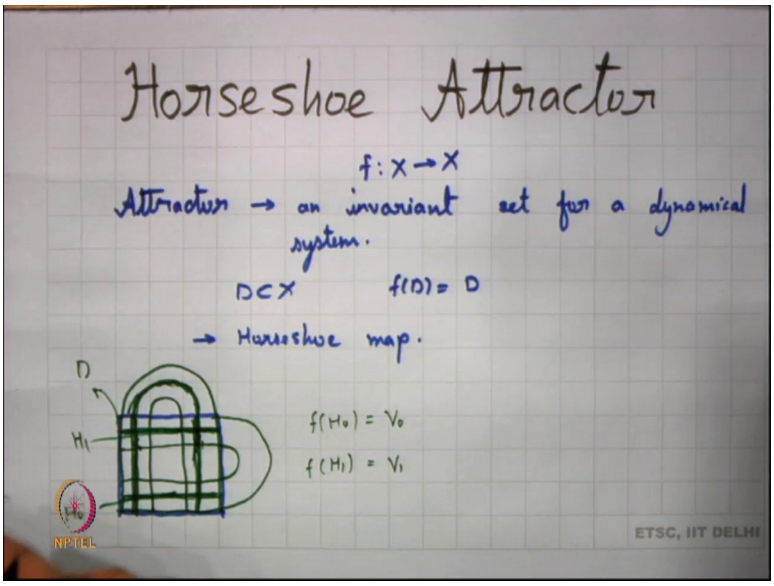
Chaotic Dynamical Systems
Prof. Anima Nagar
Department of Mathematics
Indian Institute of Technology, Delhi

Lecture - 11
Horseshoe Attractor

Today we start up talk we will be talking about Horseshoe Attractor.

(Refer Slide Time: 00:28)



Now what do we mean by an attractor? So, we know that we can have a dynamics, right. F acting from x to x. And then what happens is usually there will be some orbits of x which converge to some particular set I am not talking of points, but they will be converging to some particular set. And there will be some orbits which will sort of disappear. So, attractor is basically an invariant set for a dynamical system. So, basically if you have D subset of x, right. Such that f of D is D, then we say that D is an attractor.

Now, if you recall last time in previously we have seen; what is a horseshoe map. So, we have seen a horseshoe map, and we had noticed that the horseshoe map is something maybe I can say that it is of this form, right. You have a unit interval and the unit interval you take the transformation, right. You elongate the interval, right. Expanding x direction contacting y direction putting sorry, expanding y direction contacting x direction putting it back onto itself. And what you get is something of this form. And what happens in this phase is that you have horizontal rectangles H naught which is mapped by the function f to V naught, and you have horizontal rectangle H 1, which is mapped to the vertical rectangle V naught. So, this is our V naught this is over V 1. And similarly, you can have the inverse transformation also, where you can see that V naught is mapped onto H naught and H naught is mapped onto V naught.

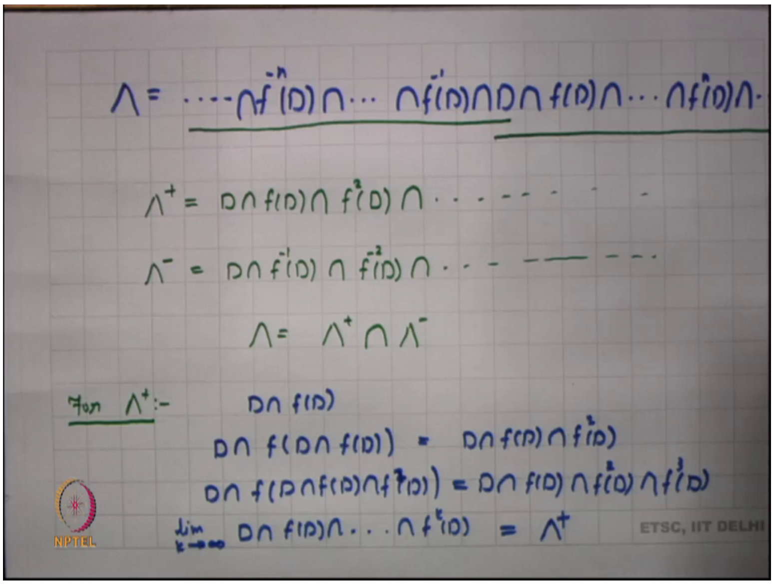
So, basically this is our H naught and this is our V naught H 1. So, V 1 is mapped to H 1 image, and V naught is mapped to H naught under f inverse. So, we have seen that we have both these transformations. And another property of the transformation, that we had seen is that if I have any vertical rectangle. So, any kind of vertical rectangle of course, it need not be subset of V naught any vertical rectangle. Then under f it is basically mapped to some it is basically mapped any vertical rectangle is mapped under f 2 some vertical rectangle.

So, what happens is you get the image of a vertical rectangle is 2 vertical rectangles in the plane D. This is what our call our D. So, in the D we find that a vertical rectangle is mapped to 2 distinct vertical rectangles and similarly, if we take a horizontal rectangle. Then the horizontal rectangle under f inverse is mapped to 2 horizontal rectangles.

So, we had say then this in the previous section that this is what is basically the dynamics of the horseshoe map when we as we had seen last time also, that we are interested in looking into what exactly is the dynamics of the horseshoe. So, basically, we have this horseshoe attractor. So, we are looking into all those points, which remain inside D for all possible iteration under all possible iterations of f. So, we are looking into the invariant set of f, and for this invariant set, we are interested in looking into what is the dynamics theory.

So, we are essentially interested in looking into this invariant set lambda, which we call as a horseshoe attractor. We again define our lambda.

(Refer Slide Time: 05:10)



So, let me define this lambda. So, this lambda I can say that this is so, f minus nd, right. Intersection f inverse D intersection D intersection f of D intersection f n D intersection so on, right.

So, you will find that we are looking into this particular intersection. Now this is our invariant set. So, what happens over here is we can try to think about this we can try to see visualize this lambda by looking into 2 things separately so on. One hand we have this negative iterates, right. We can try to see this negative iterates. And on the other hand, we have the positive iterates right. So, we have this particular positive iterates over here. And we try to see consider lambda, right. Consider these negative iterates and positive iterates separately. And then intersect them to find out the lambda.

So, for us we will be looking into 2 sets. So, we are looking into this kind of lambda plus, which I can say is nothing but D intersection f D, right. Intersection f square D and so on right. So, you looking into the positive iterate. We again have lambda minus which is D intersection f inverse D, right; intersection f minus 2 D and so on. And then we know; that what is our lambda our lambda is nothing but lambda plus intersect with lambda minus. So, we can try to think in this particular aspect. And it will be easier for us to construct to make this kind of geometrical construction.

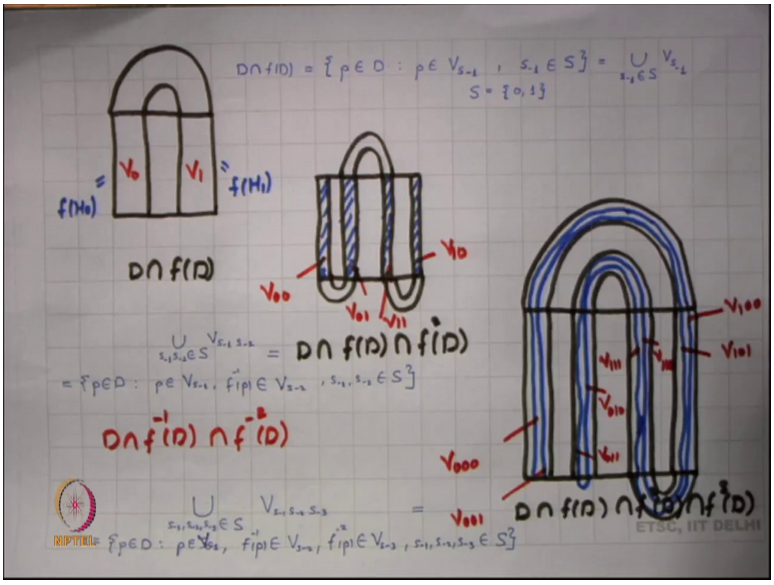
Now, as again I said that our geometrical cons construction just depends on how we had taken up our horseshoe map, and how we had geometrically seen how we could construct this attractor. And the concept that vertical rectangles are mapped to vertical rectangles and horizontal rectangles are mapped to horizontal rectangles. We will be looking into this concept.

So, we first start with our lambda plus. So, we start with lambda plus. What happens in the case of lambda plus. Now we are interested in constructing lambda plus. So, for lambda plus, we try to what scheme we are going to follow is, we are going to look into this set say D intersection f of D, right. And then after looking into D intersection f of D, we again take D intersection. So, again what we do is we take D intersection f of D intersection f of D. Now we know that this is going to be nothing but this is D intersection f of D intersection f square D right. So, what nexted it in the next step we will be only looking into what is D intersection f of; then again, I have this concept D intersection f of D the set fs f square D.

And this is going to be our set D intersection f of D intersection f square D, right intersection f cube D and then after having looked into say n times, right. What we can do is we can see that our lambda plus is nothing but say supposing I have this D intersection I already have found out what is D intersection f D, right. N intersection f k D. If I have this set with me, then we know that what we can do here is we can take this set we can take this limit as k tends to infinity, right. And this is what is going to be our lambda plus.

So, we are going to try to visualize D intersection f D, D intersection f D intersection f square D and so on. And once we have constructed k steps. It will be easy for us to visualize what happens when we take the limit as k tends to infinity. So, let us try to visualize; what is D intersection f of D.

(Refer Slide Time: 10:00)



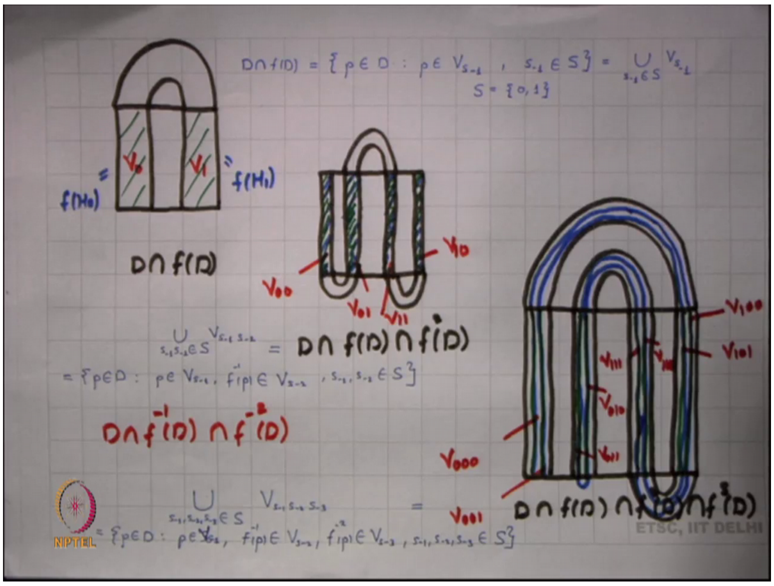
So now we take what is D intersection f of D. Now we already know what is D intersection f of D right. So, we have seen that this is nothing but my V 0 union V 1, right. Thinking recall from what was done last time. This is just V 0 union V 1 and when I state this V 0 union V 1. I can start with an indexing set. So, I am starting with this indexing set s to be equal to 0 1. And starting with this indexing set s equal to 0 1, I can say that this is nothing but this is my union of; now I am indexing it in a very special manner, I am saying that s minus 1 belongs to S V s minus 1. So, this is my D intersection f of D.

Now, we can look into what is D intersection f of D intersection f square D. So, if you try look looking into this aspect this is nothing but now I am looking into D intersection f of this part right. So, I can index it I can simply say that this is nothing but this is union of s minus I belonging to s, right. Where I am taking my I to be equal to 1 and 2 and I have f of V s minus 2 intersection V s minus 1 which I am writing now as. So, I am indexing them as s minus I belonging to s, where I is 1 and 2. And I can write this as V s minus 1 s minus 2. What does this set? So, if you try to look into this set I am essentially looking into all the points p in D, right such that p belongs to the vertical triangle. V is minus 1. And then we know that the image of vertical triangle is a vertical triangle. So, basically, I am looking into f inverse p belonging to V s minus 2. And again, here my si belongs to s. i should say s minus i belongs to s, right. And I am taking my I to be equal to 1 2.

So, I am looking into essentially these points. So, this makes it easier for me to guess what is D intersection f of D intersection f square D intersection f cube D. So, again I know that it will be D intersection f of this part right. So, you can say that this is union of s minus I belonging to s, right. I goes from one to 3. I now have f of V s minus 2 s minus 3, right intersection V s minus 1, which I can write it down as union of s minus I belonging to s, right I going from 1 2 3 which is V of s minus 1 s minus 2 s minus 3. And if you try to look into what are these points. So, these are exactly those points p and D n such that p belongs to V s minus 1, right. F inverse p belongs to V s minus 2, and fs minus 2 that is of p belongs to V s minus 3. Where again my s minus i belongs to s and my i happens to be 1 2 3. So, I know that this set happens to be of this particular form.

Now, if you recall that we had seen this figure in terms of figure we had seen it will last time. So, I am just trying to recall this once again.

(Refer Slide Time: 14:31)

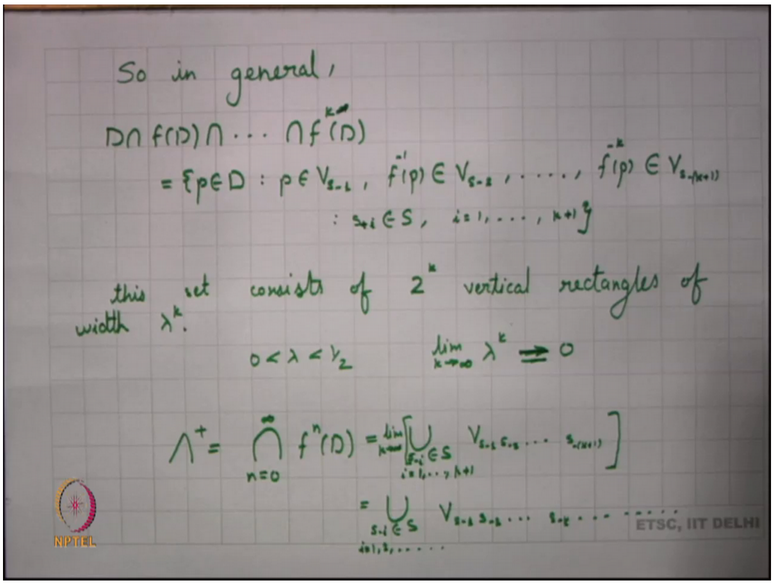


So, let us now look into what is our action of f. So, we had seen that our D intersection f of D, right happens to be this V naught and V 1, right. So, you have this V naught n right. So, this is basically your V naught and V 1 this is D intersection f of D. And here we know that D intersection f of D is the set of all p in D, such that p belongs to V s minus 1, right. And we can again denote our D intersection fd intersection f square D, which is basically nothing but it is a union of 4 vertical rectangles. So, if I look into what are these vertical rectangles, right. I can see that this is basically the set of all p in D such that p belongs to V s minus 1. And f inverse p belongs to V s minus 2, for s minus 1 s minus 2 belonging to s.

And again, if I look into vd intersection f of D intersection f square D intersection f cube D, right. What I get here is I get this kind of I get this 8 vertical rectangles. These are 8 vertical rectangles that we get. And this 8 vertical rectangles are basically the set of all p in D, such that p belongs to V s minus 1, f inverse p belongs to V s minus 2 f minus 2 belongs to V s minus 3, right. For s minus 1 s minus 2 s minus 3 belongs to s.

So, if we had seen this last time, and this is what is our description. So, in general what can we say about any k. So, if you start with that part right.

(Refer Slide Time: 16:30)



We can say that in general, we have D intersection f of D intersection f k minus 1 D or I can say f k D may be what is that equal to. So, that happens to be the set of all p in D n, right. Such that p belongs to V s minus 1, right. F inverse p belongs to V s minus 2 and I can start this by saying that f minus k p belongs to V s minus k minus k plus 1 here. And such that si s minus I belongs to s and my I varies from 1 to k plus 1, right. So, this is what happens at the general instant when we are taking k iterations. And now what can we say about D intersection f D intersection f minus k D.

Now, let us get back to a figure again. Now what had happened was here, right. That our width here if you recall this part our width had the width of the rectangle V naught was lambda, right. Now here the width would have gone down to lambda square, right. The width again your contacting the width right. So, the width here was lambda, the width here goes down to lambda squared the width here goes down to lambda cube. So, what can we say in terms when we are looking into the k-th part?

So, this set consists of so, we can say that this is a set. And this set consists of how many rectangles do we have here? 2 to the power k, right. So, we have 2 to the power k rectangles vertical rectangles of width lambda to the power k. So, we now at the k-th stage, we have this construction.

Now, we basically want to see what happens width k tends to infinity. So, we recall here that our lambda was a quantity which was between 0 and half. Lambda was something which was less than half. So, with this quantity, right we know that is k tends to infinity lambda to the power k tends to 0. So, or basically this is limit. So, this is equal to 0. So, limit of lambda to the power k is 0.

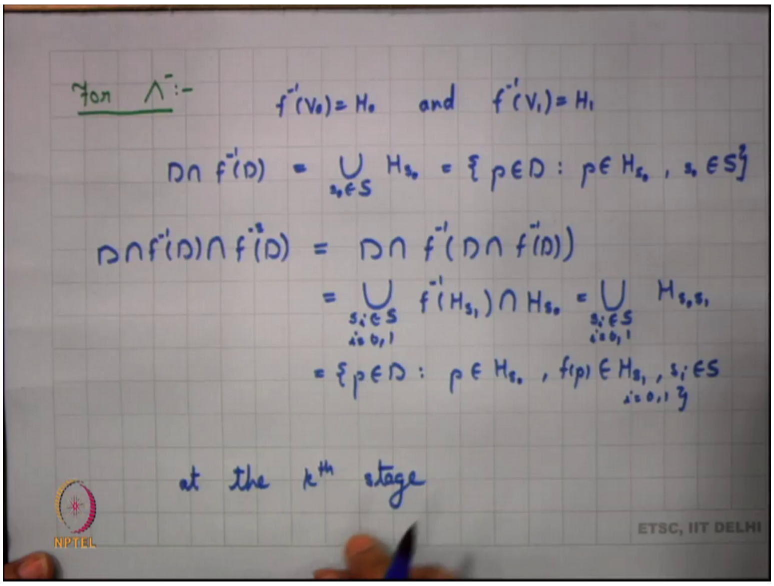
So, that means, now as we go on or applying f again and again, right. these rectangles are slowly gradually going to lose that width. And 0 width means they become a straight line. So, what happens at the infinite stage here? So, there infinite stage here, we find that lambda plus which is basically my intersection n goes from 0 to infinity fn of t is nothing but this is union of, right. I am now looking into this part this is limit or I should say that this is union of say s minus i belong to S, i goes from 1 to k plus 1, right. I have V s minus 1 s minus 2, right S minus k plus 1. I have this fact here, right

I am looking into this particular union, and I am taking this limit as k tends to infinity. And if I look into this limit if I just see this part what we have actually, what we obtain is we obtain vertical lines now, because the rectangles no longer exists because the width tends to 0. So, we get vertical lines here, and I can say that this would be union of all s minus I belong to s where I varies from one to infinity, right. And what I get is here V minus 1 V s minus 1 s minus 2, right S minus k and so on.

So, what happens here is; for each sequence that you can get from s, now our s is a set 0 1. So, from each sequence that you can get in s you get a vertical line which is labeled by that particular sequence, right. And the union of all these vertical lines happens to be your lambda plus. So, lambda plus is now the union of vertical lines, right. And this vertical lines are labeled by a sequence of 0's and 1's

So now what is this? This is a sequence of infinite number of. So, this is basically this lambda plus consists of infinite vertical lines, and each of this vertical line is labeled by a sequence of 0's and 1's. So, let us now look into how do we construct our lambda minus.

(Refer Slide Time: 22:37)



So, what happens for lambda minus? Now if we try to see what happens for lambda minus for lambda minus again we have a same factor here. So, we know that what f minus f inverse does, f inverse maps horizontal rectangles to horizontal rectangles. And we know that what is f inverse it maps V naught to H naught, right. And it maps V 1 to H 1. And in general, it is mapping horizontal rectangles to horizontal rectangles.

So, we know that f inverse of V naught is H naught. And f inverse of V 1 is H 1. And what is our D intersection f D, f inverse D? Again, it is just H naught union H 1, right. D intersection f inverse D is just this H naught union H 1 V. And so, we can say that this is union of all and I am saying that this s naught belonging to S, I have H of s naught, right. And this I can say is this; this the set of all p and D such that p belongs to H of s naught, right for s naught belonging to S.

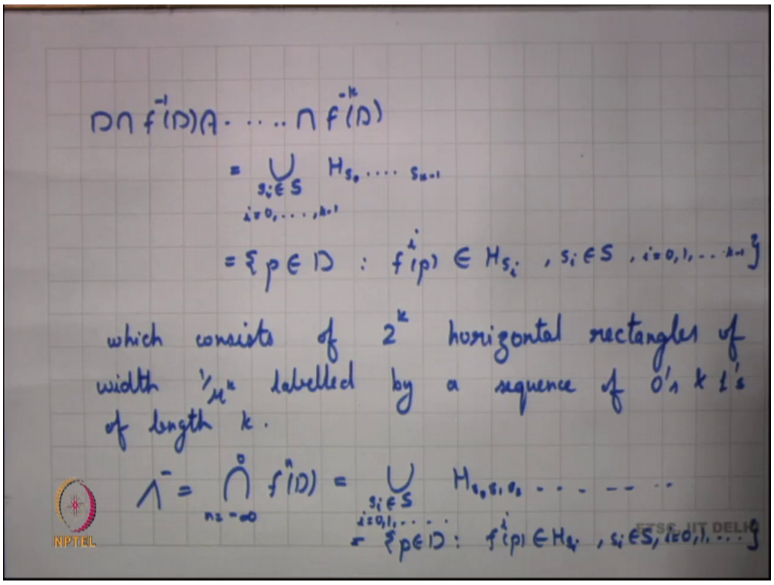
Now, again the factors similarly the kind of construction that we have seen earlier, what is like D intersection f inverse D intersection f minus 2 D. What is this part? So, if you try to look into this part this is nothing but I can say that this is D intersection, f inverse of D intersection f inverse D. And so, I can say that this is nothing but this is my union of s i belong to s where my I varies from just 0 and 1. And I can say that this is f inverse of

H s 1 intersection H s 0; which I can write it as union of s i belonging to s i is 0 1, right. I can say that this is my H s 0 s 1.

So, I have sort of horizontal rectangles here, which are labeled as s naught s 1. Where s naught again is from s and s 1 is also from s. And I can write it in terms of p as this is the set of all p D, such that p belongs to H of s naught, and f of p belongs to H of s 1. Where again by si belongs to s and i is 0 1. So, this is my D intersection f inverse D intersection f minus 2 D. And I could similarly take D intersection f inverse D intersection f minus 2 D, right. What we are what we are interested in what happens at the k-th stage.

So, what happens at the k-th stage?

(Refer Slide Time: 26:09)



So, our D intersection f inverse D; sorry, intersection f minus k D what happens at this k-th stage? What I get here is I get this to be union of s i belong to s, right. I have H of s naught up to s k minus 1, right. And I have this I varying from 0 to k minus 1 and if we try to look into this factor. This is nothing but this is the set of all p belonging to D, such that f I of p belongs to H of s i, right. That is what we get here right.

We think try to think of that part. D intersection f inverse D intersection f minus 2 D happened to be the set of all p in D such that p belongs to H of s naught f p belongs to H of s 1. So, the next stage I will get H of s naught s 1 s 2, right. Such that p belongs to H of s naught f p belongs to H of s 1 and f square p belongs to H of s 2. And similarly, what happens at the k-th stage here is; we get H naught s 1 H naught s 1 s minus k such that p belongs to H of s naught f p belongs to H of s 1, right. And f I belongs to H of s I, right. For all s i belonging to s and my I varies from 0 1 up to k minus 1.

So, this is what happens at the k-th stage. And again, we have seen that horizontal rectangles are mapped to horizontal rectangles. And so, what happens over here is that we get 2 to the power k. So, this consists of so, the set consists of 2 to the power k horizontal rectangles of width, yes can somebody tell me what will be the width here. Now we know how f inverse maps, right. It goes to 1 by mu, isn't it? And I have you have already started with mu greater than 2, right.

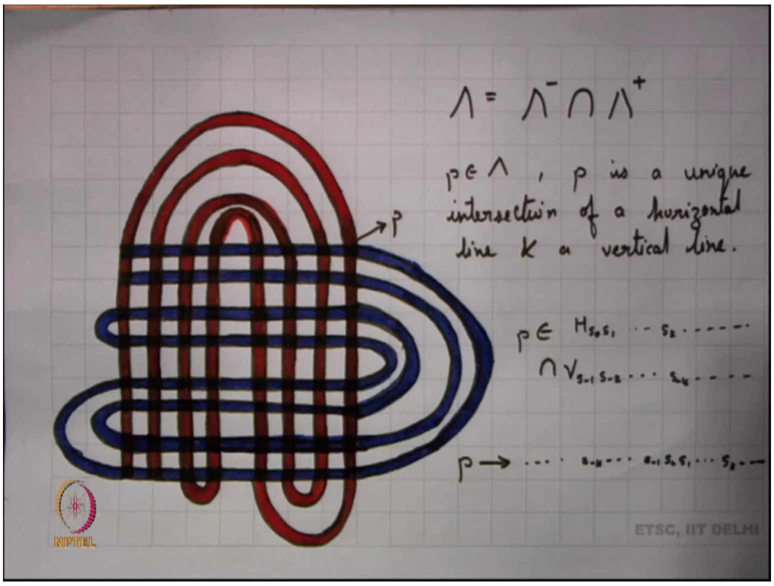
So, at the k-th stage my width happens to be 1 upon mu to the power k, right. And how is it labeled? So, this we have 2 to the power k horizontal rectangles of width 1 upon 2 to the power k, right; labeled by a sequence of 0's and 1's of length k. So, I am taking a sequence of length k of 0's and 1's. And I find how many of because I am taking 0 1 again. So, how many sequence of length k do I get? I get again 2 to the power k sequences, right. And if we look into what happens over here is that we get 2 to the power k horizontal rectangles. So, we have unique we have sort of a unique association of a sequence of length k, with each of the horizontal rectangles.

And now it is just as before it is easy to say what will be your f lambda minus. So, what is my lambda minus, I can say that this happens to be my intersection, n going from minus infinity to 0, right f and D. And I can take this limit as n tends to infinity in the k-th stage. So, what I get here is this happens to be the union of all s I belongs to s, where my I varies from 0 1 onwards, right. And I get here sequences H s naught s 1 s 2 and so on.

Now, with the k-th stage when we are taking the limit we know that mu is greater than 2. So, 1 upon mu to the power case again going to tend to 0; so at the infinite stage what will we get is and infinite seek or basically an infinite horizontal lines. So, what we get is infinite horizontal lines. Now each of this line is measured by is each of this line has a unique association with a sequence of 0's and 1's. And there are infinitely such horizontal lines. Also, we can say that this would be basically the set of all p and D, right. Such that f i of p belongs to H of s i where my si belongs to s. And my i varies from 0 1 onwards.

So now we are having this particular picture for the forward and the backward. How exactly can we see that? May be let us try to see this figure. So, let us try to say this particular figure.

(Refer Slide Time: 31:47)



Now, if we tried to look into the forward iterates right. So, the forward iterates basically comprise what is the red region here. So, you have this kind of vertical rectangles, right. Each vertical rectangle is associated with a sequence of 0's and 1's. And now you can imagine what happens at n tends to infinity, you will get infinitely many vertical lines. And each of these vertical lines is uniquely associated with one infinite sequence of 0's and 1's. So, you have vertical lines here. And if we try to look into what happens to the negative side, right the backward iterates. Then you find that you find this. So, basically this picture gives you the horizontal rectangles.

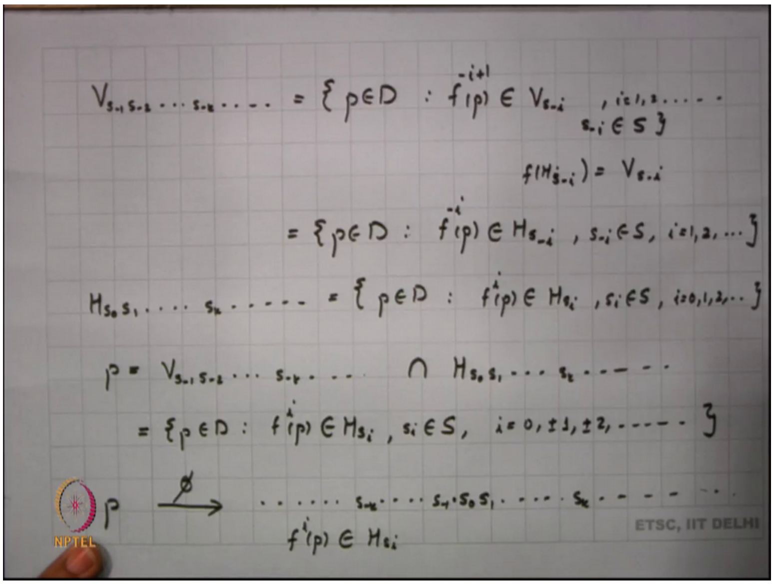
Now, the horizontal rectangles are also uniquely associated with the sequence of 0's and 1's. So, what you get here is; you get that you have at the infinite stage you get infinitely number of horizontal lines, right. And each horizontal line is uniquely associated with a sequence of 0's and 1's. So now, what happens to our lambda here. If when asks for our lambda is nothing but it is lambda minus intersection with lambda plus. So, what is that aspect?

Now, if we can see what is our lambda minus right. So, if you see what is our point p here. So, I can say that what is my point p here. So, this is a sequence of all 0's and this is a sequence basically, this is a set of all horizontal lines. This is a set of all vertical lines right. So, you have horizontal lines and vertical lines. So, if I take any point p belonging to lambda, it is basically in the intersection of a horizontal line with a vertical line right. So, p is a unique intersection of a horizontal line and a vertical line. So, it take any point p here. So, supposing I want to say that this particular point p is in lambda then this particular point p has to be a unique intersection because we know that anywhere 2 lines will intersect only at one point right. So, we find that this is a unique intersection of a horizontal line and a vertical line.

Now, the more interesting aspect comes up over here. What is the horizontal line that p that is that we are considering. So, supposing the horizontal line that we were considering, right turns out to the form of H s naught s 1, right s k and so on, right. This is the horizontal line, the unique horizontal line. So, that means, that there is a sequence s naught s 1 s k a sequence of 0's and 1' which labels this particular line. So, this is our unique horizontal line. And my p happens to be the intersection of this horizontal line with a unique vertical line. So, my vertical line I am labeling again by a sequence of 0's and 1's. So, for unique sequence of 0's and 1's you have a unique line. So, supposing this was V s minus 1 s minus 2 s minus k and onwards. So, this was a unique sequence over here. So, what we get here is that associated to p, we get a unique by infinite sequence of 0's and 1's.

So, I can say that my p here can be uniquely associated, right. Uniquely corresponds to this by infinite sequence, I can write it in this form s minus k s minus 1 s 0 s 1, right s k and so on. So, this any point of my horseshoe attractor, right is can be basically can be associated to bind a unit by infinite sequence of 0's and 1's. But this is giving us some kind of an interesting picture here. So, let us try to analyze this once again.

(Refer Slide Time: 36:27)



So now we look into what is our vertical rectangle; so vertical sorry a vertical line. So, my vertical line is f V s minus 1, s minus 2, s minus k and so on which essentially is the set of all p in D, right such that f minus i plus 1 of p, right. Belongs to V s minus i, right for i going to 1 2 and all and my si s minus i belongs to S, right. So, this was basically my vertical line. And if I try to look into this vertical line, we recall here that f of H i, right whatever be or maybe I can take s minus i, right. Will be nothing but it will be V s minus i.

So, I am trying to lake take this association. We know that V maps H maps to the horizontal rectangle maps to the vertical rectangle in this manner, because again our indexing is just 0 and 1. And we know that H naught is mapped to V naught. And H 1 is mapped to V 1. So, if I try to put this equation inside what we get here is this is the set of all p in D such that f minus i of p belongs to H s minus i, right. Where again my s minus i belongs to s and my i varies from 1 to so on.

Now, I can easily say what is my point p. And we already know that what is our H s naught s 1, right. S k. What is this line equal to? So, this is my vertical line which I can now have has this representation. And this is my horizontal line we already know what (Refer Time: 38:19) what representation it has it is the set of all p in D, such that f I of p belongs to H s i. Again, my s i belongs to s where I varies from 0 1 2 and so on.

So, I can say my p the point p, that we have which is the unique point of intersection of this horizontal line with this vertical line, right. This can be written in the form of which basically let me write it in this form this is a unique ripped intersection s minus 1 s minus 2 s minus k and so on. Intersect with H s naught s 1 s k and so on. So, this is basically a singleton element, and this singleton element is basically the set p in D, such that f I of p belongs to H of s i where my s i belongs to s, right. And now where is my i varying from. So, I is 0 plus or minus 1 plus or minus 2 and so on.

So, if we try to analyze this part, right. What happens here is what is the dynamics of p. So, that means, we are interested in looking into what happens to the iterate of p, what happens to the orbit of p? So, to understand the orbit of p, it is very interesting that we all have to look into what is this horizontal rectangle. So, at the i-th stage whether we go into the negative or the positive. It is going to stay in this rectangle H si and here what we have here is we have p in that case because this set is a singleton, right. This p is associated let me call phi to be that association. This p is uniquely associated to this by infinite sequence s minus k s minus 1 s 0 s 1 sk and so on.

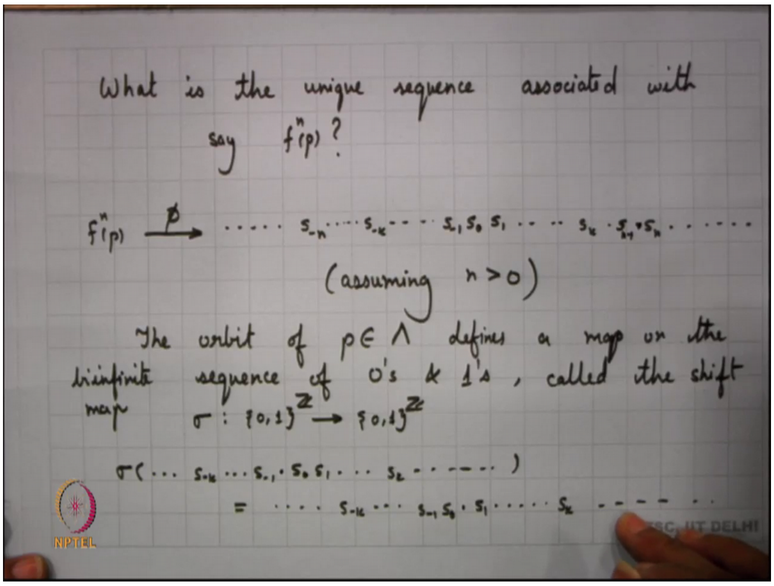
So now I have this by infinite sequence. So, I will always try to look into what happens to the negative iterate what happens to the positive iterate. So, we always write a by infinite sequence with a decimal point. To separate out the negative iterates and the positive iterates right. So, we have p under phi being associated to this by infinite sequence of 0's and 1's.

So now let us try to understand the dynamics on the horseshoe attractor, what happens on the horseshoe attractor? So, on the horseshoe attractor let me again come back to this part figure on the horseshoe attractor. Now I am looking into this I am looking into this picture again and I am trying to lay we just imagine what happens at infinity. So, at infinity we have those vertical lines infinitely many vertical lines and infinitely many horizontal lines. And we find that each point of intersection is associated with a sequence.

Now, one thing is clear to us that this association is in such a manner that whenever I am saying that p is associated with this particular sequence, right. I have I have this in mind that f i of p always belongs to H of s i whatever be my i, right. My i could be negative or positive this belongs to H of s i.

So, what our conclusion here? Is that in order to understand the dynamics of p, it is enough to understand what is the dynamics of the by infinite sequence.

(Refer Slide Time: 42:13)



Or in other words let us say that what will be what is now we already know what is p at (Refer Time: 42:15) right. So, p is associated with this particular sequence. So, we would be interested in what is the sequence what is the unique sequence associated with say f fnp. So, at the nth stage what happens here? What is the sequence which is associated with fnp? Because we are looking at ultimately, we are trying to analyze the orbit of p, right

So, what happens at fnp? Now if I try to look into what happens at fnp, and if I go and back into this construction here right. So, what would happen to fnp here right. So, I know that my fnp belongs to H of s n. My fnp here, right from this factor it is very clear that my fnp belongs to H of s n; that is if my n happens to be positive, right. What happens if my n tends to be negative. Again, I can say that it belongs to H of s minus n in that sense, right if n is negative right.

So, in all cases I have that my association is given in this particular manner and so, when I am looking into this particular sequence. So, I am again looking into this particular sequence. So, let me write the sequence s minus n, right. S minus k s minus 1 s 0 s 1, right s k s n and so on. So, what happens to fnp if I want to note? What happens to my fnp, right?

Since my fnp will be like at the 0th point at the 0th part it should belong to H of s n right. So, we can say that this association can be given in terms of now when we look into the by infinite sequence, right; if we look into this particular by infinite sequence, where it was mapped to, right. We know that fnp belongs to H of s i, right. My decimal point was here.

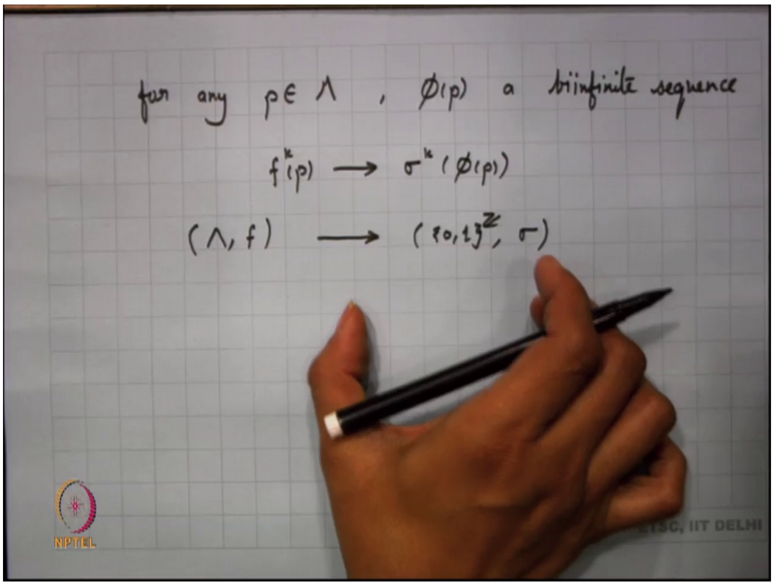
So, when I am looking into what happens to fn of p what is the association. Then it is going to be it is going to remain in the same sequence. The sequences the sequence of 0s and ones is going to remain the same. Excepting what happens is if n is positive, my decimal point moves n steps to the, right. If my n is negative, right. My decimal point moves n step to the negative side to the left hand side, right till the point which is adjacent to the decimal becomes s n right. So, it keeps on moving. So, what happens here is if we try to look into the orbit of p the orbit of p noting, but involves in moving this decimal points to the left or to the, right.

So, we see that what is my fnp, right basically let me assume. So, assuming my n to be positive, right, my decimal point puts up here. So, I have an s n minus 1 here and my decimal point is here. So, this is again a unique sequence of 0s and ones. So, my orbit of p can be easily seen to be concurrent with the orbit of 0's and 1's, right. And this is just where my map is just moving a decimal point to the left or the, right.

So, this moving of decimal point to the left and is basically called a shift map; so what we find out is that the orbit of p in this horseshoe attractor, right. It defines a map on the sequence of 0's and on the by infinite sequence, which is called a shift map, where I will define the shift map, right. Now what I have is I have basically my sequence of 0s and ones, right. These are by infinite sequence which they move on both the sides. So, you can say that 0 1 to the power z, right to 0 1 to the power z and I am defining this as sigma of s minus k s minus 1 my point lies here s naught s 1 s k. If I am looking into this part my shift is only looking taking the decimal point one point to the, right. Right

So, this is nothing but my s minus k, right. S minus 1 s naught I have a point here, I have s 1 and then sk. Now this sigma is an invertible map one can see look into the part this is an invertible map. And so, my sigma inverse would take the decimal point one point to the left right. So, this is a unique kind of association of the dynamics of f on the horseshoe attractor with the dynamics of the shift on the set of all by infinite sequences, and we find that any p for any p n p in the horseshoe attractor I have phi p, right.

(Refer Slide Time: 48:03)



A by infinite sequence, and then what is my fk of p; what is this association fk of p. Then this fk of p is nothing but it is sigma to the power k, right. I am applying shift k times to phi p.

So, we have a direct relation between the dynamics. So, I (Refer Time: 48:37) I am looking into the dynamical system sigma, sorry lambda and f which is my horseshoe attractor the dynamics on the horseshoe. Then there is a direct association of this with I have this sequence of 0's and 1's as I said, right and sigma. So, we see that there is a direct association over here. Can we come back from here to here? Maybe first of all we will have to analyze this part. So, the analysis of this part is what is what turns out to be again a very important branch of dynamics which we call symbolic dynamics. And this is something which we will be taking up next.

So, today we end up over here.