Chaotic Dynamical Systems
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Lecture – 10
Nonlinear Systems

Welcome to students. So, today we will be dealing with some Nonlinear Systems. Now linear systems have comparatively very simple dynamics. So, if you observe linear systems, you will either find that the dynamics are taking some kind of exponential decay or there is some exponential growth. On the contrary with that when we look into nonlinear systems, nonlinear systems are seen to having some property which is neither exponential decay, nor exponential growth and it is also observed that they are not even stable in terms of the dynamics. So, this is where you can observe some kind of chaos.

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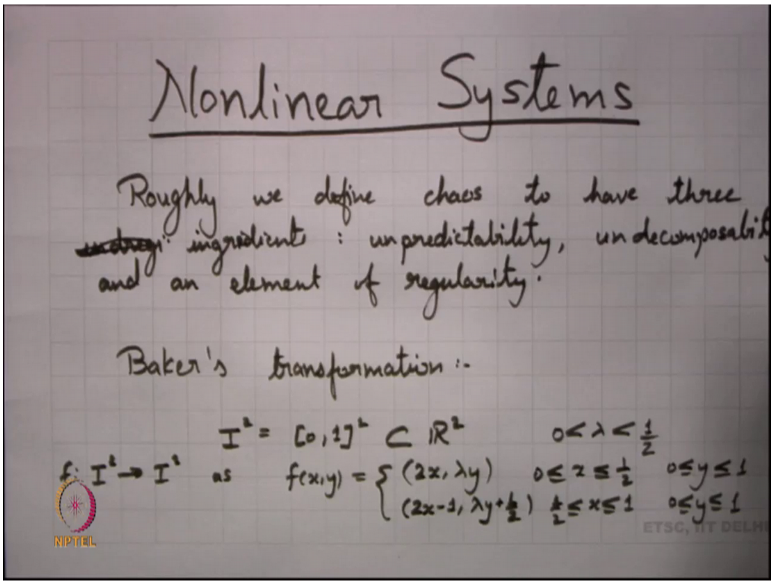

So, roughly we define chaos to have these 3 ingredients, and these 3 ingredients are unpredictability, un decomposability and an element of regularity. Now if you try to see these aspects, we can look into we can try to look into some kind of examples here. So, the first example that we take here will be Baker's transformation.

Now, what is Baker's transformation? So, if something like how the Baker needs the floor, the Baker wants to make some cake or makes want to make some bread, and for making a bread he needs to dough. So, when he is doughing what he does is he stretches the dough right folds it folds its back right, and then again the combined thing is stretches it up and then again folds it back. So, this is how the Baker has been doing his like for year for many years, this is how the Baker has been making his dough.

So, something similar we can try to imitate it in terms of mathematical definition. So, we start with say I equal to. Let me get I squared to be 0 1 square and we are all working in terms of R square. We let our lambda b between 0 and half, and we define f from I square to I square as f of xy is say 2 x lambda y for 0 less than or equal to x less than or equal to half, and 0 less than or equal to y less than or equal to 1. And on the other part we define it to be 2 x minus 1, and again lambda y plus half, for half less than or equal to x less than or equal to 1 and again 0 less than or equal to y less than or equal to 1.

What can we think about this kind of transformation? So, we can say that this transformation is something which stresses I square now. I square is basically a 1 cross 1 rectangle. So, it kind of stretches are I square into a 2 cross lambda rectangle. So, the y coordinate is basically diminished right. So, the y side is diminished to lambda and the x side is stretched to double right. So, it is starched in that manner and then we cut it. So, this is basically not a continuous transformation, because we are cutting the rectangle into 2 pieces. So, we cut the rectangle into 1 cross lambda 2, 1 cross lambda rectangles and then we place 1 over the other right, with a gap of 1 minus 2 lambda in between we are replacing 1 rectangle over the other.

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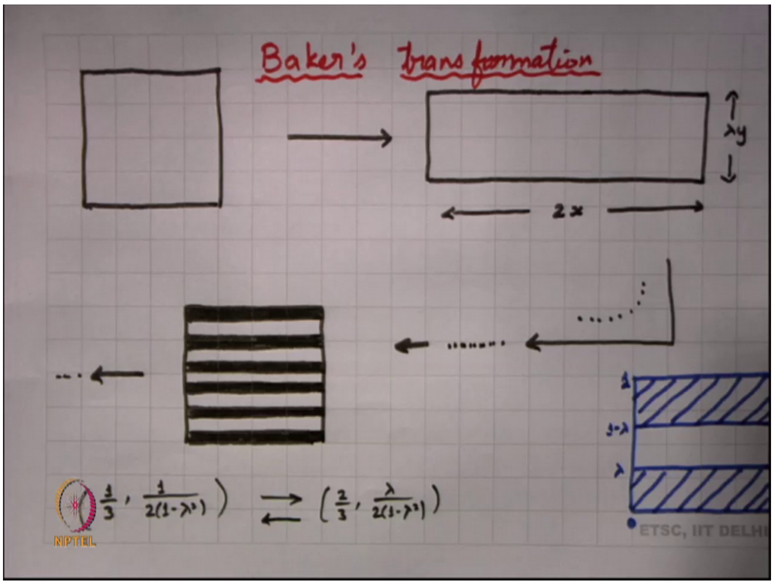

So, how does this transformation look, maybe we can look into this picture. So, we had this 1 cross 1 rectangle or basically a square over here, which we sort of elongated from the x direction to 2 x right. So, we applying 2 x here, and we applying lambda y here. So, there is a contraction in the y direction, there is an expansion in the x direction and then what we are doing is, we are trying to cut this into 2 parts right. So, we are cutting it into 2 parts, we are place 1 part over here. So, basically your 0 to half right; this is 0 to half which is placed over here.

So, for x being between 0 and half replace this rectangle here, y is against shrinked to lambda. So, this is placed over here and since we cut it over here, we are pasting the other half above it right. So, we place it in the other half in this particular term. So, what we have is the other half let is from x equal to half 3 x equal to 1 right we have placed it over here. So, this gives us the Baker transformation and then we try to look into the iterations of this. So, we are basically not only iterating a point, basically we are iterating the entire domain. So, we try to iterate this and maybe at some kind of iterations we will get these kinds of steps.

Now, you can well imagine what happens when we keep on repeating this process. So, whenever we keep on repeating this process, every time the length of the y or whatever the y length is right that gets reduced by lambda, and since I m already assumed my lambda to be less than half right. So, lambda to the power n tends to 0 as n times to infinity. So, what do you find is, that towards the end right you will have some sort of cantor set here right. A line basically you will have straight lines right which are almost like a basically on or you can say on the y coordinate you will have a cantor set. So, this is cantor set cross straight lines that you can see over here.

So, you will find this array of straight lines right which almost appear like cantor set. So, this is one kind of a fractal that you can see here. Now the question is, is this unpredictable? Definitely unpredictable because if you see that points which are very very close by right there almost going up very far apart. And then after so many steps we do not know where these points are landed. So, points which are very close they move apart.

The second thing what we observe here is, that you can decompose it right because if we try to take composite of course, this and that part is made of from this one right. So, we cannot think of these in 2 separate pieces of dynamics right, all the dynamics is basically involved in each other. Set every step what you get is, you get 2 rectangles here then again the rectangle from here comes here a rectangle from here goes there. So, this kind of dynamics is basically involved in each other right. So, we do not find them to be decomposed into 2 pieces and the other thing which we find is constantly some kind of regularity.

So, what is this regularity that we see here? So, we can easily check using the equations that if I take this point 1 by 3 and this point 1 by twice, 1 minus lambda square. If I look into this particular point, this particular point gets mapped to 2 by 3 lambda upon twice 1 minus lambda square, and again when I want to map this particular when I take the function again applied to this particular point, I find that this gets map that to itself.

So, we find that there is a periodic point of period 2 and in fact, it is not very difficult to see that this will have periodic points of all periods. So, we do have some kind of regularity here, because there are periodic points and we know that periodic points are fixed orbits. So, there are periodic points and they are going to have their orbits are going to be stable in the sense that their orbits are going to be very regular right, they occur at each interval you will find them after finitely many steps coming back to themselves. So, you find a sense of regularity here also.

So, we look into next example. So, we take this example of solenoid. So, let us look into this example of solenoid.

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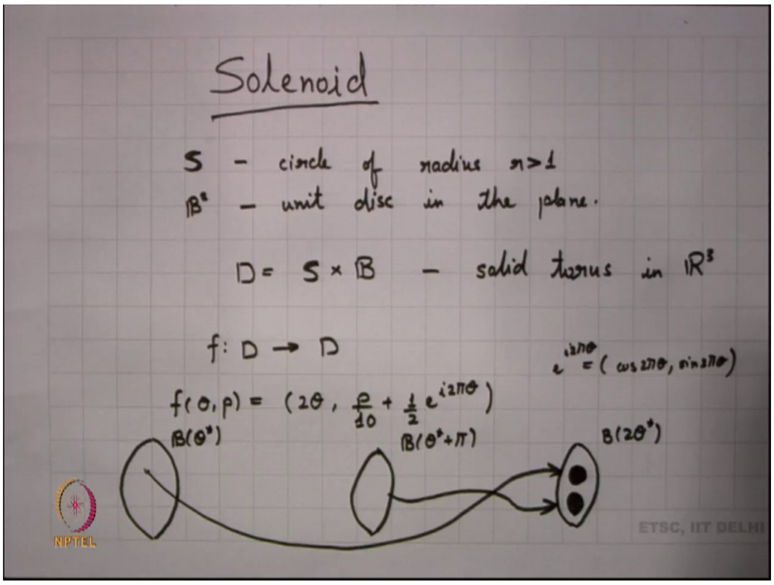

So, what is the solenoid? So, let me take this circle S right. So, this is a circle of radius r greater than 1, and we want to take this unit disc. So, I am taking B square which is a unit disc in the plane. Now I am defining this D to equal to S cross B. So, this gives me a solid torus in R cube. So, we have a solid torus in R cube, and now we want to define some kind of a map on this.

So, we define a function f from D to D as, where again we know what is meaning of e to the power i 2 pi right theta right just the points cos 2 pi theta and sine 2 pi theta. How do we basically describe this dynamics? Now think what happens in the theta coordinate. So, there are 2 coordinates here, one coordinate coming from the circle s 1 coordinate coming from the unit disc.

Now, what happens to the coordinate from circle S? So, the coordinate from circle S right then its just period doubling here right, you find that people goes to 2 theta right. So, its just the end I doubling here right. So, angle gets doubled over here what happens to the coordinating B? Now we know that B is basically a unit disc since a disc of radius 1; what we are doing is that for this disc of radius 1 right any point there would be the disc of radius one. So, it would be something like within this radius.

What happens is it becomes less shrinking it up right. So, we are shrinking. So, the whole disc gets shrink by say here we have taken 1 by 10. So, it is getting shrink by 1 by 10 and plus what do we mean by half e to the power i 2 theta, that it is not remaining at the center right we are also dislocating the center right by say 2 pi theta is full circles. So, we have just dislocating it by a certain angle right. So, what happens here is that, if I now want to look into the dynamics here. So, let us try to see what happens over here. So, supposing now this is a solid torus right. So, I am just taking as cross section here, what happens to the cross section here. So, I have a cross section here right suppose in this cross section I have this v theta star right at the point theta equal to theta star, I am looking into this cross section here.

Now, I already know that what happens any point over here, this since theta star is mapped to be twice theta star right. So, this cross section basically would be mapped to something, which I can denote here it as B twice theta star and what happens now to the whole disc here. Now this is a disc of radius 1 right. So, we find that this disc of radius 1 is now shrunk to a disc of radius 1 by 10 right. So, wherever the point p is right may basically looking into the magnitude basically decreases by 1 by 10, and then this whole disc right is also revolving. So, we would also giving it some kind of a push it is not centered it is not where it was centered, but now all the points are also moving around the circle and write with this particular term.

So, what we find is that this particular disc in B twice theta star, in this particular cross section is map somewhere over here. Now if I look into what happens to B of theta star plus pi what happens to this cross section? Then this cross section is also mapped on to me twice theta star right. If I look into this cross section this cross section was mapped here right in B twice theta star and if I look into what happens to this cross section. So, this cross section is also mapped into B twice theta star, because that is how it goes right, but again I am mapping here is half e to the power i 2 pi theta. So, again that gets map, but then here the angle here is dislocated by half on 1 side, but because of this presence of pi, it is again this located on by half on the other side.

So, what you get here is that this angle turns out to be something of this order. So, this disc gets map. So, basically I can say that this disc was mapped to this particular part, and this disc gets maps to this particular part. So, what you find is that within that cross section if you want to look into the image right, the image you will find that there are the cross section you will find 2 discs right, which are a little bit apart from each other they do not intersect there apart from each other and that is why we have reduce the radius here you reducer the radius considerably. So, that they do not intersect.

So, what do we actually get in this term? So, if we try to say in this particular terms right what happens to the entire torus now.

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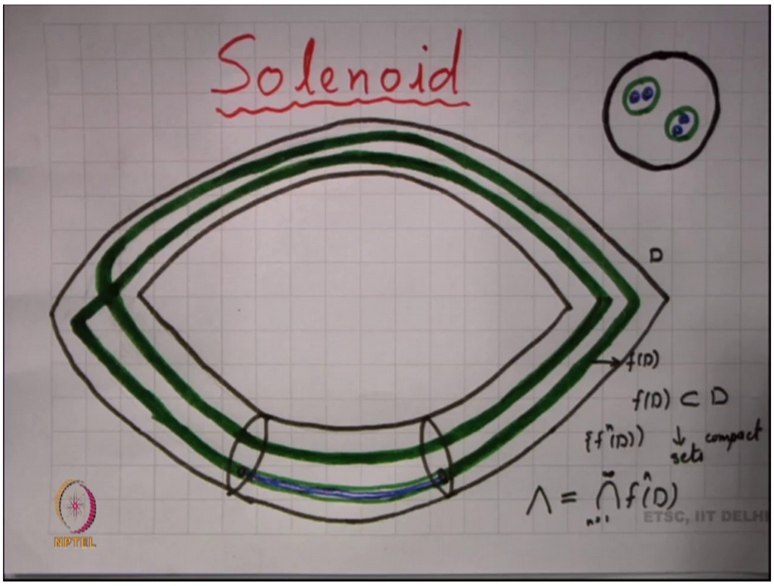

So, we want to look into what happens to the torus now. So, supposing we say that this is our torus right. Now in the torus this is our circle right S and then this is our disc. So, basically there are 2 components here right. This is basically our circle s right and this is what our disc walls. Now we know what happens in the disc gets double, but circle what happens here is because this is elongated twice, but again we are putting that back into itself.

So, what you get here is that this particular torus is mapped into again a bigger torus right we are basically reducing the what to say we are contacting the diameter here. So, if the contract the diameter is contracted, but the length is elongated and then after elongating the length, what we try to do is we try to push that back on itself right. So, what we do is, we are arranging the disc something like this. So, we have the disc something like this coming up or I could maybe use another color to show this. So, what happens here is that, you find that this particular disc was contacted back on itself. Now this you can easily see that, this transformation where not cutting anywhere right we just folding it and put it in putting it back to itself.

Now, what happens? So, this is how you can see the picture here and what happens now if you take one more iteration? So, what happens when you are taking one more iteration, again you are looking into the torus now you have 2 bands of circles right. So, in both the bands now you are elongating the 2 bands right, you are again folding that and putting it back to itself right. So, the global picture looks something like this. So, here you have 2 circles here right you had these 2 bands.

And now I am looking into what happens, it cut over here and look into the 2 bands what happens here is you can find that there are 2 discs right. What you find is that there are again 2 discs. So, there is again 2 discs folded into itself, and in the cross section you can see that right. This was the original disc right the original disc if you see a cross section now we have in this cross section we find 2 discs, but again if you take what happens inside it you can find 2 small discs right and again if you see what happens in the 2 small discs right you will again find that there are smaller discs here right. You will find some smaller disc here again 2 parts over here.

So, if we try to look into this fact what happens towards the end? So, basically if we try to you keep on iterating it, iterating it right where does it lend to? So, find that this will be a highly fibrous kind of picture right, if you see what happens towards the end. Its highly fibrous kind of picture, where you see lot of line segments right and each cross section you will find that this highly fibrous figure that will get in towards the end right if you take a cross section of that what you find here is a cantor set. So, this fibrous picture is basically what is called a solenoid and that is one more example where you can see that see some kind of unpredictability.

Now, let us look into this picture once again. So, this was of our D right what I see the green part here is your f of D, and what we find here is that f of D is a subset of D. Now since f of D is a subset of D what you find here is that f square D will be a subset of f of D. So, if you look into the sequence right f to the power n D right this happens to be a decreasing sequence of compact sets. So, this is a decreasing sequence of compact sets and so, the total intersection will be non empty.

So, our solenoid is basically lambda which we can describe as the total intersection of these discs. This is our solenoid and we see that this is a non empty structure and again if we try to look into what is the dynamics on the solenoid. Now we know that since, but the way if the solenoid is defined right the solenoid is an invariant set. So, f of lambda will be same as lambda right this is an invariant set. So, what happens to the dynamics of f on the solenoid now?

So, we try to look into the dynamics of f on the solenoid, you will again find this kind of regularity right, you will find some kind of unpredictability, and you will find and undecomposibility. So, you would not be able to decompose it into 2 parts, it remains as it is because we know that points are again in the solenoid the points are going to move around right according to the rule f. So, this solenoid and this bakers map that we have seen, these are basically examples which are giving us some kind of chaos using some kind of stretching and folding. So, we using these concepts of stretching and folding to get these examples, and we are not getting into details into mathematics here, but we will study all these properties in detail in another prototype model.

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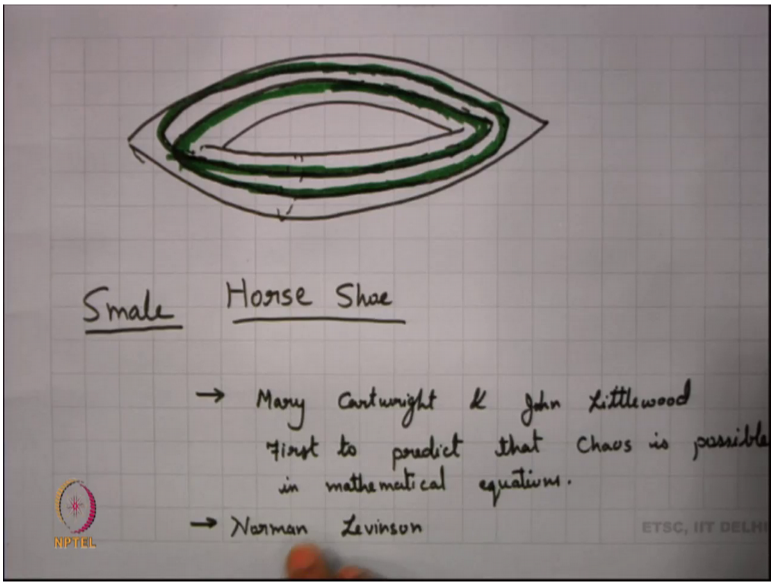

So, what is the prototype model? So, we look into this prototype model which we call as horse shoe. Now what is this model, how did it basically term up. So, this is not just an ordinary horse shoe, we always call it a smale horse shoe. Now it was something around say 19th century when people started discussing about chaos weather chaos is possible of course, people had thought of some kind of structures like bakers, map etcetera, etcetera, but when chaos has been discussed, many times it was considered that in mathematics this is not possible. The concept of chaos right cannot be described mathematically it is not possible to have it in mathematics. And it is more kind of an imaginative concept than a kind of practical concept, because if you want to look into something practical concept, you should have equations describing them.

So, it is more of imaginative thing than a practical thing it is not possible, but there were 2 British mathematicians. So, this is Mary Cartwright and John Littlewood. Now this was the time of world war and they were studying some kind of equations related to radio waves that were being used in the world war. So, during such a kind of study, they came across some kind of equations. And when they try to analyze these kinds of equations, there were the first one to predict that Chaos is possible in mathematical equations.

Of course, they gave their own theory and then it was later Norman Levinson he tried to analytically study these equations. Now when Norman Levinson was analytically studying these equations he did talk to few other mathematicians and 1 of them was Stephen Smale. So, Stephen Smale was originally a topologist and he is best known for his work on point carries conjecture.

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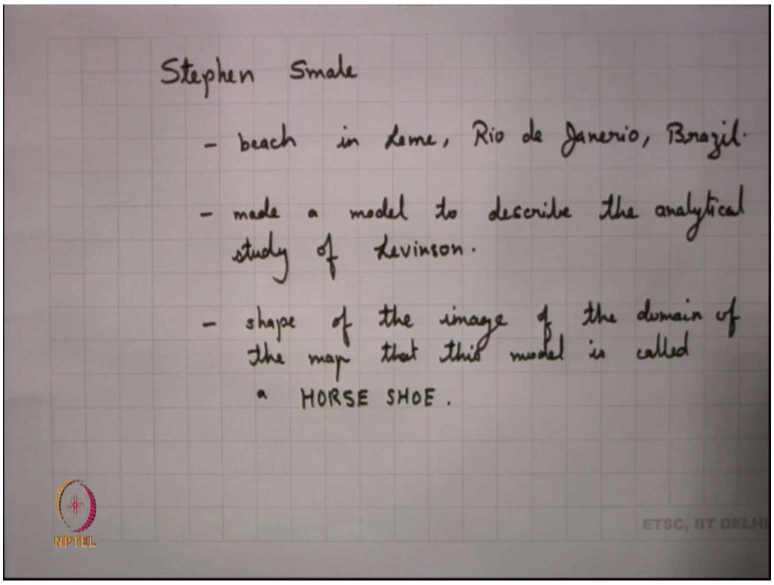

So, Stephen Smale was a topologist who was working in some kind of geometric topology and Norman Levinson he send him his analytical details of looking into these equations and looking into this concept of chaos.

So, Stephen Smale thought of applying some kind of his geometric ideas to these analytical equations that he had seen, and in trying to establish those of course, there is a big story behind it, then at that time Stephen Smale was an sabbatical and he was basically on a beach in. So, it he was basically on a beach in Leme at Rio. So, he had this concept in mind and he was trying to look into how to genetically describe this analytical study of Levinson, and at that time what he realized was of course, what you do at a beach is always you go to the beach and try to draw something, and that is what he tried to do. And he came across a mathematical model of this analytical study.

So, he made some kind of mathematical model. So, he made a model to describe the analytical study of Levinson. So, he tried to make this particular model and it was because of the shape of this graph that he had taken up the image. So, it was because of the shape of this image of this of the domain of the map, that this model is called is called horse shoe.

So, let us try to study this horse shoe, let us now look into this model its basically called Smales horse shoe. So, let us look into this model Smales horse shoe.

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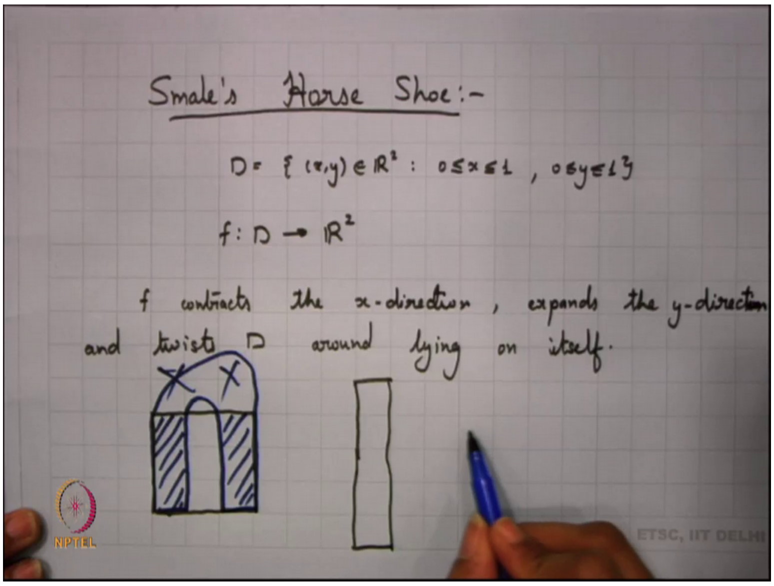

Now, how do we describe this mathematically? So, we again take our D to be the set of all x y in a R square right where 0 is less than x is less than 1, 0 is less than equal to y less than or equal to 1. So, you are taking basically again the rectangle, the square basically unit square and we define f from D to R square.

Now, the description of this f is not very mathematical, what ideally you do is that f contacts the x direction, it expands the y direction and it twists D lying on itself. So, roughly what how fx here is that you have this kind of a unit square here, now what is f doing is, it is contracting the x direction right and it is expanding the y direction. So, I can say that what f does here is that this x direction is being contacted right, see you contacted the x direction here expand at the y direction, and you are expanding in such a way, but now you want to twist the whole image right and put it back over where D was. So, what you are doing here is exactly, you are twisting this image right. So, you are turning this image around and you want to put that back on itself.

So, you put it back around itself. So, what you get here is, get some part of it is lost right if you look into what happens in to D right you have twisted it and put on D right. So, you find that there is some part of D which is again coming back into D, and there is some part of D which is lost right, because this part is completely lost. So, you lose this part and some part comes into D. So, ideally we can define f not exactly completely because this is very nonlinear kind of description that is taken up, where we cannot have mathematical equations to describe this, but we can define it on some kind of horizontal rectangles. So, let us take this to be our horizontal rectangles. So, we try to take up maybe. So, we take this unit square.

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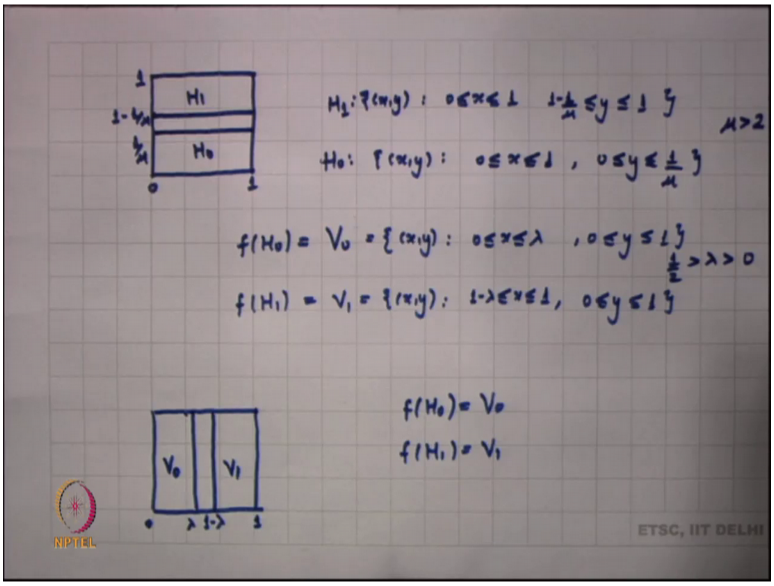

And try to define these horizontal rectangles H 0 and H 1, and we define this on this map affine we define an affined description define it a finally, on this horizontal rectangles, when you are looking into this horizontal rectangles what is H 0 and what is H 1. So, that H 0 happens to be the set of all x y such that 0 is less than or equal to x is less than or equal to 1 0 is less than or equal to y is less than or equal to 1 by mu.

So, we want to know what is our mu. So, we assume our mu to be greater than 2. So, here this is 0 this is 1 and we define this to be H 0 to be from by going from 1, 0 to 1 by mu. So, we have 1 by nu here and what is my H 1. So, my H 1 happens to be equal to again all x y such that 0 is less than or equal to x is less than or equal to 1 and what is your y. So, your y lies between 1 minus 1 by mu less than or equal to y is less than or equal to y. So, this is your H 1. So, this is 1 minus 1 by mu and this is 1.

Now, your defining if a finally, on this right; so you define it in such a manner that f of H 0 is V 0 and if I want to say what does V 0, and you find my V 0 to be equal to x y such that 0 is less than or equal to x, is less than lambda and 0 is less than or equal to y is less than or equal to 1, then the question is what is lambda. So, we take lambda to be greater than say its greater than 0, but maybe it want of a lambda to be less than half. So, we start this is my lambda. So, my lambda is less than half here, and we define f in such a manner that f on H 1 happens to be equal to V 1, where my V 1 is the set of all x y such that 1 minus lambda is less than or equal to x is less than or equal to 1 and 0 is less than or equal to y is less than or equal to 1.

So, I want to take this in such a manner. So, if I am looking into this equation. So, I am looking into this fact what happens here is that, I am looking into this particular part right this rectangle and we want to say that our V 0 is such that right. So, x lies between 0 and lambda right and y lies between 0 and 1. So, this is basically my V 0 and I want to define my V 1 right where x lies between 1 minus lambda 2 1 right and y goes between 0 and 1 and we are mapping it in such a manner that f of H 0 is equal to V 0, and f of H 1 is equal to V 1.

We can precisely state this in terms of equations and what are this equations. So, I am taking this H 0.

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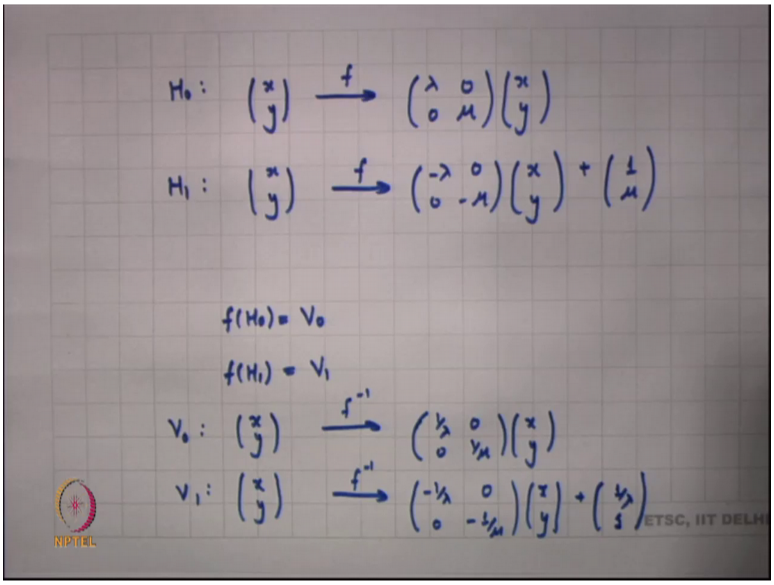

And on this H 0 I am defining my x y. So, my f is defined in such a manner, but this vector fy its map by f xy and on H 1 I am defining my xy map by f 2 minus lambda 0 0 minus mu xy plus 1 mu.

Now, why did we take the minus sign over here? So, as you have noticed this, what we are doing is we are stretching this right and then we are putting it back on itself. So, when we put in the back on itself right we are basically changing the orientation right. So, the 0 part is being mapped onto the one part right. We are change in the orientation because we are putting it back on itself in a horseshoe form. So, we have this negative sign here. So, we can a finally, define this map on H 0 and H 1 and we can easily see that f of H 0 will be equal to V 0 and f of H 1 is equal to V 1, where in V 1 right it is going on like the increasing x right what to find is that the increasing y is being mapped onto the decreasing x.

So, this is; what is basically a figure here. So, you have your H 0 and H 1 right and your H 0 and H 1 you are defining your f in such a manner, that basically this is elongated right.

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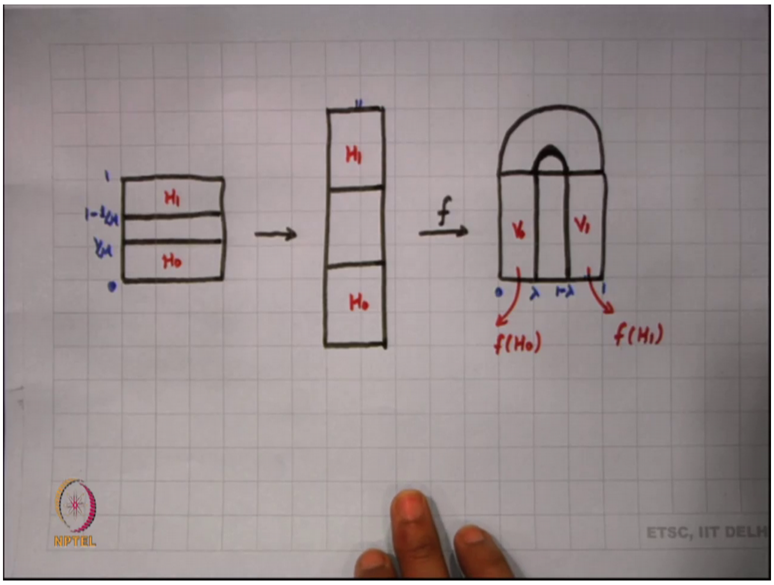

So, you get not H 0 here you get H 1 here because you are contacting the x direction with expanding the y direction and since you have put this in form of a horseshoe right you start with here you put this here right you get your V 0 here, this is the missing part here goes out of the rectangle out of the square basically. And what you find is H 1 now is mapped in such a manner that this particular portion is map here right. So, you get it mapped into this particular manner, and this is basically your f of H 0 which is V 0 and your f of H 1 which is V 1.

Now you could also think of defining this in the reverse manner. So, you can also think of f inverse acting on D. So, we find that f inverse acting on D right is defined in such a manner that if I take my V 0 right which is basically xy, what happens under f inverse. So, under f inverse right this is mapped in terms of say 1 by lambda mu. So, this is mapped in terms of 1 by lambda 0, 0 1 by mu times xy and V 1 is mapped again because we have a twisting here you have a negative sign here.

So, define f inverse also similarly and if I now along to look into f inverse now, f inverse is defined not completely, but f inverse as defined on V 0 and V 1. So, if we try to now look into 1 2 S inverse is defined.

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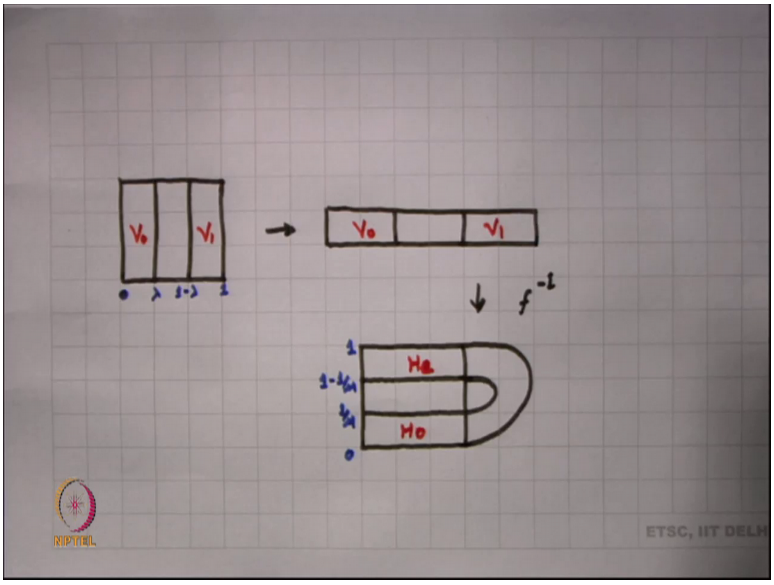

So, this is how f inverse gets defined. So, we find that you have V 0 which is basically a rectangle lambda cross 1 right of dimension lambda cross one. So, you find that V 0 right what happens what does this map do is, that it is basically expanding the x direction and contracting the y direction right, and now the y direction is contracted in such a manner that you map V 0 is mapped onto H 0 by f inverse right and V 1 is mapped into H 1 right it goes in this particular direction.

What I should say it should be taken in the other way round sorry this is line H 1 this is my H 0 and this is my H 1, it is mapped into this particular direction right your V 0 is mapped into H 0 you place it over here then you take this part you twist this part which we do not care of right and then V 1 is mapped into H 1 right it place it over. Now what happens if you look into this map, what is the mathematical significance of this map or basically what is geometric significance of this map?

So, we can see a small lemma here.

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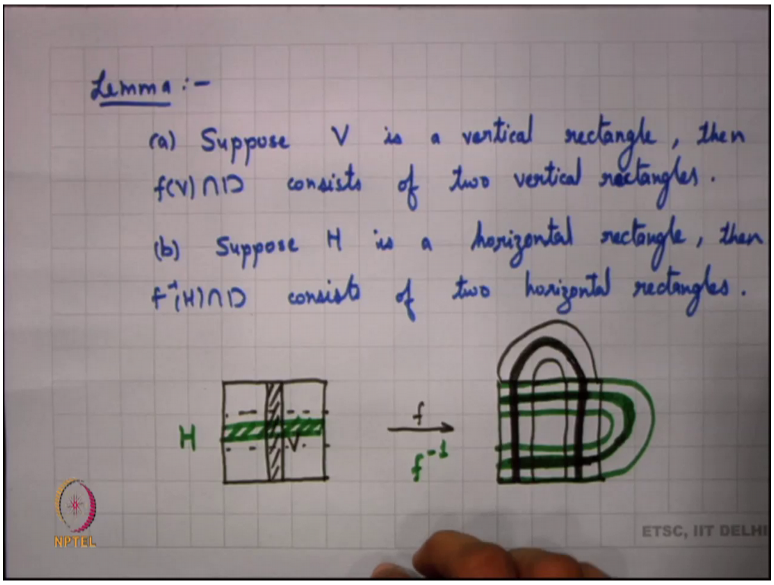

So, let us look into a small lemma here, I should not say it is a lemma it is just an observation, but you look into this observation it says that suppose V is a vertical rectangle then f of V intersection D consists of no vertical rectangle, and on the other hand suppose H is a horizontal rectangle then f inverse H intersection D consists of 2 horizontal rectangles.

Now, let us try to see this in the picture here. So, what I have here is, I have my D here now in this particular D, I am looking into a vertical rectangle here. So, let me try to see let me call this as my vertical rectangle. So, this is my vertical rectangles V. Now how is this mapped? So, we know that what we are trying to do is right how does f act here is right that we are trying to elongate the y part right and then putting it back onto itself right. So, what we get here is under f what you find here is, this is how the whole D is mapped because here I can divide this into say H 0 and H 1 right and then we know that this H 0 is part is mapped onto this one the H 1 part is mapped onto this one. So, how is this vertical rectangle being mapped to?

So, this vertical rectangle is being mapped like this. So, you find that this vertical rectangle right its not something in this manner, and then what is f of V intersection D. So, we find that f of V intersection D is again 2 vertical rectangles right again what happens to H. So, if I take any horizontal rectangle. So, I can think of some horizontal rectangle say I take a horizontal rectangle here, and I call it H right how is f inverse being mapped to? Now we very well know about how f inverse maps right it maps V 0 to H 0 it maps V 1 to H 1. So, what you find here is that this horizontal rectangle would be mapped in such a manner. So, we know that this is basically mapped how is this map. So, this is map something like this form.

So, how is this horizontal rectangle being mapped? So, this horizontal rectangle would be mapped something in this particular form and so, if I take a f inverse H. So, this is like how my f inverse maps right. So, this is how f inverse is mapping. So, if I take my f inverse of H and I will signed map it intersected with D, I get 2 horizontal rectangles. So, horizontal rectangles; so vertical rectangles and their f of map to vertical rectangle then we get 2 vertical rectangle there of course, the diminished are basically they are contacted right and for a horizontal rectangle under f inverse we find that a horizontal rectangle is not to 2 horizontal rectangles and again they are contracted.

So, how does my f function? So, bigger concept here is we can see this part figure how is f acting now?

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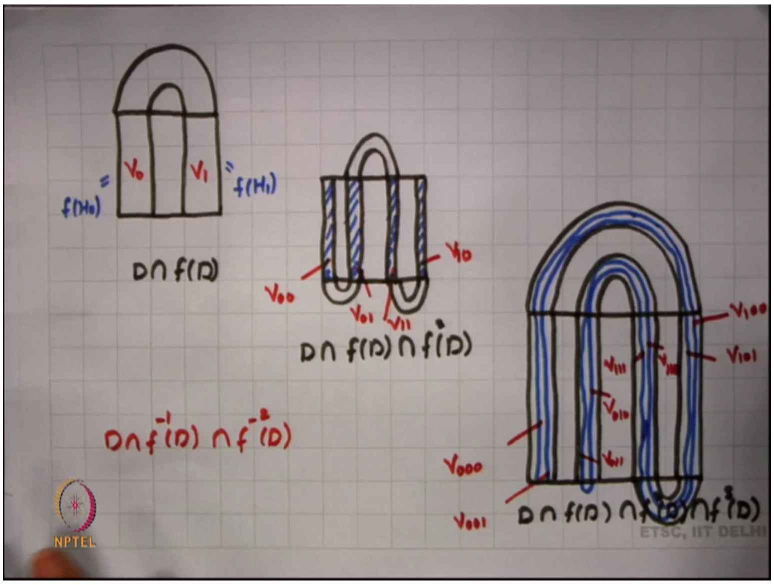

So, we start looking to into this figure right now when I am looking into what is my D intersection f of D, we know that we had 2 vertical rectangles here V 0 and V 1 right because that is how H 0 is not because my V 0 is nothing, but it is basically f of H 0 and my V 1 is nothing, but it is my f of H 1 right. So, this is how we had mapped H 0 1 H 1.

Now, I particularly now want to look into how does D intersection f D intersection f square D workout. Now we know that we have vertical rectangles V 0 and V 1. So, V 0 will give rise to 2 vertical rectangles right under f it will give rise to 2 vertical rectangles and if I look into V 1. So, V 1 is going to again give rise to 2 vertical rectangles over here. So, we can say that this vertical rectangles we can call them as V 0 0 right V 0 once basically V 0 0 is mapped here V 1 1 is mapped here, V 0 1 is mapped here v 1 1 gets mapped here and this is V 1 0.

So, we find that this gives rise to 4 now what I have is in D intersection f D intersection f square D I get 4 vertical rectangles right. So, these are the 4 vertical rectangles that you can get here. So, this is one rectangle this is another one right this is the third one and this is a fourth one. And I can name it from where it is coming from; so from V 0 if I apply again V 0 into. So, V 0 will have 2 parts. So, one will be V 0 0 and one will be V 0 1 and V 1 will have 2 parts which is V 1 1 and V 1 0.

Now, what happens if I take it further? So, if I take it further right what will I get here? So, let us look into this part now I am just stretching up this part right. So, this is what is my f of D, D intersection f D intersection x square D right when it stretch it further we get again 2 parts here. So, I am just trying to draw this part over here. So, now, if I look into each of this part what happens is, each vertical rectangle here is giving rise to 2 vertical rectangles.

So, we can say that here I find so many parts I can say that this is my V 0 0 0right this happens to be my V 0 0 1 right this was originally my part of V 0 1. So, we find that this has 2 parts here. So, this part is V 0 1 0, I find this part which I can name as V 0 1 1 right I find this part which was originally my V 1 0. So, we find that this is V 1 0 0 and this part I can say that this was 1 0 1 right and this particular portion, which was originally my V 1 1 we find it as V 1 1 0 and this is my V 1 1.

So, what I find is that at each step right we are getting vertical rectangles, now if we look into the vertical rectangles right basically if you look into their width right their width is basically decreasing. And you can imagine what happens if you continue this process right.

So, since again here I have my right this is a subset this is a subset of this one right. So, again we have decreasing sequence of compact sets and so, the total intersection will be non empty and that is; what is the Smale horseshoe. But the Smale horse shoe is a little bit more than that, because here we are also looking into what is f inverse. So, I can think of D intersection f inverse D right.

So, we can also think of D intersection f inverse D right and then I can again look into intersection f minus 2 D, where now what will happen is D intersection f inverse D is going to give me horizontal rectangles right. So, I keep on getting more and more horizontal rectangles right. How do we perfectly define the Smale horseshoe is something which will be seen next time right because that still involves a lot of analysis. So, today we stop here.