Chaotic Dynamical Systems
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Lecture - 01
The Beginning

Welcome to the students. And today we start the first topic that is the Beginning. So, in this course of chaotic dynamical systems, let us start looking into; how did this entire subject begin. So, to say that how did it all begin, right. We can first look into the name of Ptolemy.

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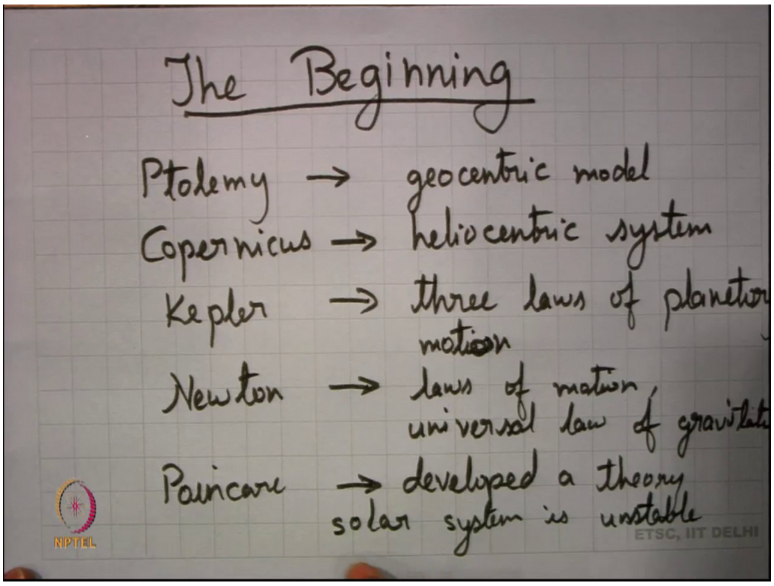

Now, you all know about this Greek mathematician, and what was his contribution was the geocentric model of solar system.

So, he had actually hypothesis that the solar system, in the solar system the earth is at the center and the sun and the planets all are revolving around it. Later on, this model was contradicted by Copernicus, who gave the heliocentric model. So, he described the heliocentric system, which said his model was that the sun is at the center of the solar system, and all the other planets are moving around it. Now in between ptolemy and Copernicus in India we did have Aryabhata giving up his model of solar system.

So, Aryabhata and Madhavaram had described some kind of a system. Now after that we had Kepler who gave the 3 laws of motion. And after Kepler, we had again Newton who essentially gave the mathematical laws of motion. So, he essentially described the laws of motion in terms of mathematics. And also discussed the universal law of gravitation with Newton he basically described the 2-body problem; that is, how does the earth move around the sun or how does the moon move around the earth. He not only gave the 2-body problem, he basically described the 2-body problem in terms of mathematical equations. He solved those mathematical equations, and he deduced that most of the solar system most of the dynamics that you see around is integrable. So, he used the concept of integration to look into the solution to look into the solution for systems.

Now, after that it was generally believed that all systems all kind of solar systems or whatever dynamical systems you come across, they are tan tending to be integrated systems. And looking into the integrated systems, then people try to devise more and more systems around it. And the logical term came up to the 3-body problem. So now, we have a sun the earth and the moon. So, what happens to the 3-body problem? And to solve the 3-body problem actually for a long time nobody could succeed in solving it. And it was later Poincare who gave us a he devised he developed a theory to say that the 3-body problem cannot be solved. And that the solar system is unstable.

Now, this is what was the beginning of Chaos.

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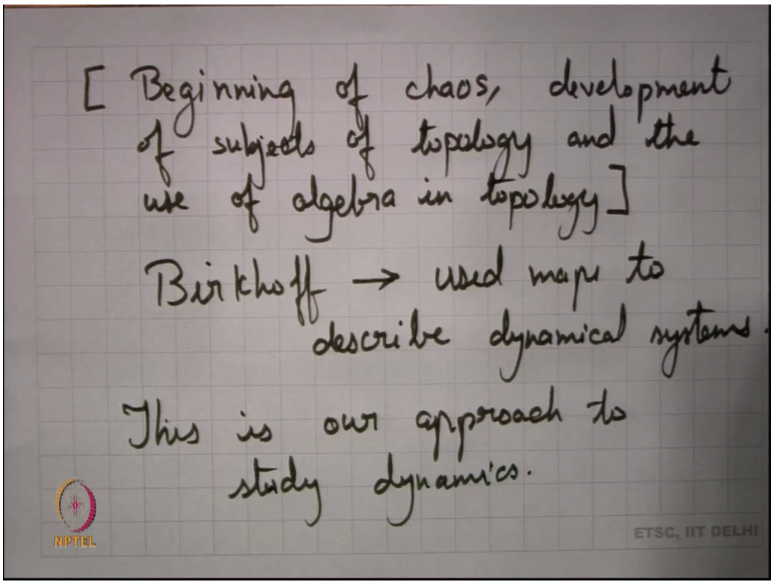

This was also the beginning of the subject this was also the development of subjects; the subjects of topology, and the use of algebra in topology. After Poincare developed this it was one of a student's Berkoff. Who started basically used maps to describe dynamical systems. And he started solving dynamical systems with the use of maps.

Now, it was after it was this approach of Berkoff that we shall be using throughout this lecture. So, this is our approach to study dynamics. How do we briefly summarize this theory of dynamical systems? So, if we start from the starting, right. This theory has come up with 2 great revolutions.

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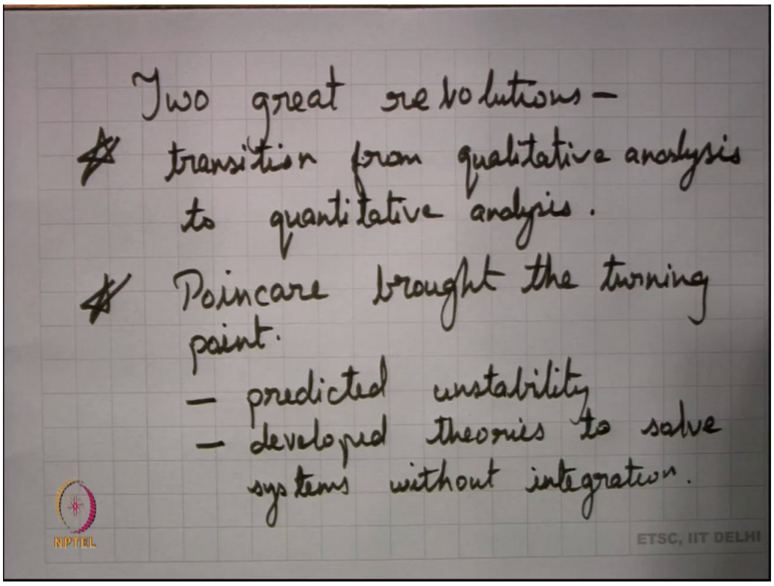

So, this theory came up with 2 great revolutions, if I look into the first revolution. So, the first revolution was the transition from qualitative this was the transition from qualitative analysis to quantitative analysis.

Now, what is what do we mean by saying that we have a transition from qualitative analysis to quantitative analysis. So, what happened essentially was that when Kepler studied. So, the loss given by Kepler etcetera what Copernicus discussed. So, these were all theories. So, it was like believed that it happens like this it believed that happens like this. With Kepler giving up his 3 laws of motion and with Newton exactly describing it in terms of differential equations, this qualitative theory was actually quantized. So, you had a quantitative way of looking into how dynamics happens.

Now, with this kind of quantitative part then after Newton it was basically Euler LaGrange Hamilton etcetera, they tried to develop this theory further and further using this quantitative methods. So, it was generally every system was put up in terms of differential equations. And then integration methods were used. So, integration methods develop very fast. And integration methods were used to solve all this differential equations.

Now, what happened as an essence to it was that people generally went from. So, all this entire mechanical system which existed in the world got a very nice mathematical theory. Now using this mathematical theory people tried to solve all sorts of system, but then the turning point came up with Poincare. So, Poincare brought the turning point. Now what was Poincare's contribution? So, basically Poincare developed a method as I said to show that the solar system is unstable.

So, Poincare predicted he predicted basically unpredictability, unstability. He developed theories to solve systems without using any integration. Now what happened as a result? So, as a result the entire subject of topology was born he developed topological methods he developed analytical methods, and that is how he came up with showing that the system happens to be unstable the system is unstable the system cannot be solved. And there are other methods which can be applied to study dynamics.

Now, this was the second revolution for system for the study of dynamical system, because then again, all the quantitative methods again became qualitative methods and later on Berkoff etcetera. They tried to implement this method for studying of dynamics. So, starting from the point of integration of a system, Poincare introduced the subject of topology for answering such questions by just not looking into the differential equation involved, but looking into what is the nature of solution, and trying to see how one can predict the nature of solution using whatever initial conditions are given up.

So, they are almost all stages then there was a need that in all stages we should be able to find we should be able to predicts systems such that, they have a given properties at almost all times. So, not all times very important, but almost all times was important, and that led to the subject of measure theory. So, measure theory the subject of measure theory was born with by this basically to study the statistical properties, which are associated with the systems with the solution of these kinds of dynamical systems.

So now one can essentially ask a very important question what is a dynamical system.

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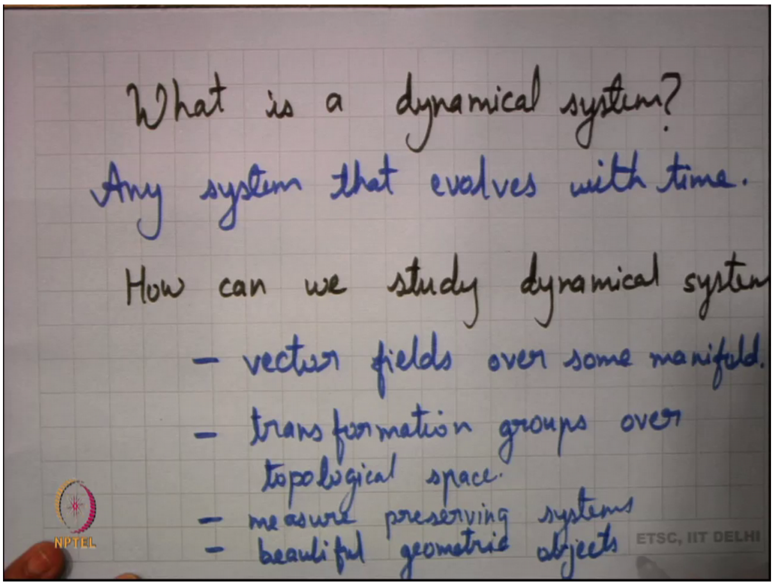

And if one has to answer it, well a very simple answer would be any system that evolves with time. Now any system evolving with time; that means, we are looking into time as a parameter time continuously changes, and any system which evolves with time is can be considered to be a dynamical system.

So, here one can also ask the question, how can one study dynamical system. So, how can one study dynamical system? If one looks into that, right. One can study them as vector fields of so, vector fields over some manifold. And what essentially one studies over here is that one studies the kind of smoothness of the curves, the integral curves, right. One studies what are the critical points that one can find over there. And that is essentially; what is the behavior of the nearby points, right; at some particular instant. So, this is what one studies if one has to look into that as a vector field or if one looks into the differential aspects of the dynamics.

The other study can be looked into as the study of transformation groups over topological spaces. So, one need not get into the differential structure, one can simply look into the continuity of the system. And one can look into transformation groups acting on topological spaces, and study the topological aspects of the system.

Then there is also a way of studying, where one can look into without if one is not much concerned with the continuity then one can look into what are the measure preserving measure preserving systems on say measurable spaces. Apart from that there is also a way to study where one can study beautiful geometric objects, which result from the dynamics. And these objects as you all know are called fractals. So, one can also look into the aspect or look into the analysis of fractals because this is where the dynamics happens to be chaotic. So, one can look into this aspect also for in study of dynamical systems.

Basically, we can say that the function precisely any function along the trajectories the time average for any function along the trajectories the time average is equal to the space average.

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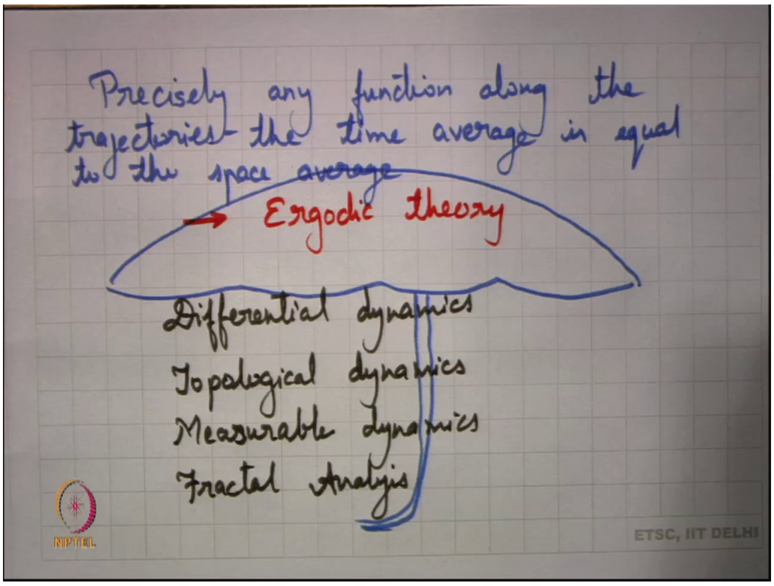

This is essentially the concept of ergodic theory, which results in the subject of ergodic theory. So, once looks into this entire concept from all points of view even looking into the statistical part.

Now, what does essentially everything sum up for. So, one can say that when we are looking into the differential aspects of the dynamical system, one is essentially studying what is called differential dynamics. When one is studying one is looking into the topological aspects of the system, one is basically studying what is called topological dynamics. When one is looking into what is basically without looking into the statistical property of the system one is essentially studying what is called measurable dynamics. And when one is looking into what are the objects generated, right. Basically, one is looking into fractal analysis.

And all this topics basically come under the whole umbrella of ergodic theory. So, these topics basically tend to be subtopics of the mathematical subject of ergodic theory. And what we shall be looking into is since we shall be looking into very elementary systems, we shall be touching almost basic concept in each one of this without getting much into details.

We let us look into how did we actually come how did we actually transform our system from the quantitative part to the qualitative part. So, to get from the quantitative part to the qualitative part we first looked into the concept of differential equations. So, what happens when one is starting from differential equations?

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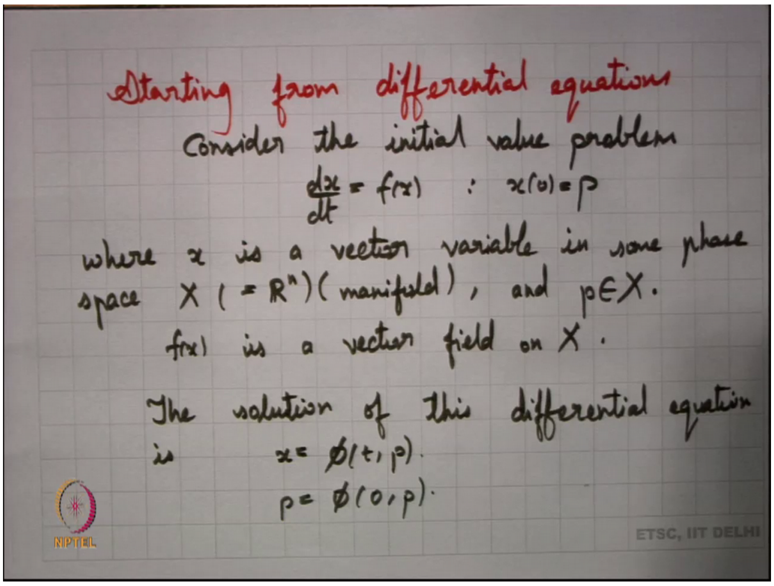

Now, when one is starting from differential equation, we first consider an initial value problem. So, we consider this initial value problem. So, my initial value problem can be said as say dx by dt is f of x and x at 0 is my point p, right. Where my x is a vector variable in some phase space x, where my x could also be R n, right. Or it could be any manifold if one looks into the general aspect of this. And my p happens to be some point in x, what is fx in that says?

So, in that case my fx is a vector field on x. Now what is the solution of this differential equation, since we have a differential equation? We can think of solving it. Now suppose this differential equation is solvable, right. Then the solution of this differential equation is I can look into this as x equal to say a function phi, which is a function of t and of course, the function of the point initial point p.

We all know that solutions of differential equations are integral curves. And each integral curve all this in different set of integral curves are disjoint, right. And each integral specifying a point through which the integral curve passes, right. One can just specify the solution at that particular point. So, when we are looking into an initial value problem, we are basically looking into the integral curve or basically we are solving the integral curve at an initial condition at that particular point.

Now we can say that this happens when my p is nothing but phi of 0 p. So, at the point 0, right, your integral curve lies at the point p. Now this is what had been looking into. So, whenever we look into a system, right. We are trying to solve the equation, we find a set of integral curves, and an integral curve passing through the point p is basically what we look into the solution at p.

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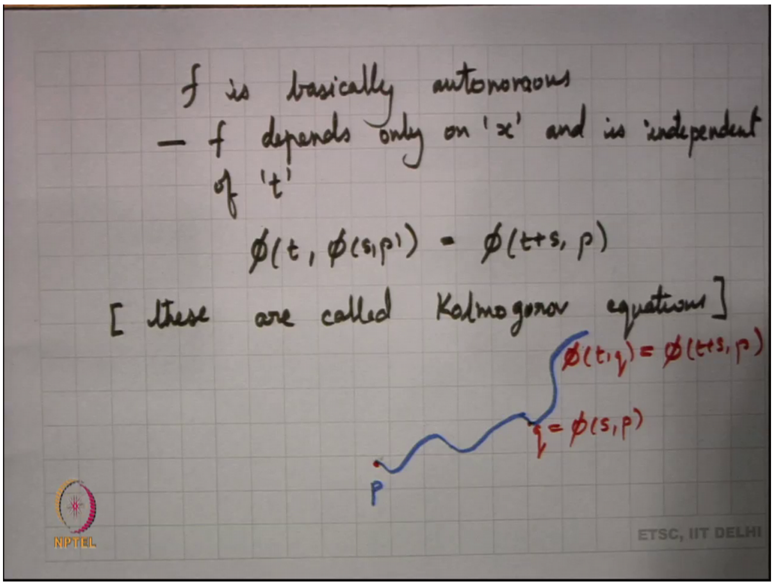

This integral curve can also be looked as the orbit of p, right. Because p happens to be phi 0 p we start at p, right. There is an integral curve passing through p.

Now, in most cases we work with our system f, right which is autonomous. So, our f is basically autonomous. And what do we mean by it being autonomous is that f depends only on x and is independent of t. So, it depends on your variable x, and it is independent of t. When we have an autonomous system in autonomous differential equation and autonomous system, and when we are looking into the solution of this differential equation, we find that this solutions the satisfy certain semigroup identities, and that semigroup identities is phi of t or phi of s p is basically same as phi of t plus s and p. And these identities are also called as Kolmogorov identities equations or identities.

Now, what does mesa say essentially, what does that mean? So, what we have is that we have this particular point p, right. And I am trying to solve this equation at this particular point p. So, suppose now, I have solve this equation for say time equal to s what happens time equal to s, right. What happens where does p move, right. After time equal to s, and supposing that it reaches a point q. So, this is my p and one finds that it reaches this point q, right after time s. So, basically my q happens to be equal to phi of s p; now, starting from p when starts from p and then reaches say perhaps q at this particular instant.

Now I again want to solve this same equation considering the initial point as q. So, considering the initial point as q, we solve the same equation, right. And maybe after time t, you reach another point which I could frame this as phi of t q, right.

Because after all we get the same solution there what changes is the initial point, then once when we reach this particular point, right. Then if we start from p and we solve this equation at the time t plus s, right. We would have basically reached the same point. So, basically, we reach this particular point t q, which is same as what the solution we would have got by starting from p and looking into the time t plus s.

So, what we get here is the Kolmogorov identities which say that; this particular solution has to satisfy this identity. So, irrespective of what is your starting point, right. After a given time t your integral curve satisfies the same or basically your integral curve is following a definite path; so the solution of dynamical system or the solution of differential equation, right. It basically satisfies the same path moving across this thing.

So now comes where we would like to look into dynamics once again think of that I am now looking into this part the solution, right. Now the solution I can consider that p happens to be a parameter here, right. Because again if I am describing the solution, right. My solution is described in terms of what is my initial point. So, my point p the initial point p happens to be a parameter here. And if I look into what is my t, my t happens to be a variable here. So, looking into t and looking into p we find that this is a parameter the initial point turns out to be a parameter, and time happens to be a variable. What happens supposing, we want to look at the same system, and we try to switch what is the notion of parameter and what is the notion of variable. We know that when we are looking into some kind of family of curves, right. Because after all your integral curves divide the whole system, right. Across every point there is one integral curve, and no 2 integral curves will intersect.

So, you have both the things, right. The parameter as well as the variable, right. They are both changing continuously as you switch into points. So now, if we look in the whole system, right. We have something called parameter we have something called variable and both of them are not stagnant they both keep on changing. So, what we try to do here is now for our study what we try to see is we try to look into this again. And we switch the rules let us switch the rules. So, let us look into our parameter which was our points, let us now take them as variables. And let us look into time which was our variable now as a parameter.

So, that means, now we have a union of integral curves for the union of integral curves when we are changing our time, right our variable x that is our position vector, right becomes a variable there because then my time is kept fixed. So, at a particular time t, right; I will get a series of or and get a series of function points x, right they need not be a very smooth picture need not be a very continuous picture need not be a continuum in that sense, but we get that as our variable.

So, if you look into that aspect, what we get now is the switch over. So, let us know the switch.

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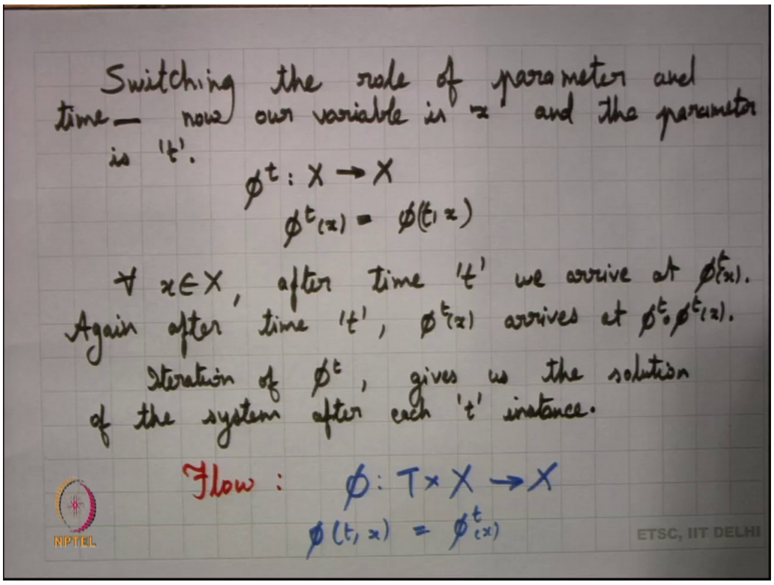

So, switching the role of parameter and time, what do we get here is now. So now, our variable is x, right and the parameter is t.

So now with this part we can think of keeping t fixed, and changing x. Now if you keep t fixed and change x, what we get is a curve phi t, right which varies from x to x, right. And I can describe this function as phi t of x, right. It is nothing but it is basically my phi of t x which I had started initially. So, phi t of x is nothing but the original integral curve that we had taken up phi. So, it is phi tx, right which is solution of that particular equation.

So now I get a map here from x to x. So, starting with a differential equation, we are now looking into this now as a map. So, we get a map now, now what does this map tell me? So, there is map tells me that after t units of time, right. Starting from x after t units of time where do we land up right. So, this is loud looking into the position of the point, right well, where does the system land up after a given time t.

So now if we look into that aspect, right for every x in x, right. We start with every x in X fine what happens is after time t we arrive at phi t of x. So now, we are looking into the system not in a continuous manner, but in jumps of time t. Because now if I want to look into what happens to phi t of x, right. Then after t instant of time phi t of x goes to what is I can say the iteration of it phi t composite phi t of x, right.

So, again after t instance what we have now is an iteration of this function phi t and this function phi t the iteration of this function phi t gives us how we are looking into how the system appears. So, gives us it gives us solution of the system after each t instance. So, after each t instance now we are getting a solution and that is given by the iteration of the map phi t.

Again, how have we moved on. So, if you look into this aspect this gives us something which is called a flow. Now what is a flow? So, a flow is basically I can think of this as a map phi from t cross x to x. Where I can simply say that this map is given as phi of t x is nothing but it is phi t of x. So, this gives me something in terms of a continuous factor what happens is that I can get a map basically not on x, but now I am looking into the product t cross x I get a map which is nothing but it gives me what is here phi tx.

Now, again what happens to such a kind of a flow if one thinks of then for such a kind of a flow also we find that phi t of x satisfies this also satisfies the identity the Kolmogorov identity.

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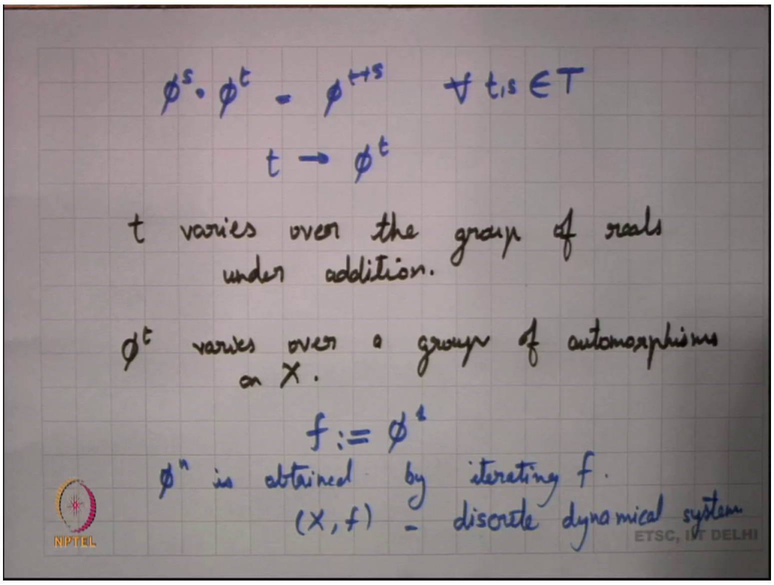

That phi of s composite with phi of t, right is nothing but phi of t plus s and this is true for every tns belonging to t.

So, we have this identities tns coming up with t. So now, what have we achieved right? So, what we have achieved is we have achieved something we have achieved an association t goes to phi t. Let us try to understand this association. Again, what is t? T is varying over the reals, right. So, I can say that my t varies over the group of reals, right under addition. So, t varies over the group of reals under addition. And what is phi t? Now if I look into my phi t my phi t is basically it is a map from x to x right. So, I can think of this as sort of an automorphism, right on x. So, phi t is basically varies over a group of automorphisms, now what do we basically mean by automorphisms we are basically looking into because each phi t. Now try to understand phi t, right each phi t for unique x, right. It is going to give us a unique y, right.

So, if I look into this association this phi t is a surjection. This phi t is injective right. So, it is basically phi t happens to be bijective mapping on x, and not only it is a bijective mapping on x, right. It is also continuous can I think of it is inverse. So, if I look into phi t; that means, I am moving forward t steps, right. Moving forward t steps I can also move t steps backwards, right if I move t steps backwards or basically when I look into my time t to be negative, right. What we are doing is we are just moving backwards now; that means, what we are doing is we are tracing the curve, right in a backward manner. Now when we trace the curve in a backward manner, right where phi t happens to be something like an invertible map, right.

We also know that it is bijection, it is continuous. And it also comes up from or one can say that it is inverse is also continuous. So, this phi t is essentially a homeomorphism on x, and since I am talking of a homeomorphism it is a self-homeomorphism on x, you can think of phi t to be an automorphism. So, what phi t forms, right. It is basically one can say that all this phi t basically it forms a group of automorphism, it is a group, when once when things of automorphism, right. It forms a group under the natural operation of composition right. So, this forms a group. So, one can say that your phi t which you get here, right. It forms a group of automorphisms on x.

Now, if I think of this phi t also to be differentiable, right. Then we know that homeomorphism which are also differentiable. Of course, when we talk of differentiability we keep in mind that we are working with not in any abstract things, but our x has essentially happens to be having a manifold structure.

So, once when we talk of phi t as to be differentiable also we can say that this is a group of diffeomorphisms on x right. So, essentially what we have done here is that we have associated each time t, right with a group of automorphisms on x. So now, look into the fact what has happened here is we had a differential equation, we tried to solve the differential equation, right. Now this differential equation gives gave us disjoint integral curves and each of this looking into each of these integral curves changing the concept of this parameter and variable, right. We could end up with associating time with a group of automorphisms.

And now essentially what are we now looking into the study. So, our steady now is concentrated on what happens to the automorphisms on x. So, we start with our x maybe our x was a manifold maybe our x was R n, right. And what we stride was we associated a group of diffeomorphism to x, and now what we are essentially going to study is the group of diffeomorphisms.

Now, this variable t no longer matters to us. Because what we have done is we have done this identification of t with this group of automorphisms. So, instead of looking into how t varies and looking into how does the function behave and t varies, or how does the curve behaved at t varies, how is the system behaving when t varies. We are now essentially just looking into how does the group of automorphisms behave, right as your x varies, right. You are looking into all the automorphisms coming up from the space x.

So, this is essentially how you can convert the quantitative theory to the qualitative theory starting from differential equations, now what we get is sort of a discrete system, right. Or I could say a difference system. So, what happens here is that one. So now, I am saying what happens now if I am looking into the fact that I am pushing this 1 to t to be equal to 1, what happens at one instant?

So, if I am looking into phi of one, right. I can say that by definition I call this a map f. So, what now happens is f is nothing but it is phi of 1. So, instead of looking into t as t varies over all the reals. Now we want to look into t only when t varies over all integers. We are discretizing time right. So, instead of looking time as a continuum R, we are now looking into the discrete z, right. We are looking into the terms of integers. So, supposing I call f to be equal to phi of 1, then f equal to phi of one gives me basically a map. F is just a map phi of 1 is a map right. So, f is basically a map it is a map from x to x, right. And what we have done is now we have studied we have discretized this part, and then what happens is I can get phi of n, right this is obtained iterating f. And we get a discrete dynamical system x f, right. We get this discrete dynamical system xf.

Now look into this discrete dynamical system. This is what was the model proposed by Berkoff that most systems can be studied in this manner. Take any system there is a way to convert it into a discrete system. What happens now? I want to study some system which evolves with time in order to study some system which evolves with time I start with the continuous of course, I need to study a flow. A flow means I am looking into time as a continuous variable, right. A discrete flow is something when I am looking into time as a discrete variable.

So now essentially time is always a continuous variable. Time is not a discrete variable, but when we try to take observations our observations basically come from a discrete. We are not observing all the time maybe we are taking some observations whether it is an experiment or whether it is a system, right we are taking an observation using discrete time. So, maybe we take a maybe essentially, we could take observations every R or maybe we could take it every day or maybe we could take it every week.

So, when we try to make such observations we are essentially, looking where our observations are always discrete. And that is that gives a more emphasis to study each system in terms of discrete states. Our system cannot pick a week we do have continuous system all our systems are continuous. In fact, we cannot solve discrete system using our integration, because our integration method itself is continuous.

So, if we want to integrate a system and study it definitely we need a flow, but if we do not need a flow if we if we want to lick look into our observations, our observations are discrete. Many times we have over discrete observations which we interpolate and make it into a continuous observation, right.

So, we are looking into this discrete system, and henceforth we will all be looking into this discrete model that Berkoff had proposed to study dynamics. Because essentially, any property any nature of dynamics can be studied using such discrete systems. So, if we try to look into this discrete system, right. We can think of this factor and we can call this as a discrete dynamical system.

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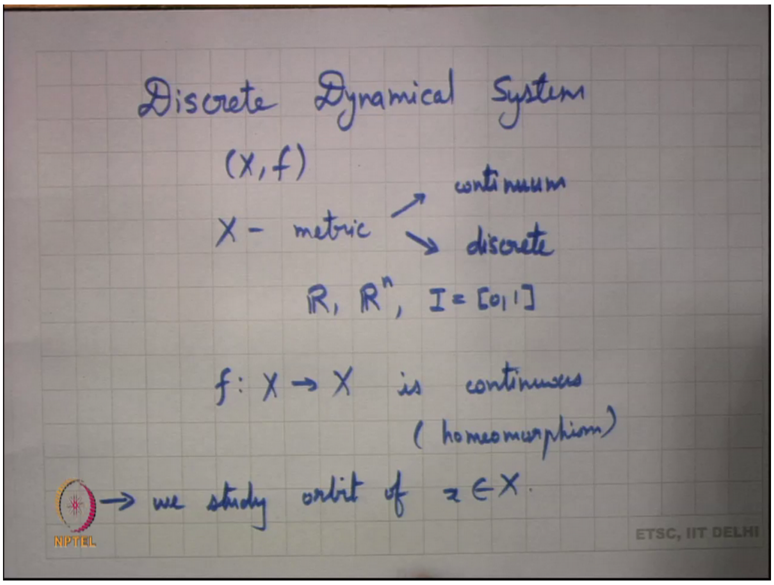

Now, what is the discrete dynamics? So, in discrete dynamics, we have a pair x f. Now what is x? Now to be very frank, we do not have any differential equation in our background. So, what we try to do is we can just think of x to be any space. So, when I think of x as any space, I could well work out with a metric space, right. Of course, we need to define a function on it. So, it needs to it is a space. So, it should have some kind of a structure right. So, we can work out with x to be a metric space, maybe our x can have some kind of structure over here our metric space can have some kind of structure over here. Maybe it is a continuum or maybe our metric space is discrete.

So, maybe we can have we can work with some non-metric spaces in the non-metric spaces can turn out to be R. Or I could say the n where n dimensional R n, right. Or we could just work with the unit interval 0 1, right this gives us a continuum.

So, when we started with that with study of discrete dynamical systems. We are basically working out with these aspects, and then what is our f? So, my f is just a function from x to x and this is continuous. Yes, I could think of this to be a homeomorphism in that sense I want my f to be invertible also. So, this could have this property of being a homeomorphism depending on what my x is supposing I give some kind of a manifold structure to x, I can think of this to be also diffeomorphism, right.

Supposing I do not want my x of course, it is like trying to I do not want my f to be continuous, right. Just looking into we just want some to study some kind of statistical properties, we need not have f to be continuous, right. In that case we can think of f to the measure preserving, right. And then your x happens to be measure space, right. You have some kind of probability measure defined on x, and your f is nothing but it is basically a measure preserving transformation.

So, we start with that and what essentially do we study over here is the orbit of a typical point x in x. So, basically, we study so, what is orbit right. So, one can define this factors you start with point x.

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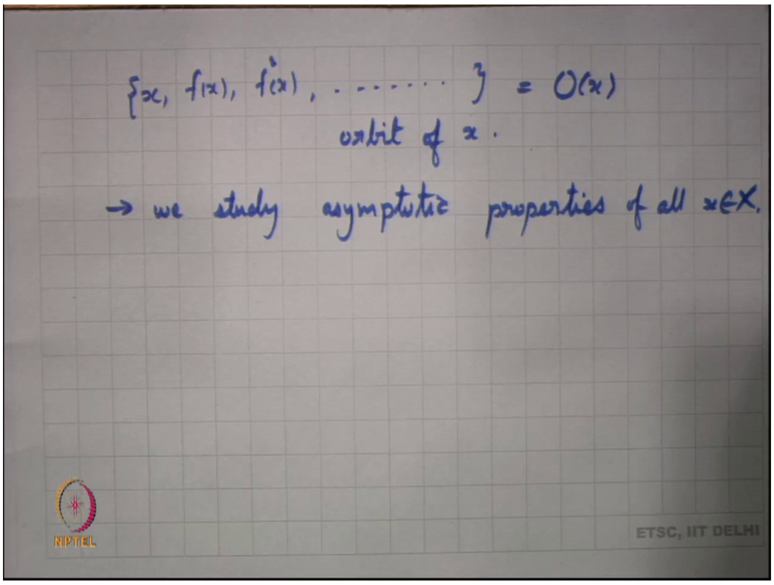

We know that under of under the action of f x moves to f x. Then again under the action of f it moves to f square x that is the iteration of f with itself and so on. If I look into the sequence of this points this is essentially, what is called the orbit of x which we denote as ox. So, this is basically the orbit of x and what do we study in dynamical systems we study the asymptotic properties of all x in X.

Now, this is the asymptotic property, and maybe in the next lecture we would start with looking into what kind of properties. And how do we study these kinds of properties in the context of a very general sense and very abstract sense of metric spaces. So, we end our talk today here.