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NPTEL ONLINE CERTIFICATION COURSE

Stochastic Processes - I

Module 7: Brownian Motion and its Applications

Lecture-04

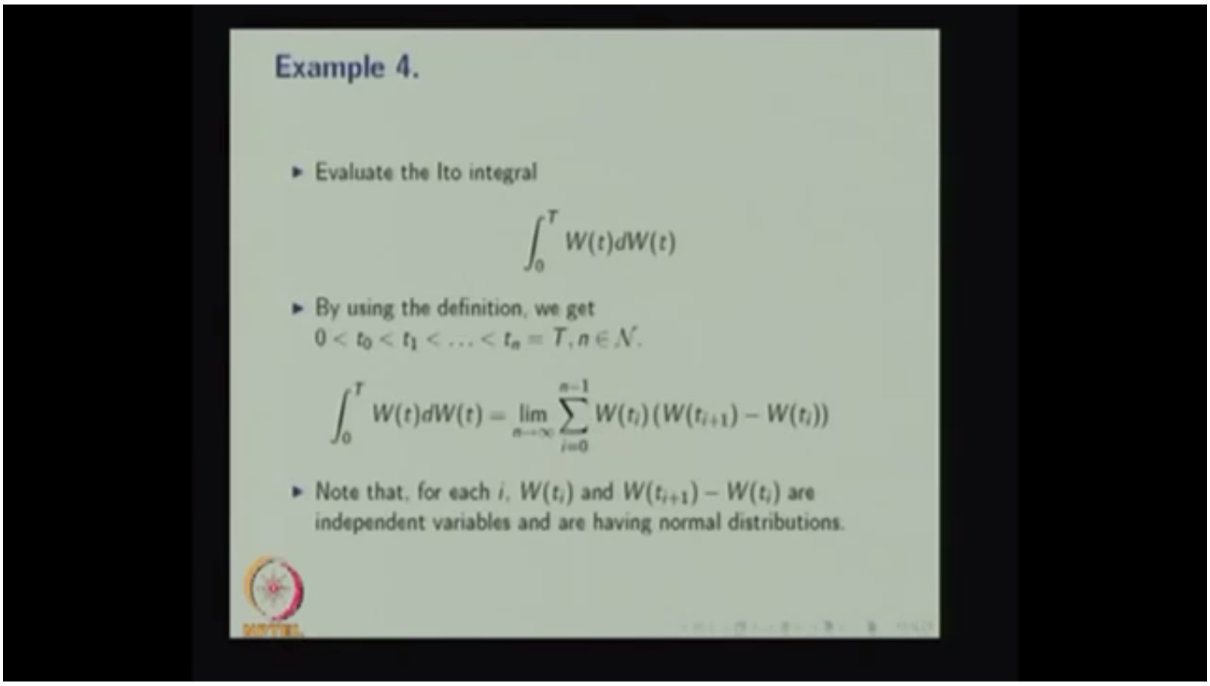
Ito Integrals contd

With

Professor S. Dharmaraja

Department of Mathematics

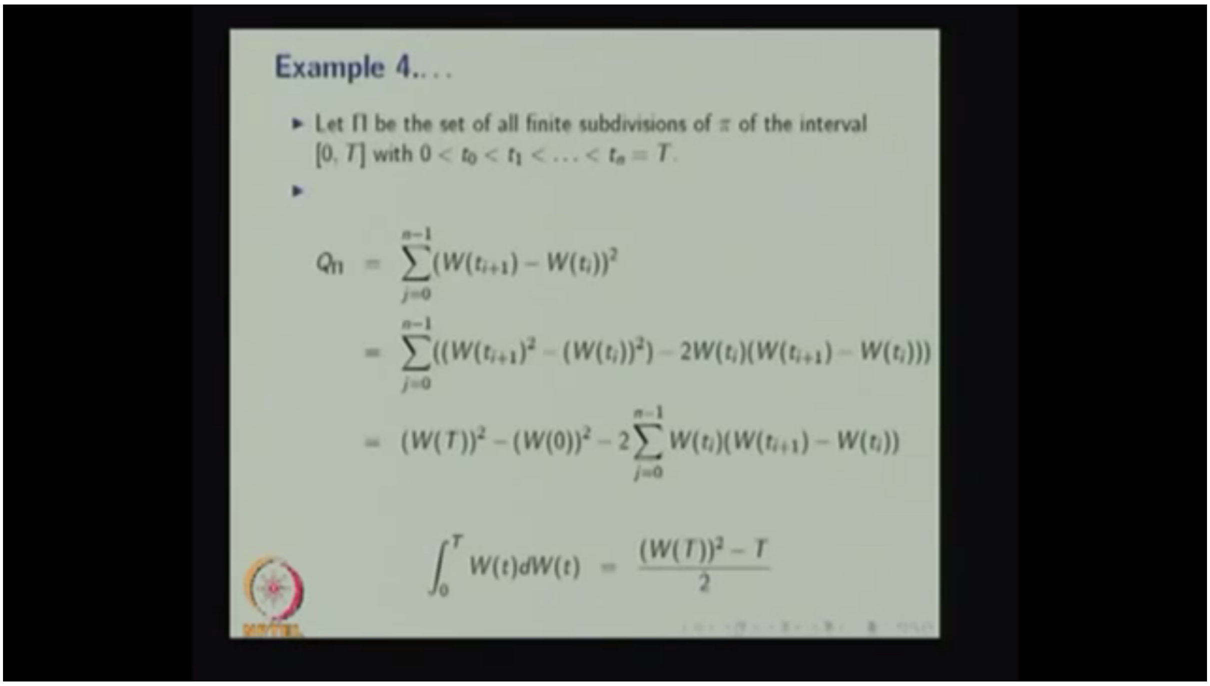
Indian Institute of Technology Delhi



We are moving into the next example, evaluate the Ito integral, $\int_0^T W(t)dW(t)$ that means integrant is W(t), integration with respect to W(t). So you can use the partition in the interval 0 to T into n pieces, n parts. Then as limit n tends to infinity, this integration is nothing but $\lim_{n\to\infty} ❑ \sum_{i=0}^{n-1} W(ti)$ and the difference of Ws. So the same definition we have given in the Ito integrals also.

So here, the Ito integral is well defined because W(t) that is integral which is adapted and also the mean square integral. Therefore, this is Ito integral. So the I(t) = $\int_0^T W(t)dW(t)$ that integral is Ito integral. But here, we are going for the upper limit is T, not the variable limit. So for each I, the way we have written $\lim_{n\to\infty} ¿$ summation with the of w Ws, I and Ws, and this is a W at the point t_i whereas this one is a $W(t_{i+1}) – W(t_i)$). Therefore, these two are the non-overlapping Ws. So the increments are independent. Therefore, $W(t_i)$ and $(W(t_{i+1}) – W(t_i))$ are the independent random variables or having the normal distributions.

We are using the property of Brownian motion therefore, $W(t_i)$ and $(W(t_{i+1}) – W(t_i))$ are independent random variables.



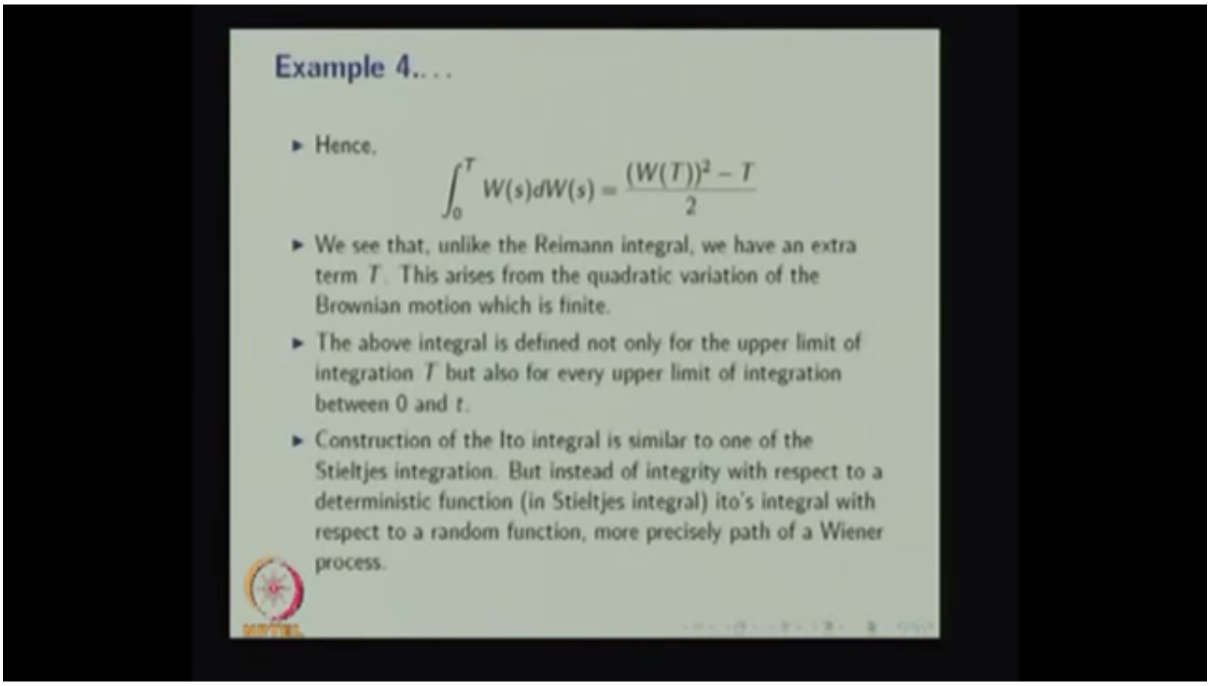
So what I am going to do here, let π be the set of all finite subdivisions of π of the interval 0 to T. Therefore, first we'll find out what is the Q_π that is different from what or what we want is different. What we want is $\lim_{n \to \infty} \sum_{i=0}^{n-1} W(ti)$ with the difference. But what we are defining now, Q_π is nothing but the difference of whole square. but what we want is this $W(t_i)$ with $W(t_{i+1}) - W(t_i)$. So we started with Q_π that is equal to the difference of whole square. The difference of whole square is the same as suppose you treat this as a, this as b. So $(a-b)^2$ is same as $a^2-b^2-2(a-b)-2b(a-b)$. So if you simplify, you'll get $(a-b)^2$. But if you expand this summation now, this will be the only the last term will be there and all other terms vanishes. So therefore, you will $(W(T))^2-(W(0))^2$. The last term and the first term will exist and all other terms will vanish because of minus squares.

Whereas this one, $-2 \sum_{i=0}^{n-1} W(ti)$ $(W(t_{i+1})-W(t_i))$. That's what we want as $n \to \infty$. But we know the result, limit norm of $\pi \to 0$ of Q_π will be T. That we have discussed in the lecture 1, when we discussed the quadratic variation of Brownian motion. In the lecture 1, we have discussed quadratic variation of Brownian motion in that Q_π is defined limit norm of $\pi \to 0$ Q_π will be T.

Therefore, what we want is a limit $n \to \infty$ of this quantity and that is in the third term in this Q_π. So now you apply a limit $n \to \infty$ in equation; so the left-hand side becomes the T, the right-hand becomes $(W(T))^2 - (W(0))^2$ you know that W(0) is 0 for a standard Brownian motion. Therefore, we will get only the first term; second term will be 0, third term is the unknown. That's the integration. $n \to \infty$ of summation that is nothing but the I(T).

Therefore, $\int_0^T W(t)d(t)$ that is nothing but in the left side we got T, therefore, we will get $(W(T))^2 - T$ is equal to, since it is -2 times, therefore it will be right.

Hence, the integration $\int_0^T W(t)\,d(t)$ becomes $\frac{(W(T))^2 - T}{2}$. So here we have used the quadratic variation of Brownian motion is T that we have used for the the interval 0 to T. Therefore, the integration 0 to T so we have used the Brownian motion quadratic variation between the interval 0 to T that is T. Therefore, the integration is going to be $\frac{(W(T))^2 - T}{2}$.



Hence, this integration, $\frac{(W(T))^2 - T}{2}$, we see that unlike Reimann integral, we have extra term T. If the integration is with respect to bounded variation differentiable function then you won't have the term T here. Just it is $\frac{(W(T))^2}{2}$. Suppose you replace W(s) by x, therefore you'll get

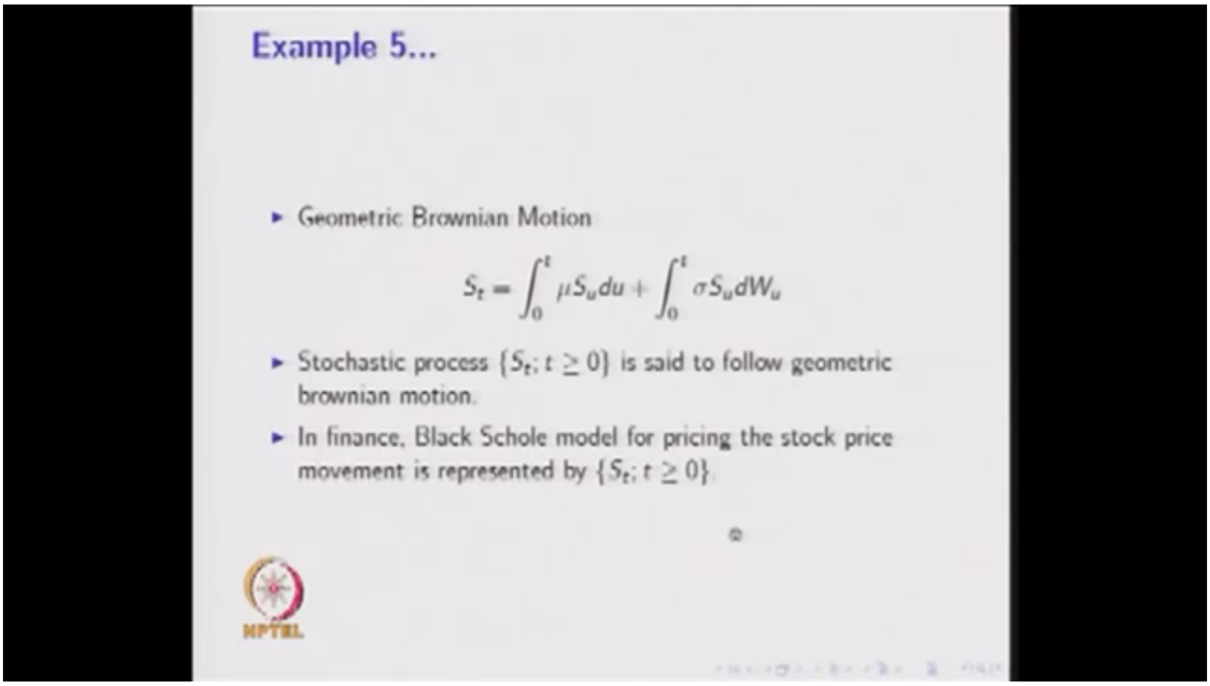
$\int_0^T X(x)\,d(x)$ that is nothing but $x^2/2$ whenever the integrant is bounded variation as well as differentiable function. That is nothing but the Riemann integral. So in the Riemann integral you won't have the extra term T/2, but here we have the extra term minus T/2.

So this arises from the quadratic variation of a Brownian motion which is finite whereas if it is a real value function which is bounded then the quadratic variation is going to be 0. But here, the quadratic variation which is finite value, therefore you are getting minus -T/2.

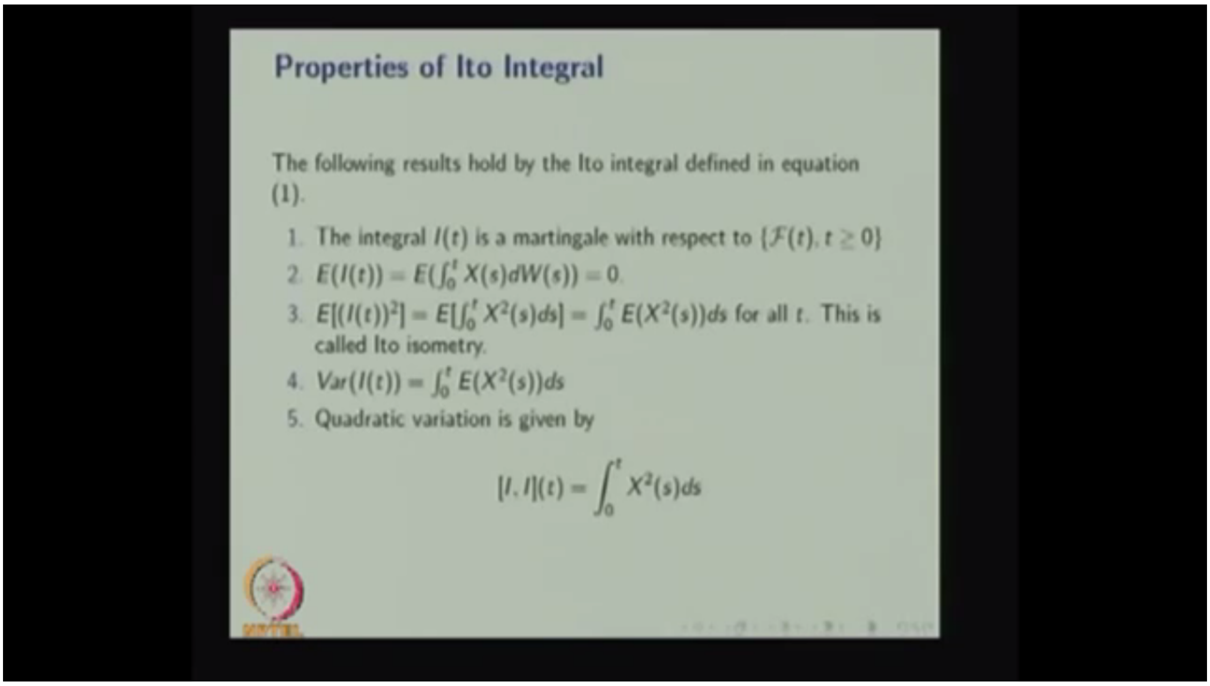
So the next remark, the above integral is defined not only for the upper limit integration, T, but also every upper limit of integration between 0 to T. In this integration, we have done the upper limit as T, therefore we used the quadratic variation of Brownian motion between the interval 0 to T that is T. But instead of that, we can go for variable, t, also. In that case, here you have to use the quadratic variation of Brownian motion between the interval 0 to t and that will be t itself. That's a variable t.

Therefore, this integration will be $\int_0^t W(s)\,dW(s)$ is nothing but s is nothing but $\frac{(W(T))^2 - T}{2}$ where both the ts are small variable.

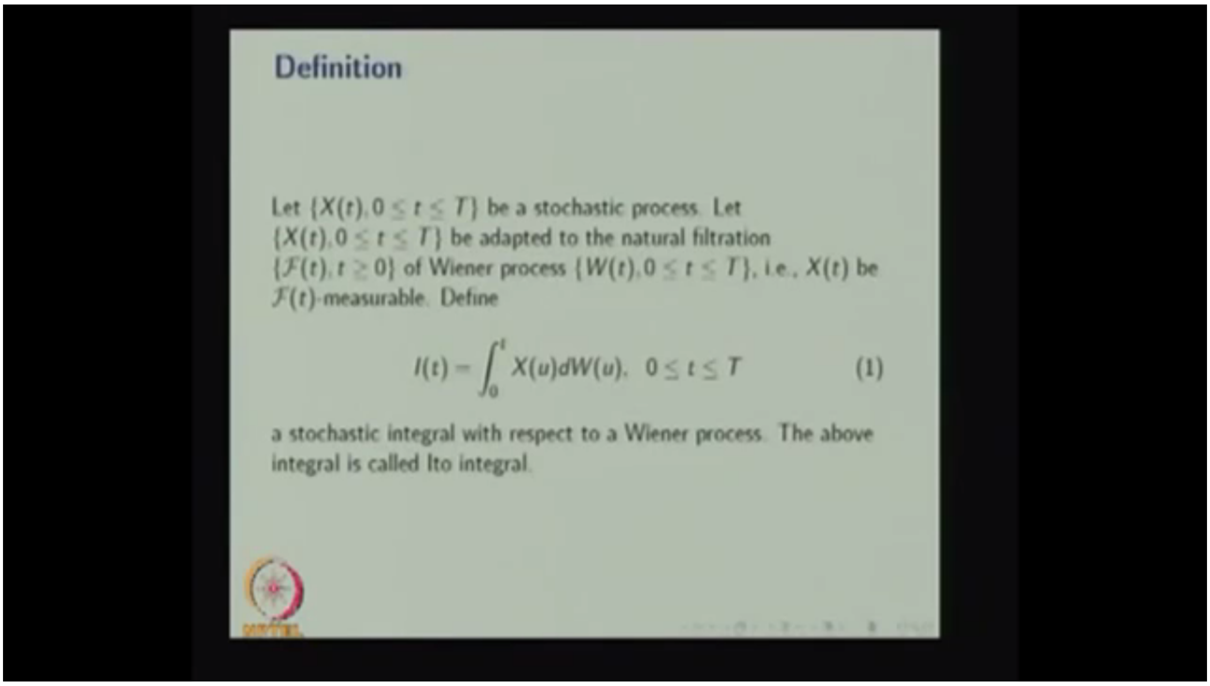
You see the construction of Ito integral is similar to one of the Stieltjes integration, but instead of integrating with respect to the deterministic function the Ito integral with respect to the random function most precisely path of Weiner process. So that is a difference between the Ito integral and the usual integral. The usual integral is with respect to the deterministic function whereas in the Ito integral, the integration with respect to the path of Brownian motion or Weiner process which is unbounded variation and nowhere differential. Therefore, the whole integration is different.



Consider this stochastic integral. In this, first integral is a Riemann integral of stochastic integrant whereas the second integral is a Ito integral. This equation is very useful in finance. In the Black Schole model for pricing, the stock price is seeming to follow geometric Brownian motion.



Now we are discussing the few properties of Ito integrals. Still we have used a few properties in the examples, but now you can understand how the properties will be used in the examples. The following results holds by the Ito integral defined in the equation 1.



In the equation 1, we have discussed – yeah, this is the Ito integral which we have discussed in the first equation. So this we are referring. Yes, the first property, the integral I(t) T is a martingale with respect to the filtration F(t). This is a very important property. If you have a Ito integral that means the integrant is adopted and the mean square integral and the Ito integral is a stochastic process and that stochastic process is having the martingale property with respect to the filtration F(t). The same F(t) the integrant is adopted also. With respect to the same filtration, the integrant is adopted also.

So the proof is if you want to verify the stochastic processes martingale you have to check three properties; the first property is it has to be the stochastic process for fixed T, it has to be integrable. Then third property the conditional expectation has to satisfy the equality property. So if these three properties are satisfied by the stochastic process then we say the stochastic process has the martingale property. So here I(t) is a stochastic process for t≥0.

So first, we have to verify it is integrable. So you can find out the expectation of I(t) or every T that is going to be a finite value. Then the second one, I(t) has to be -- for fixed T, I(t) has to be a F(t) measurable that is adopted. So for fixed already the integrant is adopted, the integration with respect to W(T) and W(T) is also adapted to F(t) one can prove, the I(t) is also adapted to the filtration F(t).

The third condition, expectation of I(t) given F(s) where s less than t that is same as I(s). You have to prove expectation of I(t) given F(s) where s is less than t that is equal to I(s) if that is proved for all t.

Since these three properties of martingale is satisfied, I(t) will be a martingale with respect to the filtration F(t). Here, I am not discussing the proof, here I am not giving the proof, but I have discussed whatever all the properties has to be verified.

Second property, expectation of I(t) that is same as a expectation of 0 to t, integration the, definition that will be 0. Since it is a martingale it will have a constant mean. Therefore, expectation of I(t) is same as expectation of I(0) and you know how to evaluate expectation of I(0) that will be 0.

And the third property, expectation of a $(I(t))^2$ that is same as expectation of $\int_0^t X^2(s)\,ds$ that is same as $\int_0^t E(X^2(s))\,ds$ So the expectation and the integration interchange the place. This is called Ito isometry. So this is a very important property of second order expectation that is the integrant and earlier you have expectation of $(I(t))^2$ that is nothing but $\int_0^t X^{2ds}$ this is nothing but the Reimann integral.

For fixed s, the expectation of $X^2(s)$ is the function of s. So you are integrating $\int_0^t E(X^2(s))\,ds$.

So this is nothing but the Reimann integral.

Now based on the property number 2 and 3, you can conclude Var(I(t)) is nothing but $\int_0^t E(X^2(s))\,ds$ because the E(I(t))=0, therefore $Var(I(t)) = E(I(t)^2) - E(I(t))^2$ that is same as

$X^2(s)ds$

$E¿$

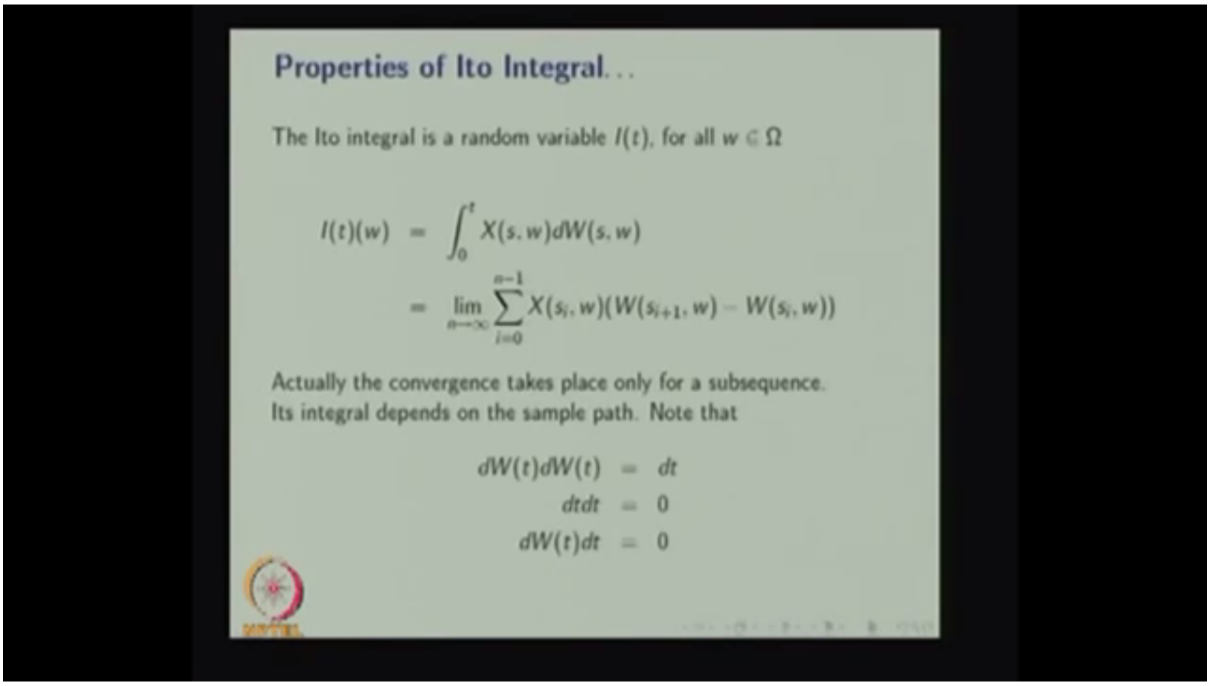
$\int_0^t ¿$.

One can find the quadratic variation of I(t) also because I(t) is a stochastic process. So you can find out the quadratic variation between the interval 0 to t. So [I,I] between the interval 0 to t, that means a second order variation between the interval 0 to t for the function I(t).

I(t) is a Ito integral whereas the quadratic variation is nothing but integration $\int_0^t X^2(s)ds$.

So this is the Reimann integral.

So I have now given the proof of the fifth property also. We are just stating the results of Ito integral and we have used a few properties in the examples.

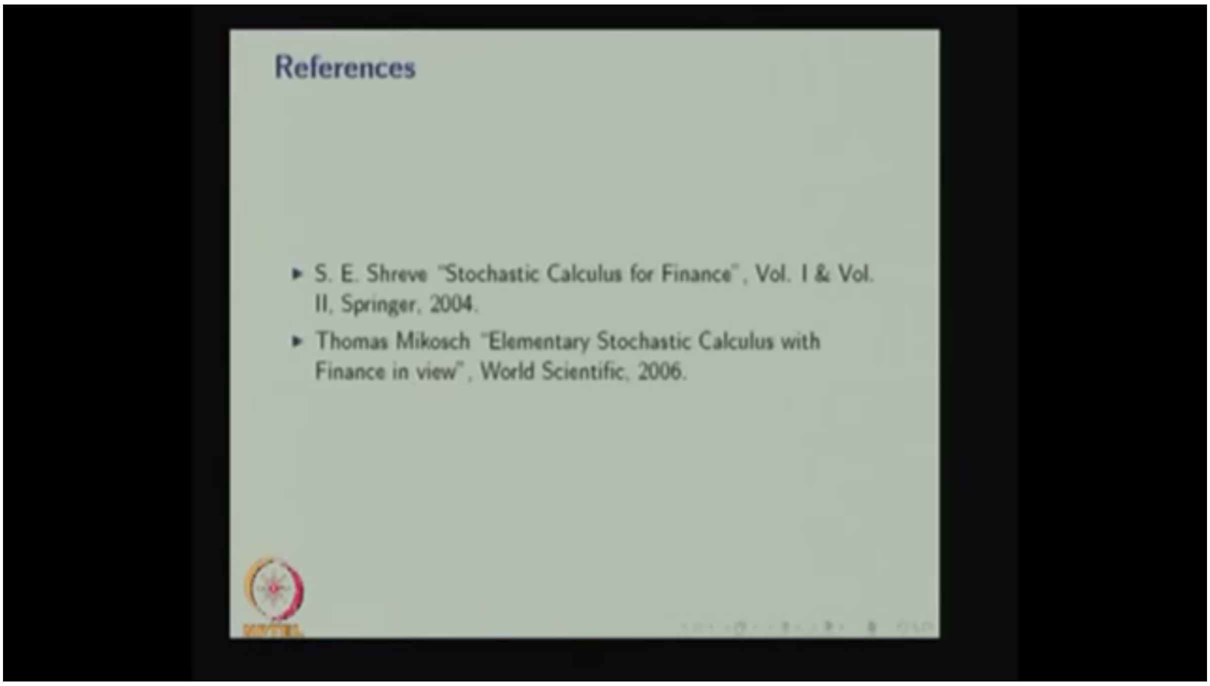


The Ito integral is a random variable for all T, for all W belonging to Ω. Therefor,e you can write down, I(t)(w) that is nothing but limit $n \to \infty$, the value $X(s_i)$, the difference of Ws. You know that X is a random variable, the difference of Ws is also a random variable. Therefore, for fixed n, this summation is equal to 0 to n-1 will be nothing but a random variable.

We are finding limit $n \to \infty$ of this summation. Therefore, actually the convergence takes place only for the subsequence. That means I(t) is a random variable and that random variable is nothing but the convergence of the right-hand side, limit $n \to \infty$ of this summation and its integral depends on the sample path because you are finding the difference of $W(s_{i+1})$-$W(s_i)$ therefore, this integral depends on the sample path.

Also, we need these following properties the dW(t)dW(t) will be dt that is nothing but the quadratic variation of Brownian motion is nothing, but a finite value if you are finding the increment between the interval 0 to t then the quadratic variation is small t, whereas if it is a real value function, which is a differentiable, then dtdt, the quadratic variation of the function t will be 0. And the mixed variation, cross-variation dW(t)dt that will be 0.

So if you find out the quadratic variation with the W(t) that will be the t, whereas a quadratic variation with the function t will be 0 and the cross-variation will be 0. So this result will be used when you are finding the I(t).



Here are the important references for the Ito integrals. In the next class, we will consider the Ito formulas.